Turk. J. Math. Comput. Sci.
17(1)(2025) 243–263
© MatDer
DOI : 10.47000/tjmcs.1677588

A Novel Interval-Valued Intuitionistic Fuzzy Neutrosophic Framework for Addressing Income Inequality in Multi-Criteria Decision-Making

Nazli Büşra Karaoğlu[1,2], Mehmet Ünver [2], Murat Olgun[2,*]

[1] *Graduate School of Natural and Applied Sciences, Ankara University, Ankara, Turkey.*
[2] *Department of Mathematics, Faculty of Science, Ankara University, 06560 Ankara, Turkey.*

Received: 16-04-2025 • Accepted: 11-06-2025

Abstract. Income equality plays a fundamental role in ensuring sustainable development and social welfare globally. The increase in economic inequality threatens not only the living standards between individuals but also societal harmony and economic stability. In many developed and developing countries, injustices in income distribution exacerbate social tensions and negatively affect long-term economic growth. This study introduces a novel multi-criteria decision-making framework based on interval-valued intuitionistic fuzzy neutrosophic sets (IVIFNSs) to evaluate and compare income equality across selected Organisation for Economic Co-operation and Development (OECD) countries. The IVIFNS model improves traditional fuzzy systems by representing truth, indeterminacy, and falsity degrees with interval-valued intuitionistic fuzzy values, offering a more nuanced approach to uncertainty. Algebraic operations are defined using triangular-norms and triangular-conorms, and new aggregation operators are developed. The proposed theory is applied to rank 25 OECD countries based on key income inequality indicators: Gini coefficient, Palma ratio, P90/P10, and P90/P50, using the Weighted Aggregated Sum Product Assessment (WASPAS) method. A comparative analysis demonstrates the method's effectiveness in capturing complex, uncertain data and producing robust country rankings for policy evaluation.

2020 AMS Classification: 03E72,03E75,91B06

Keywords: Interval-valued intuitionistic fuzzy neutrosophic set, WASPAS, multi-criteria decision-making, income equality.

1. Introduction

Income equality is a critical concept that involves ensuring fairness in the distribution of income among individuals within a society. It plays a central role in determining the economic and social well-being of nations, as the extent to which income is distributed equitably significantly impacts both individual lives and societal structures. Income inequality, characterized by a widening gap between the rich and the poor, can have detrimental effects on economic stability and social harmony. It often leads to increased poverty rates, diminished access to essential services such as education and healthcare, and an overall decline in living standards. Moreover, high levels of income inequality can perpetuate disparities in opportunities, creating barriers to upward mobility for disadvantaged groups [44]. Achieving income equity is not merely a matter of economic growth, but it is deeply intertwined with broader social justice principles, such as equal opportunity, protection of individual rights, and the promotion of fairness in societal systems.

*Corresponding Author
Email addresses: nbkaraoglu@ankara.edu.tr (N.B. Karaoğlu), munver@ankara.edu.tr (M.Ünver), olgun@ankara.edu.tr (M. Olgun)

A fair income distribution enables individuals to meet their basic needs, contributing to the overall prosperity and long-term sustainable development of a society. Policies aimed at fostering income equity often involve strengthening social safety nets, reforming taxation systems to ensure fairness, and ensuring equal access to employment opportunities. In this regard, the pursuit of income equity is essential for maintaining economic stability and societal cohesion[46].

Furthermore, income equity plays a significant role in improving social equality and reinforcing democratic values. High levels of income inequality often exacerbate class divisions, leading to social polarization and weakening social trust. On the other hand, equitable income distribution fosters solidarity among individuals, improves living conditions, and enhances overall societal welfare. Therefore, income equity is not only an economic necessity but also a key factor in ensuring social and political stability. The realization of income equity is thus crucial for both policymakers and society as a whole, as it is integral to fostering a more just, stable, and prosperous future [40].

As illustrated in Figure 1, Kaasa [27] identified key determinants of income equality, grouping them into five categories: economic development, demographic factors, political factors, cultural and environmental factors, and macroeconomic factors.

- **Economic Development:** The level of economic growth and industrialization significantly influences income distribution. A well-developed economy with diverse employment opportunities and high productivity levels tends to promote income equity, whereas economic stagnation or reliance on a single sector may exacerbate disparities.
- **Demographic Factors:** Population characteristics such as age distribution, education levels, urbanization rates, and migration patterns affect income equity. A younger and more educated workforce typically contributes to a fairer income distribution, while aging populations and high dependency ratios can create financial pressures that lead to greater inequality.
- **Political Factors:** Government policies, taxation systems, labor laws, and social welfare programs play a crucial role in determining income equity. Progressive taxation, strong social safety nets, and policies promoting equal opportunities can reduce disparities, while weak regulatory frameworks and corruption can exacerbate income inequality.
- **Cultural and Environmental Factors:** Social norms, cultural values, and environmental conditions influence income equity by shaping labor market participation and access to resources. Gender roles, discrimination, and environmental degradation can create barriers to income distribution, affecting vulnerable groups disproportionately.
- **Macroeconomic Factors:** Inflation, interest rates, trade policies, and monetary policies impact income equity by influencing wages, employment rates, and access to financial resources. Economic instability, high inflation, and trade imbalances can widen income gaps, while stable macroeconomic conditions support more equitable income distribution.

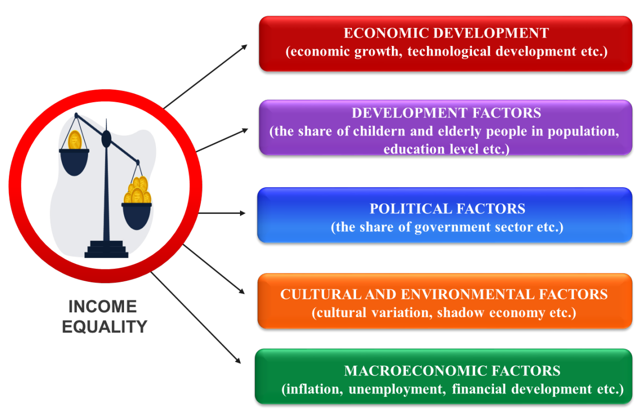

Figure 1. Fundamental factors affecting income equality

Income inequality has been increasing in many developed countries in recent years. Figure 2 shows that in the 1980s, the average disposable income of the top 10% in Organisation for Economic Co-operation and Development (OECD) countries was nearly seven times greater than that of the bottom 10%. This ratio increased to eight times in the 1990s and nine times in the 2000s. Today, income inequality has reached its highest level in the past 30 years, with the richest 10% earning 9.6 times more than the poorest 10% [29].

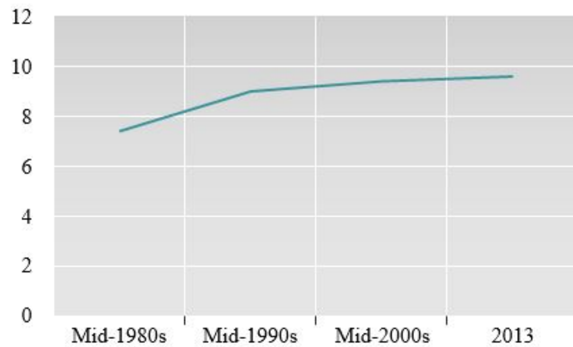

Figure 2. Income ratio between top and bottom deciles in OECD countries

A summary of various case studies focused on measuring income inequality across different years, environments, and methods is presented in Table 1. The studies span from 1995 to 2023, with a mix of fuzzy and crisp set approaches employed in the analyses. The methodologies used include fuzzy logic, panel data analysis, statistical analysis, econometric analysis, fuzzy nonlinear regression, and correlation analysis. Authors from various backgrounds, contributed to these studies, which explore different dimensions of income inequality, ranging from its measurement and causes to its impact on social factors such as poverty, information and communications technology (ICT) and life expectancy.

Table 1. Literature review on income equality/inequality studies

Year	**Case Study**	**Environment**	**Method(s)**	**Author(s)**
1995	Measurement of income inequality	Fuzzy sets	Fuzzy logic	Ok [38]
2003	Income inequality in OECD Countries	Crisp sets	Panel data analysis	Atkinson [7]
2010	Examining income equality through poverty	Fuzzy sets	Fuzzy logic	Giordani and Giorgi [20]
2011	Addressing how income inequality is measured	Crisp sets	Statistical analysis	Cowell [15]
2015	Measuring inequality in Tunisia	Fuzzy sets	Fuzzy logic	Hasnaoui and Belhadj [23]
2015	Causes and consequences of income inequality	Crisp sets	Econometric analysis	Dabla-Norris et al. [16]
2017	Investigating the dynamics of income inequality	Crisp sets	Panel data analysis	Alvaredo et al. [1]
2021	Evaluating the impact of ICT on income distribution	Fuzzy sets	Fuzzy nonlinear regression	Ashraf Ganjoei et al. [3]
2023	Explaining the relation between life expectancy and income inequality	Fuzzy sets	Correlation analysis	Rutkowska et al. [43]

The concept of fuzzy set (FS), introduced by Zadeh [61], provides a mathematical framework to represent and reason about vague or imprecise information. In contrast to classical set theory, which categorizes elements in a binary fashion—either belonging to a set or not—fuzzy sets enable partial membership, effectively capturing the uncertainty and vagueness found in real-world situations. The concept of intuitionistic fuzzy set (IFS), which is an extension of the concept of FS, was introduced by Atanassov [4]. This extension provides a more refined way to represent uncertainty, making IFSs especially useful in decision-making contexts that involve hesitation or uncertainty. Since Zadeh introduced FSs, many extensions of the FSs have been presented by researchers, such as interval-valued fuzzy sets (IVFSs) [62], neutrosophic sets (NSs) [45], Pythagorean fuzzy sets (PFSs) [59], spherical fuzzy sets [2], and linear diophantine multi-fuzzy soft set [28]. In real-world scenarios, uncertainty is not always confined to precise values but may instead exist within a range of possibilities. Therefore, Atanassov and Gargov [5] further generalized IFSs and IVFSs, defining interval-valued intuitionistic fuzzy sets (IVIFSs). Traditional a FS assigns a single value to represent the degree of membership of an element in a set in the range $[0, 1]$. However, an IVIFS addresses this limitation by allowing for the representation of membership degrees as intervals, offering a more comprehensive depiction of uncertainty.

IVIFSs are primarily utilized in the realm of decision-making, pattern recognition, and expert systems, where uncertainty and imprecision play a significant role. Wang et al. [52] introduced a methodology aimed at addressing decision-making challenges within the interval-valued intuitionistic fuzzy (IVIF) framework, employing interval-valued intuitionistic fuzzy values (IVIFVs) to represent all pertinent information, especially when attribute weightings are incomplete. Ye [60] developed the IVIF weighted arithmetic aggregation and geometric aggregation operators, along with an accuracy function presented for IVIFVs. Nayagam et al. [36] proposed a new accuracy function for ranking IVIFVs. Liu and Xie [32] defined a weighted score function and a weighted accuracy function of IVIFSs. Bai [8] introduced an improved score function designed to facilitate the efficient ranking of IVIFSs, alongside an IVIF Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method and this approach addresses decision-making scenarios where criterion weightings are fully disclosed. Xu [56] introduced some arithmetic and geometric aggregation operators and developed an approach for decision-making with IVIF information using these aggregation operators. Xu and Chen [57] defined some distance measures and similarity measures for IVIFSs. Xia et al.[55] studied the exploration of intuitionistic fuzzy aggregation operators founded on Archimedean triangular-norm (t-norm) and triangular-conorm (t-conorm), extending the concepts of Algebraic t-norm and t-conorm. Meanwhile, Wang, Liu and Qin [54] alongside Wang [50] introduced IVIF aggregation operators, also rooted in Archimedean t-norm and t-conorm principles. Raj et al.[41] explored the concept of IVIF graph and its self-centered properties, evaluating the potential applications of these graphs in areas such as defuzzification and fuzzy control.

Smarandache [45] proposed NS theory as an extension of fuzzy set theory, incorporating three membership functions: truth, indeterminacy, and falsity. Unlike fuzzy sets, there are no constraints on the sum of these membership functions. Later, Wang et al. [51], Wang et. al [53], and Ye[60] restricted the three membership degrees within the neutrosophic set framework to single-valued and interval membership degrees. This led to the creation of distinct subclasses, namely the single-valued neutrosophic set, interval neutrosophic set, and simplified neutrosophic set. Ünver et al.[49] introduced the concept of intuitionistic fuzzy valued neutrosophic multi set (IFVNMS) by merging NSs and IFSs with multi sets. After that, Bozyiğit et al. [12] defined the concept of Pythagorean fuzzy valued neutrosophic set with the help of NSs and PFSs.

Multi-criteria decision-making (MCDM) is a fundamental decision-making approach that seeks to identify the most suitable alternative by evaluating multiple criteria simultaneously. This approach assists decision-makers in choosing the optimal option by taking into account various factors such as cost, time, quality, and other relevant parameters. MCDM techniques provide a structured framework to evaluate and prioritize alternatives based on their performance across each criterion. MCDM encompasses a wide range of tools and methods that can be applied in diverse fields, from finance to engineering design. By utilizing MCDM, decision-makers are able to make more informed, rational, and balanced decisions, leading to improved outcomes in complex situations. The Weighted Aggregated Sum Product Assessment (WASPAS) method, introduced by Zavadskas et al. [64] in 2012, is a technique used to solve MCDM problems. This method combines the Weighted Sum Model (WSM) and the Weighted Product Model (WPM). By merging the advantages of these two models, WASPAS allows for a balanced utilization of the different characteristics from both methods. Chakraborty and Zavadskas [14] applied the WASPAS method to solve some manufacturing decision-making problems. Ghorabaee et al [19] applied an extended WASPAS method for green supplier selection in a green supply chain and demonstrated that it provides more reliable results compared to other multi-criteria decision-making methods. Urosevic et al. [47] utilized this method to address the personnel selection problem for the sales manager position in the tourism sector. Mishra et al. [33] used the WASPAS method to examine the performance of mobile phone service providers. Similarly, Mishra and Rani [34] applied the WASPAS method to address the problem of selecting a health waste disposal site. Hussain et al. [25] provided a different perspective by evaluating hospitals with advanced artificial intelligence technologies for robotic surgery and medical diagnosis using the WASPAS method.

The aim of this study is to rank some of the OECD countries, given in Figure 3, based on income equality. To achieve this, an MCDM approach is performed using interval-valued intuitionistic fuzzy neutrosophic sets (IVIFNSs). In the MCDM process the WASPAS methodology is employed. Within this framework, the goal is to rank the decision alternatives in the most optimal way and generate accurate and reliable results. In the present study, 25 OECD countries are selected as alternatives. The criteria used for evaluation include the Gini coefficient, Palma ratio, p90/p10 ratio, and p90/p50 ratio (see, e.g., [35]).

- **Gini Coefficient:** The Gini coefficient [21] is a widely used indicator of income inequality, derived from the Lorenz curve. It is calculated as the ratio of the area between the Lorenz curve and the line of perfect equality to the total area beneath the line of perfect equality (see Figure 4). A Gini coefficient of 0 signifies

FIGURE 3. OECD countries

perfect income equality, where everyone earns the same, while a value of 1 indicates extreme inequality, with all income held by a single individual. As a key metric, the Gini coefficient offers critical insights into how income is distributed within a society.

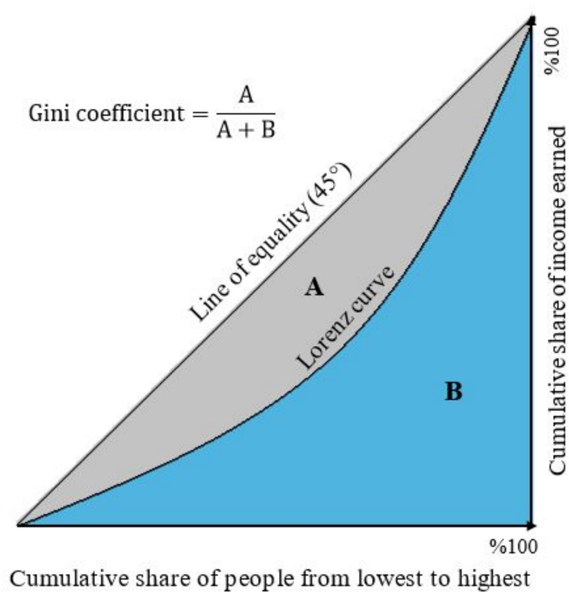

FIGURE 4. The Lorenz curve

- **Palma Ratio:** The Palma ratio [39] measures income inequality by comparing the income share of the richest 10% of the population to that of the poorest 40%. It provides a more focused perspective on income inequality, highlighting the disparity between the richest and the poorest segments of the population. A higher Palma ratio indicates a larger gap between the rich and the poor.

- **P90/P10 Ratio:** The P90/P10 (see e.g., [26]) ratio is the ratio of the income of the 90th percentile to the income of the 10th percentile. This metric shows how the income of the top 10% compares to the income of the bottom 10%. A higher P90/P10 ratio suggests a greater income disparity between the wealthiest and the poorest segments of society.
- **P90/P50 Ratio:** The P90/P50 ratio (see e.g., [13]) compares the income of the 90th percentile to the income of the 50th percentile, effectively comparing the income of the top 10% to the middle of the income distribution. This ratio offers insight into how income is distributed between the richest and the middle class, with a higher ratio indicating a larger income gap between the top earners and the median income.

These four criteria provide a comprehensive and multidimensional assessment of income distribution within a country by considering various aspects of income inequality. Each criterion unveils different dimensions of income disparities, highlighting not only the overall inequality but also the disparities between key income groups in the distribution. This approach improves the accuracy and reliability of cross-country comparisons of income inequality, enabling a more precise understanding of the variations in income distribution and the identification of crucial policy implications.

This paper presents two principal contributions:

- **Theoretical contributions:** The concept of IVIFS is extended using NS theory to introduce IVIFNSs and interval-valued intuitionistic fuzzy neutrosophic values (IVIFNVs). Through the use of IVIFNS, which consider IVIFSs for truth, indeterminacy, and falsity degrees, more sensitive assessments can be made in decision-making problems. In other words, these NSs enable the evaluation of truth, indeterminacy, and falsity degrees by modeling uncertainty with the use of IVIFVs, facilitating more accurate decision-making in situations involving uncertain data. Then we give some fundamental algebraic operations with the help of IVIFS fuzzy t-norms and t-conorms. Subsequently, a weighted arithmetic aggregation and a weighted geometric aggregation operator are defined for integrating IVIFNV classes using algebraic operations.
- **Applied contributions:** This study examines an MCDM problem directly related to income equity based on criteria. The WASPAS method is employed to assess income equity in selected OECD countries, ranking them accordingly. Moreover, the proposed fuzzy-based framework can be readily applied to a wide range of decision-making problems, spanning from the healthcare sector to the manufacturing sector.

The remainder of the paper is structured as follows: Section 2 presents some basic concepts used in the paper. Section 3 introduces the concept of IVIFNS. Moreover, some algebraic operations for IVIFNSs are defined using IVIF t-norm and t-conorm and a score function is introduce. In Section 4, a weighted arithmetic and a weighted geometric aggregation operator for IVIFNVs are presented. In Section 5, an MCDM problem on income equality of some OECD countries is addressed using the WASPAS method within the framework of IVIFNSs. In Section 6, a comparative analysis is conducted to evaluate country rankings from different perspectives. In the final section, Section 7, the study's findings are analyzed, conclusions are drawn, and suggestions for future research are offered.

2. Preliminaries

This section outlines essential definitions and fundamental properties that form the basis of the study. Throughout this paper, we assume that $X = \{x_1, \dots, x_n\}$ is a finite set and $\xi[0,1]$ is the set of all closed sub-intervals of $[0,1]$

Definition 2.1 ([5])**.** [5] An IVIFS A defined on X is given by

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x_i \in X\},$$

where $\mu_A, \nu_A : X \to \xi[0,1]$ for all $x \in X$, with the condition

$$0 \le \sup \mu_A(x) + \sup \nu_A(x) \le 1$$

for any $x_i \in X$. The lower and upper boundaries of $\mu_A(x)$ and $\nu_A(x)$ are denoted by $\mu_A^L(x), \mu_A^U(x), \nu_A^L(x)$, and $\nu_A^U(x)$, respectively.

Remark 2.2. Wang et al. [54] defined a partial order on the lattice

$$\mathcal{L} = \{([u,v],[x,y]) : [u,v],[x,y] \in \xi[0,1],\ v + y \le 1\},$$

with the ordering relation

$$([u_1, v_1], [x_1, y_1]) \sqsubseteq ([u_2, v_2], [x_2, y_2]) \iff [u_1, v_1] \leq_I [u_2, v_2] \text{ and } [x_2, y_2] \leq_I [x_1, y_1],$$

where the top and bottom elements of the lattice are given by

$$\top = ([1, 1], [0, 0]) \quad \text{and} \quad \bot = ([0, 0], [1, 1]),$$

respectively. The interval order relation $\leq_I$ used above is defined as

$$[u_1, v_1] \leq_I [u_2, v_2] \iff u_1 \leq u_2 \text{ and } v_1 \leq v_2,$$

as discussed in [10].

Now, we recall the notions of IVIF t-norm and t-conorm.

Definition 2.3 ([58])**.** An IVIF t-norm is a binary operation $\mathcal{T} : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$, which satisfies the following axioms for all $\psi_1, \psi_2, \psi_3 \in \mathcal{L}$:

(T1) $\mathcal{T}(\psi_1, \top) = \psi_1$,
(T2) $\mathcal{T}(\psi_1, \psi_2) = \mathcal{T}(\psi_2, \psi_1)$,
(T3) $\mathcal{T}(\psi_1, \mathcal{T}(\psi_2, \psi_3)) = \mathcal{T}(\mathcal{T}(\psi_1, \psi_2), \psi_3)$,
(T4) For all $\psi_1, \psi_1', \psi_2, \psi_2' \in \mathcal{L}$, if $\psi_1 \sqsubseteq \psi_1'$ and $\psi_2 \sqsubseteq \psi_2'$, then $\mathcal{T}(\psi_1, \psi_2) \sqsubseteq \mathcal{T}(\psi_1', \psi_2')$.

Definition 2.4 ([58])**.** An IVIF t-conorm is a binary operation $\mathcal{S} : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$, satisfying T2-T4 and for any $\psi_1 \in \mathcal{L}$:

(S1) $\mathcal{S}(\psi_1, \bot) = \psi_1$.

We now present the definition of IVIF negation as a generalization of classical fuzzy negation.

Definition 2.5 ([42])**.** An IVIF negation is a mapping $\mathcal{N} : \mathcal{L} \to \mathcal{L}$ that satisfies the following properties:

(N1) $\mathcal{N}(\bot) = \top$ and $\mathcal{N}(\top) = \bot$,
(N2) $\mathcal{N}(\psi_1) \sqsupseteq \mathcal{N}(\psi_2)$ whenever $\psi_1 \sqsubseteq \psi_2$, for all $\psi_1, \psi_2 \in \mathcal{L}$.

If $\mathcal{N}(\mathcal{N}(\psi)) = \psi$ for every $\psi \in \mathcal{L}$, then $\mathcal{N}$ is called an involutive negator. A typical example is the standard involutive negator $\mathcal{N}_s : \mathcal{L} \to \mathcal{L}$, defined by $\mathcal{N}_s((a, b)) = (b, a)$.

Definition 2.6 ([58])**.** Let $\mathcal{T}$ be an IVIF t-norm and $\mathcal{N}$ an IVIF negation. The operation $\mathcal{S} : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$, defined as

$$\mathcal{S}(\psi_1, \psi_2) = \mathcal{N}\left(\mathcal{T}(\mathcal{N}(\psi_1), \mathcal{N}(\psi_2))\right),$$

is called the dual t-conorm of $\mathcal{T}$ with respect to the negator $\mathcal{N}$.

Notwithstanding the existence of comparable research in [9, 17, 53], Proposition 1 and its comprehensive proof are detailed below.

Proposition 2.7. *Let T and S be any t-norm and t-conorm defined on the unit interval* $[0, 1]$*. Then $\mathcal{T} : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ defined by*

$$\mathcal{T}(\psi_1, \psi_2) = \left(\left[T\left(\mu^L_{\psi_1}, \mu^L_{\psi_2}\right), T\left(\mu^U_{\psi_1}, \mu^U_{\psi_2}\right)\right], \left[S\left(\nu^L_{\psi_1}, \nu^L_{\psi_2}\right), S\left(\nu^U_{\psi_1}, \nu^U_{\psi_2}\right)\right]\right)$$

is an IVIF t-norm and $\mathcal{S} : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ defined by

$$\mathcal{S}(\psi_1, \psi_2) = \left(\left[S\left(\mu^L_{\psi_1}, \mu^L_{\psi_2}\right), S\left(\mu^U_{\psi_1}, \mu^U_{\psi_2}\right)\right], \left[T\left(\nu^L_{\psi_1}, \nu^L_{\psi_2}\right), T\left(\nu^U_{\psi_1}, \nu^U_{\psi_2}\right)\right]\right)$$

is an IVIF t-conorm.

Proof. **(T1)** As T and S are a t-norm and t-conorm, respectively, it follows for any $\psi_1 \in \mathcal{L}$ that

$$\begin{aligned}\mathcal{T}(\psi_1, \top) &= \left(\left[T\left(\mu^L_{\psi_1}, 1\right), T\left(\mu^U_{\psi_1}, 1\right)\right], \left[S\left(\nu^L_{\psi_1}, 0\right), S\left(\nu^U_{\psi_1}, 0\right)\right]\right)\\ &= \left(\left[\mu^L_{\psi_1}, \mu^U_{\psi_1}\right], \left[\nu^L_{\psi_1}, \nu^U_{\psi_1}\right]\right)\\ &= \psi_1.\end{aligned}$$

(T2) The proof is straightforward and thus omitted.
(T3) Since T is a t-norm and S is a t-conorm, it holds for all $\psi_1, \psi_2, \psi_3 \in \mathcal{L}$ that

$$\begin{aligned}\mathcal{T}(\psi_1, \mathcal{T}(\psi_2, \psi_3)) &= \begin{array}{l}\left(\left[T\left(\mu^L_{\psi_1}, \mu^L_{\mathcal{T}(\psi_2,\psi_3)}\right), T\left(\mu^U_{\psi_1}, \mu^U_{\mathcal{T}(\psi_2,\psi_3)}\right)\right],\right.\\ \left.\left[S\left(\nu^L_{\psi_1}, \nu^L_{\mathcal{T}(\psi_2,\psi_3)}\right), S\left(\nu^U_{\psi_1}, \nu^U_{\mathcal{T}(\psi_2,\psi_3)}\right)\right]\right)\end{array}\\ &= \begin{array}{l}\left(\left[T\left(\mu^L_{\psi_1}, T\left(\mu^L_{\psi_2}, \mu^L_{\psi_3}\right)\right), T\left(\mu^U_{\psi_1}, T\left(\mu^U_{\psi_2}, \mu^U_{\psi_3}\right)\right)\right],\right.\\ \left.\left[S\left(\nu^L_{\psi_1}, S\left(\nu^L_{\psi_2}, \nu^L_{\psi_3}\right)\right), S\left(\nu^U_{\psi_1}, S\left(\nu^U_{\psi_2}, \nu^U_{\psi_3}\right)\right)\right]\right)\end{array}\\ &= \begin{array}{l}\left(\left[T\left(T\left(\mu^L_{\psi_1}, \mu^L_{\psi_2}\right), \mu^L_{\psi_1}\right), T\left(T\left(\mu^U_{\psi_1}, \mu^U_{\psi_2}\right), \mu^U_{\psi_1}\right)\right],\right.\\ \left.\left[S\left(S\left(\nu^L_{\psi_1}, \nu^L_{\psi_2}\right), \nu^L_{\psi_1}\right), S\left(S\left(\nu^U_{\psi_1}, \nu^U_{\psi_2}\right), \nu^U_{\psi_1}\right)\right]\right)\end{array}\\ &= \mathcal{T}(\psi_1, \mathcal{T}(\psi_2, \psi_3)).\end{aligned}$$

(T4) Let $\psi_1 \sqsubseteq \psi_1'$ and $\psi_2 \sqsubseteq \psi_2'$. Then we have

$$\mu^L_{\psi_1} \leq \mu^L_{\psi_1'} \text{ and } \mu^L_{\psi_2} \leq \mu^L_{\psi_2'}.$$

As T is a t-norm this inequalities yield that

$$T\left(\mu^L_{\psi_1}, \mu^L_{\psi_2}\right) \leq T\left(\mu^L_{\psi_1'}, \mu^L_{\psi_2'}\right). \tag{2.1}$$

On the other hand, we obtain

$$\mu^L_{\mathcal{T}(\psi_1,\psi_2)} = T\left(\mu^L_{\psi_1}, \mu^L_{\psi_2}\right)$$

and

$$\mu^L_{\mathcal{T}(\psi_1',\psi_2')} = \left(\mu^L_{\psi_1'}, \mu^L_{\psi_2'}\right),$$

which imply with (2.1) that

$$\mu^L_{\mathcal{T}(\psi_1,\psi_2)} \leq \mu^L_{\mathcal{T}(\psi_1',\psi_2')}. \tag{2.2}$$

Similarly we get

$$\mu^U_{\mathcal{T}(\psi_1,\psi_2)} \leq \mu^U_{\mathcal{T}(\psi_1',\psi_2')}. \tag{2.3}$$

From (2.2) and (2.3) we deduce

$$\mu_{\mathcal{T}(\psi_1,\psi_2)} \leq \mu_{\mathcal{T}(\psi_1',\psi_2')}. \tag{2.4}$$

Similarly, we have

$$\nu_{\mathcal{T}(\psi_1,\psi_2)} \geq \nu_{\mathcal{T}(\psi_1',\psi_2')}.$$

Combining that with (2.4) we have

$$\mathcal{T}(\psi_1, \psi_2) \sqsubseteq \mathcal{T}(\psi_1', \psi_2').$$

Therefore, $\mathcal{T}$ is an IVIF t-norm, and by analogous reasoning, $\mathcal{S}$ is an IVIF t-conorm. □

Remark 2.8. Additive generators play a pivotal role in constructing algebraic operators for t-norms and t-conorms. Specifically, continuous Archimedean t-norms and t-conorms on the unit interval $[0, 1]$ can be derived from such generators. These generators can also be employed to define IVIF t-norms and t-conorms. An Archimedean t-norm can be generated using a strictly increasing function $g : [0, 1] \to [0, \infty]$, where $g(1) = 0$ and $g(0) = 1$ [30, 31]. Thus, the corresponding IVIF t-norm is expressed as

$$\mathcal{T}(\psi_1, \psi_2) = \begin{array}{l}\left(\left[g^{-1}\left(g(\mu^L_{\psi_1}) + g(\mu^L_{\psi_2})\right), g^{-1}\left(g(\mu^U_{\psi_1}) + g(\mu^U_{\psi_2})\right)\right]\right.\\ \left.\left[h^{-1}\left(h(\nu^L_{\psi_1}) + h(\mu^L_{\psi_2})\right), h^{-1}\left(h(\nu^U_{\psi_1}) + h(\nu^U_{\psi_2})\right)\right]\right),\end{array}$$

where the dual t-conorm is given by

$$\mathcal{S}(\psi_1, \psi_2) = \begin{array}{l}\left(\left[h^{-1}\left(h(\mu^L_{\psi_1}) + h(\mu^L_{\psi_2})\right), h^{-1}\left(h(\mu^U_{\psi_1}) + h(\mu^U_{\psi_2})\right)\right]\right.\\ \left.\left[g^{-1}\left(g(\nu^L_{\psi_1}) + g(\mu^L_{\psi_2})\right), g^{-1}\left(g(\nu^U_{\psi_1}) + g(\nu^U_{\psi_2})\right)\right]\right)\end{array}$$

with respect to the standard negator $\mathcal{N}_s$, and $h(t) = g(1-t)$. Indeed,

$$\begin{aligned}
\mathcal{N}_s\left(\mathcal{T}\left(\mathcal{N}_s(\psi_1), \mathcal{N}_s(\psi_2)\right)\right) &= \mathcal{N}_s\left(\begin{array}{c} \mathcal{T}\left(\left(\left[\nu^L_{\psi_1}, \nu^U_{\psi_1}\right], \left[\mu^L_{\psi_1}, \mu^U_{\psi_1}\right]\right),\right. \\ \left.\left(\left[\nu^L_{\psi_2}, \nu^U_{\psi_2}\right], \left[\mu^L_{\psi_2}, \mu^U_{\psi 2}\right]\right)\right) \end{array}\right) \\
&= \mathcal{N}_s\left(\begin{array}{c} \left(\left[g^{-1}\left(g(\nu^L_{\psi_1}) + g(\nu^L_{\psi_2})\right), g^{-1}\left(g(\nu^U_{\psi_1}) + g(\nu^U_{\psi_2})\right)\right]\right. \\ \left.\left[h^{-1}\left(h(\mu^L_{\psi_1}) + h(\mu^L_{\psi_2})\right), h^{-1}\left(h(\mu^U_{\psi_1}) + h(\mu^U_{\psi_2})\right)\right]\right) \end{array}\right) \\
&= \begin{array}{c} \left(\left[h^{-1}\left(h(\mu^L_{\psi_1}) + h(\mu^L_{\psi_2})\right), h^{-1}\left(h(\mu^U_{\psi_1}) + h(\mu^U_{\psi_2})\right)\right]\right. \\ \left.\left[g^{-1}\left(g(\nu^L_{\psi_1}) + g(\nu^L_{\psi_2})\right), g^{-1}\left(g(\nu^U_{\psi_1}) + g(\nu^U_{\psi_2})\right)\right]\right) \end{array} \\
&= \mathcal{S}(\psi_1, \psi_2).
\end{aligned}$$

In this formulations, $\mathcal{T}$ and $\mathcal{S}$ are said to be generated by the additive functions g and h, respectively.

Example 2.9. Let the generator functions be defined as $g(t) = -\log(t)$ and $h(t) = -\log(1-t)$. Then, the corresponding algebraic IVIF t-norm takes the form

$$\mathcal{T}(\psi_1, \psi_2) = \left([\mu^L_{\psi_1}\mu^L_{\psi_2},\ \mu^U_{\psi_1}\mu^U_{\psi_2}],\ [\nu^L_{\psi_1} + \nu^L_{\psi_2} - \nu^L_{\psi_1}\nu^L_{\psi_2},\ \nu^U_{\psi_1} + \nu^U_{\psi_2} - \nu^U_{\psi_1}\nu^U_{\psi_2}]\right),$$

while the associated algebraic IVIF t-conorm, with respect to $\mathcal{N}_s$, is given by

$$\mathcal{S}(\psi_1, \psi_2) = \left([\mu^L_{\psi_1} + {\mu_{\psi 2}}^L - {\mu_{\psi 1}}^L{\mu_{\psi 2}}^L, {\mu_{\psi 1}}^U + {\mu_{\psi 2}}^U - {\mu_{\psi 1}}^U{\mu_{\psi 2}}^U], [{\nu_{\psi 1}}^L{\nu_{\psi 2}}^L, {\nu_{\psi 1}}^U{\nu_{\psi 2}}^U]\right).$$

These are known as the Algebraic t-norm and t-conorm.

3. IVIFNSs and Aggregation Operators

Ünver et al. [49] introduced IFVNMSs, a new fuzzy set concept combining IFSs and NSs. Inspired by this, we extend the theory by combining IVIFS theory with NSs, leading to the introduction of IVIFNSs and IVIFNVs. We then define algebraic operations for these new concepts, employing IVIF t-norms and t-conorms. Finally, we utilize these algebraic operations to define weighted arithmetic and geometric aggregation operators.

Definition 3.1. An IVIFNS A on X is defined by

$$A = \left\{\left\langle x_j, \mathbb{T}^j_A, \mathbb{I}^j_A, \mathbb{F}^j_A\right\rangle : j = 1, \ldots, n\right\},$$

where $\mathbb{T}^j_A, \mathbb{I}^j_A$ and $\mathbb{F}^j_A$ are the truth, the indeterminacy, and falsity membership degrees given with IVIFVs, respectively, for $j = 1, \ldots, n$, i.e.,

$$\begin{aligned}
\mathbb{T}^j_A &= \left(\left[\mu^L_{A,T}(x_j), \mu^U_{A,T}(x_j)\right], \left[\upsilon^L_{A,T}(x_j), \upsilon^U_{A,T}(x_j)\right]\right) \\
\mathbb{I}^j_A &= \left(\left[\mu^L_{A,I}(x_j), \mu^U_{A,I}(x_j)\right], \left[\upsilon^L_{A,I}(x_j), \upsilon^U_{A,I}(x_j)\right]\right) \\
\mathbb{F}^j_A &= \left(\left[\mu^L_{A,F}(x_j), \mu^U_{A,F}(x_j)\right], \left[\upsilon^L_{A,F}(x_j), \upsilon^U_{A,F}(x_j)\right]\right)
\end{aligned}$$

are IVIFVs. For simplicity, the truth $\mathbb{T}^j_A$ is expressed by

$$\begin{aligned}
\mathbb{T}^j_A &= \left(T^{t_j}_A, T^{f_j}_A\right), \\
\mathbb{I}^j_A &= \left(I^{t_j}_A, I^{f_j}_A\right), \\
\mathbb{F}^j_A &= \left(F^{t_j}_A, F^{f_j}_A\right)
\end{aligned}$$

Here, $T^{t_j}_A, I^{t_j}_A$, and $F^{t_j}_A$ represent the membership intervals, while $T^{f_j}_A, I^{f_j}_A$, and $F^{f_j}_A$ represent the non-membership intervals. For a given index $j = 1, ..., n$, the term $\tau_j := \langle \mathbb{T}^j_A, \mathbb{I}^j_A, \mathbb{F}^j_A \rangle$ represents an IVIFNV. When the index j is clear from the context, we simplify the notation from $\tau_j = \langle \mathbb{T}^j_A, \mathbb{I}^j_A, \mathbb{F}^j_A \rangle$ to

$$\tau = \langle \mathbb{T}_\tau, \mathbb{I}_\tau, \mathbb{F}_\tau \rangle.$$

Example 3.2. Let $X = \{x_1, x_2\}$. The expression given by

$$A = \left\{ \begin{array}{c} \langle x_1, ([0.2, 0.3], [0.1, 0.6]), ([0.1, 0.7], [0.2, 0.3]), ([0.1, 0.4], [0.1, 0.5]) \rangle, \\ \langle x_2, ([0.1, 0.8], [0.2, 0.2]), ([0.2, 0.5], [0.1, 0.4]), ([0.1, 0.1], [0.2, 0.9]) \rangle \end{array} \right\}$$

is an IVIFNS.

To facilitate the comparison and ranking of IVIFNVs, we introduce a score function. This function transforms the complex, interval-based information within an IVIFNV into a single numerical value, thereby simplifying the evaluation process. The score function is designed to consider the interplay between the truth, indeterminacy, and falsity membership degrees. The following definition formalizes this score function, outlining how it is calculated from the interval-based membership values of an IVIFNV. Critically, a higher score value indicates a better or more desirable IVIFNV.

Definition 3.3. Let $\tau = \langle \mathbb{T}_\tau, \mathbb{I}_\tau, \mathbb{F}_\tau \rangle$ be an IVIFNV. A score function is defined by

$$S(A) = \frac{2 + \mathbf{T}_\tau - \mathbf{I}_\tau - \mathbf{F}_\tau}{3},$$

where

$$\mathbf{T}_\tau = \frac{2 + \mu^L_{\tau,T} + \mu^U_{\tau,T} - \nu^L_{\tau,T} - \nu^U_{\tau,T}}{4}$$

$$\mathbf{I}_\tau = \frac{2 + \mu^L_{\tau,T} + \mu^U_{\tau,T} - \nu^L_{\tau,T} - \nu^U_{\tau,T}}{4}$$

$$\mathbf{F}_\tau = \frac{2 + \mu^L_{\tau,T} + \mu^U_{\tau,T} - \nu^L_{\tau,T} - \nu^U_{\tau,T}}{4}.$$

Now, we define some algebraic operations for IVIFNVs using IVIF t-norms and t-conorms.

Definition 3.4. Let $\tau = \langle \mathbb{T}_\tau, \mathbb{I}_\tau, \mathbb{F}_\tau \rangle$ and $\rho = \langle \mathbb{T}_\rho, \mathbb{I}_\rho, \mathbb{F}_\rho \rangle$ be two IVIFNVs, let $\mathcal{T}$ be an IVIF t-norm, and let $\mathcal{S}$ be a dual IVIF t-conorm of $\mathcal{T}$. Then

$$\tau \oplus \rho = \Big\langle (\mathcal{S}(T^t_\tau, T^t_\rho), \mathcal{T}(T^f_\tau, T^f_\rho)), (\mathcal{T}(I^t_\tau, I^t_\rho), \mathcal{S}(I^f_\tau, I^f_\rho)), (\mathcal{T}(F^t_\tau, F^t_\rho), \mathcal{S}(F^f_\tau, F^f_\rho)) \Big\rangle,$$

and

$$\tau \otimes \rho = \Big\langle (\mathcal{T}(T^t_\tau, T^t_\rho), \mathcal{S}(T^f_\tau, T^f_\rho)), (\mathcal{S}(I^t_\tau, I^t_\rho), \mathcal{T}(I^f_\tau, I^f_\rho)), (\mathcal{S}(F^t_\tau, F^t_\rho), \mathcal{T}(F^f_\tau, F^f_\rho)) \Big\rangle.$$

It is demonstrated hereafter that the algebraic operations maintain the IVIFNV structure.

Proposition 3.5. *Let τ and ρ be two IVIFNVs, and let $\mathcal{T}$ be an IVIF t-norm and $\mathcal{S}$ be a dual IVIF t-conorm of $\mathcal{T}$. Then $\tau \oplus \rho$ and $\tau \otimes \rho$ are IVIFNVs.*

Proof. Since $\mathcal{T}$ and $\mathcal{S}$ are defined via functions that have range $\mathcal{L}$, and $\tau \oplus \rho$ and $\tau \otimes \rho$ are defined via T and S, the proof follows trivially.

□

Remark 3.6. If g is an additive generator of an Archimedean t-norm and $h(t) = g(1 - t)$, then we obtain

$$\tau \oplus \rho = \left\langle \begin{array}{l} \left(\left[h^{-1}\left(h\left(\mu^L_{\tau,T}\right) + h\left(\mu^L_{\rho,T}\right)\right), h^{-1}\left(h\left(\mu^U_{\tau,T}\right) + h\left(\mu^U_{\rho,T}\right)\right)\right],\right. \\ \left.\left[g^{-1}\left(g\left(\nu^L_{\tau,T}\right) + g\left(\nu^L_{\rho,T}\right)\right), g^{-1}\left(g\left(\nu^U_{\tau,T}\right) + g\left(\nu^U_{\rho,T}\right)\right)\right]\right), \\ \left(\left[g^{-1}\left(g\left(\mu^L_{\tau,I}\right) + g\left(\mu^L_{\rho,I}\right)\right), g^{-1}\left(g\left(\mu^U_{\tau,I}\right) + g\left(\mu^U_{\rho,I}\right)\right)\right],\right. \\ \left.\left[h^{-1}\left(h\left(\nu^L_{\tau,I}\right) + h\left(\nu^L_{\rho,I}\right)\right), h^{-1}\left(h\left(\nu^U_{\tau,I}\right) + h\left(\nu^U_{\rho,I}\right)\right)\right]\right), \\ \left(\left[g^{-1}\left(g\left(\mu^L_{\tau,F}\right) + g\left(\mu^L_{\rho,F}\right)\right), g^{-1}\left(g\left(\mu^U_{\tau,F}\right) + g\left(\mu^U_{\rho,F}\right)\right)\right],\right. \\ \left.\left[h^{-1}\left(h\left(\nu^L_{\tau,F}\right) + h\left(\nu^L_{\rho,F}\right)\right), h^{-1}\left(h\left(\nu^U_{\tau,F}\right) + h\left(\nu^U_{\rho,F}\right)\right)\right]\right) \end{array} \right\rangle$$

and

$$\tau \otimes \rho = \left\langle \begin{array}{c} \left(\left[g^{-1}\left(g\left(\mu^L_{\tau,T}\right) + g\left(\mu^L_{\rho,T}\right)\right), g^{-1}\left(g\left(\mu^U_{\tau,T}\right) + g\left(\mu^U_{\rho,T}\right)\right)\right],\right. \\ \left.\left[h^{-1}\left(h\left(\nu^L_{\tau,T}\right) + h\left(\nu^L_{\rho,T}\right)\right), h^{-1}\left(h\left(\nu^U_{\tau,T}\right) + h\left(\nu^U_{\rho,T}\right)\right)\right]\right), \\ \left(\left[h^{-1}\left(h\left(\mu^L_{\tau,I}\right) + h\left(\mu^L_{\rho,I}\right)\right), h^{-1}\left(h\left(\mu^U_{\tau,I}\right) + h\left(\mu^U_{\rho,I}\right)\right)\right],\right. \\ \left.\left[g^{-1}\left(g\left(\nu^L_{\tau,I}\right) + g\left(\nu^L_{\rho,I}\right)\right), g^{-1}\left(g\left(\nu^U_{\tau,I}\right) + g\left(\nu^U_{\rho,I}\right)\right)\right]\right), \\ \left(\left[h^{-1}\left(h\left(\mu^L_{\tau,F}\right) + h\left(\mu^L_{\rho,F}\right)\right), h^{-1}\left(h\left(\mu^U_{\tau,F}\right) + h\left(\mu^U_{\rho,F}\right)\right)\right],\right. \\ \left.\left[g^{-1}\left(g\left(\nu^L_{\tau,F}\right) + g\left(\nu^L_{\rho,F}\right)\right), g^{-1}\left(g\left(\nu^U_{\tau,F}\right) + g\left(\nu^U_{\rho,F}\right)\right)\right]\right) \end{array} \right\rangle.$$

Now, we define multiplication by a positive constant and the positive power of IVIFNVs using additive generators of Archimedean t-norm and t-conorm.

Definition 3.7. Let $\tau = \langle \mathbb{T}_\tau, \mathbb{I}_\tau, \mathbb{F}_\tau \rangle$ be an IVIFNV and let λ be a positive constant. Let $g : [0,1] \to [0,\infty]$ be the additive generator of an Archimedean t-norm and $h(t) = g(1-t)$. Then

$$\lambda\tau = \left\langle \begin{array}{c} \left(\left[h^{-1}\left(\lambda h\left(\mu^L_{\tau,T}\right)\right), h^{-1}\left(\lambda h\left(\mu^U_{\tau,T}\right)\right)\right], \left[g^{-1}\left(\lambda g\left(\nu^L_{\tau,T}\right)\right), g^{-1}\left(\lambda g\left(\nu^U_{\tau,T}\right)\right)\right]\right), \\ \left(\left[g^{-1}\left(\lambda g\left(\mu^L_{\tau,I}\right)\right), g^{-1}\left(\lambda g\left(\mu^U_{\tau,I}\right)\right)\right], \left[h^{-1}\left(\lambda h\left(\nu^L_{\tau,I}\right)\right), h^{-1}\left(\lambda h\left(\nu^U_{\tau,I}\right)\right)\right]\right), \\ \left(\left[g^{-1}\left(\lambda g\left(\mu^L_{\tau,F}\right)\right), g^{-1}\left(\lambda g\left(\mu^U_{\tau,F}\right)\right)\right], \left[h^{-1}\left(\lambda h\left(\nu^L_{\tau,F}\right)\right), h^{-1}\left(\lambda h\left(\nu^U_{\tau,F}\right)\right)\right]\right) \end{array} \right\rangle$$

and

$$\tau^\lambda = \left\langle \begin{array}{c} \left(\left[g^{-1}\left(\lambda g\left(\mu^L_{\tau,T}\right)\right), g^{-1}\left(\lambda g\left(\mu^U_{\tau,T}\right)\right)\right], \left[h^{-1}\left(\lambda h\left(\nu^L_{\tau,T}\right)\right), h^{-1}\left(\lambda h\left(\nu^U_{\tau,T}\right)\right)\right]\right), \\ \left(\left[h^{-1}\left(\lambda h\left(\mu^L_{\tau,I}\right)\right), h^{-1}\left(\lambda h\left(\mu^U_{\tau,I}\right)\right)\right], \left[g^{-1}\left(\lambda g\left(\nu^L_{\tau,I}\right)\right), g^{-1}\left(\lambda g\left(\nu^U_{\tau,I}\right)\right)\right]\right), \\ \left(\left[h^{-1}\left(\lambda h\left(\mu^L_{\tau,F}\right)\right), h^{-1}\left(\lambda h\left(\mu^U_{\tau,F}\right)\right)\right], \left[g^{-1}\left(\lambda g\left(\nu^L_{\tau,F}\right)\right), g^{-1}\left(\lambda g\left(\nu^U_{\tau,F}\right)\right)\right]\right) \end{array} \right\rangle.$$

The following theorem ensures that $\lambda\tau$ and τ^λ are also IVIFNVs.

Proposition 3.8. *Let $g : [0,1] \to [0,\infty]$ be the additive generator of an Archimedean t-norm and let $h(t) = g(1-t)$. Then $\lambda\tau$ and τ^λ are IVIFNVs.*

Proof. Given $h^{-1}(t) = 1 - g^{-1}(t)$ and $\mu^U_{\tau,T} \leq 1 - \upsilon^U_{\tau,T}$, with h and h^{-1} increasing, it follows that

$$\begin{aligned} 0 &\leq h^{-1}(\lambda h(\mu^U_{\tau,T})) + g^{-1}(\lambda g(\upsilon^U_{\tau,T})) \\ &\leq h^{-1}(\lambda h(1 - \upsilon^U_{\tau,T})) + g^{-1}(\lambda g(\upsilon^U_{\tau,T})) \\ &= 1 - g^{-1}(\lambda g(\upsilon^U_{\tau,T})) + g^{-1}(\lambda g(\upsilon^U_{\tau,T})) \\ &= 1 \end{aligned}$$

which proves that $\mathbb{T}_{\lambda\tau}$ is an IVIFV. Similarly, we can prove that $\mathbb{I}_{\lambda\tau}, \mathbb{F}_{\lambda\tau}, \mathbb{T}_{\tau^\lambda}, \mathbb{I}_{\tau^\lambda}$, and $\mathbb{F}_{\tau^\lambda}$ are IVIFVs. □

4. Weighted Aggregation Operators for IVIFNVs

To compress fuzzy set data, aggregation operators are vital. We proceed to define weighted arithmetic and geometric aggregation operators for IVIFNVs, based on algebraic operations. We start with the arithmetic aggregation operators.

Definition 4.1. Let $\left\{\tau_j = \langle \mathbb{T}_{\tau_j}, \mathbb{I}_{\tau_j}, \mathbb{F}_{\tau_j} \rangle : j = 1, \ldots, n\right\}$ be a collection of IVIFNVs. A weighted arithmetic aggregation operator WA is defined by

$$WA(\tau_1, \ldots, \tau_n) := \bigoplus_{j=1}^{n} \varpi_j \tau_j,$$

where $\varpi = (\varpi_1, \ldots, \varpi_n)$ is a weight vector such that $0 \leq \varpi_j \leq 1$ for any $j = 1, \ldots, n$, and the sum of all elements equaling 1.

The WA is formulated as follows.

Theorem 4.2. *Let* $\left\{\tau_j = \langle \mathbb{T}_{\tau_j}, \mathbb{I}_{\tau_j}, \mathbb{F}_{\tau_j}\rangle : j = 1, \ldots, n\right\}$ *be a collection of IVIFNVs and g be the additive generator of an Archimedean t-norm and* $h(t) = g(1-t)$*. Then* $WA(\tau_1, \ldots, \tau_n)$ *is an IVIFNV and we have*

$$WA(\tau_1,\ldots,\tau_n) = \left\langle \begin{array}{l} \left(\left[h^{-1}\left(\sum_{j=1}^n \varpi_j h(\mu^l_{\tau_j,T})\right), h^{-1}\left(\sum_{j=1}^n \varpi_j h(\mu^u_{\tau_j,T})\right)\right],\right. \\ \left.\left[g^{-1}\left(\sum_{j=1}^n \varpi_j g(\nu^l_{\tau_j,T})\right), g^{-1}\left(\sum_{j=1}^n \varpi_j g(\nu^u_{\tau_j,T})\right)\right]\right), \\ \left(\left[g^{-1}\left(\sum_{j=1}^n \varpi_j g(\mu^l_{\tau_j,I})\right), g^{-1}\left(\sum_{j=1}^n \varpi_j g(\mu^u_{\tau_j,I})\right)\right],\right. \\ \left.\left[h^{-1}\left(\sum_{j=1}^n \varpi_j h(\nu^l_{\tau_j,I})\right), h^{-1}\left(\sum_{j=1}^n \varpi_j h(\nu^u_{\tau_j,I})\right)\right]\right), \\ \left(\left[g^{-1}\left(\sum_{j=1}^n \varpi_j g(\mu^l_{\tau_j,F})\right), g^{-1}\left(\sum_{j=1}^n \varpi_j g(\mu^u_{\tau_j,F})\right)\right],\right. \\ \left.\left[h^{-1}\left(\sum_{j=1}^n \varpi_j h(\nu^l_{\tau_j,F})\right), h^{-1}\left(\sum_{j=1}^n \varpi_j h(\nu^u_{\tau_j,F})\right)\right]\right), \end{array} \right\rangle. \tag{4.1}$$

Proof. Propositions 3.5 and 3.8 establish that $WA(\tau_1, \ldots, \tau_n)$ is an IVIFNV. Additionally, Equality 4.1 can be demonstrated using the method from Theorem 4 in [49].

□

Remark 4.3. Consider the additive generator defined by $g(t) = -\log t$. Then we obtain the Algebraic weighted arithmetic aggregation operator given by

$$WA_A(\tau_1,\ldots,\tau_n) = \left\langle \begin{array}{l} \left(\left[1-\prod_{j=1}^n(1-\mu^L_{\tau_j,T})^{\varpi_j}, 1-\prod_{j=1}^n(1-\mu^U_{\tau_j,T})^{\varpi_j}\right], \left[\prod_{j=1}^n(\upsilon^L_{\tau_j,T})^{\varpi_j}, \prod_{j=1}^n(\upsilon^U_{\tau_j,T})^{\varpi_j}\right]\right), \\ \left(\left[\prod_{j=1}^n(\mu^L_{\tau_j,I})^{\varpi_j}, \prod_{j=1}^n(\mu^U_{\tau_j,I})^{\varpi_j}\right], \left[1-\prod_{j=1}^n(1-\upsilon^L_{\tau_j,I})^{\varpi_j}, 1-\prod_{j=1}^n(1-\upsilon^U_{\tau_j,I})^{\varpi_j}\right]\right), \\ \left(\left[\prod_{j=1}^n(\mu^L_{\tau_j,F})^{\varpi_j}, \prod_{j=1}^n(\mu^U_{\tau_j,F})^{\varpi_j}\right], \left[1-\prod_{j=1}^n(1-\upsilon^L_{\tau_j,F})^{\varpi_j}, 1-\prod_{j=1}^n(1-\upsilon^U_{\tau_j,F})^{\varpi_j}\right]\right) \end{array} \right\rangle.$$

We proceed with the geometric aggregation operators.

Definition 4.4. Let $\{\tau_j = \langle \mathbb{T}_{\tau_j}, \mathbb{I}_{\tau_j}, \mathbb{F}_{\tau_j}\rangle : j = 1, \ldots, n\}$ be a collection of IVIFNVs. A weighted geometric aggregation operator WG is defined by

$$WG(\tau_1,\ldots,\tau_n) := \bigotimes_{j=1}^n \tau_j^{\varpi_j},$$

where $\varpi = (\varpi_1, \ldots, \varpi_n)$ is a weight vector whose components lie within $[0, 1]$, and sum to 1.

We now provide a formula for the WG.

Theorem 4.5. *Let* $\{\tau_j = \langle T_{\tau_j}, I_{\tau_j}, F_{\tau_j}\rangle : j = 1, \ldots, n\}$ *be a collection of IVIFNVs, g be an additive generator of an Archimedean t-norm and* $h(t) = g(1-t)$*. Then,* $WG(\tau_1, \ldots, \tau_n)$ *is an IVIFNV, and we have*

$$WG(\tau_1,\ldots,\tau_n) = \left\langle \begin{array}{l} \left(\left[g^{-1}\left(\sum_{j=1}^n \varpi_j g(\mu^l_{\tau_j,T})\right), g^{-1}\left(\sum_{j=1}^n \varpi_j g(\mu^u_{\tau_j,T})\right)\right],\right. \\ \left.\left[h^{-1}\left(\sum_{j=1}^n \varpi_j h(\nu^l_{\tau_j,T})\right), h^{-1}\left(\sum_{j=1}^n \varpi_j h(\nu^u_{\tau_j,T})\right)\right]\right), \\ \left(\left[h^{-1}\left(\sum_{j=1}^n \varpi_j h(\mu^l_{\tau_j,I})\right), h^{-1}\left(\sum_{j=1}^n \varpi_j h(\mu^u_{\tau_j,I})\right)\right],\right. \\ \left.\left[g^{-1}\left(\sum_{j=1}^n \varpi_j g(\nu^l_{\tau_j,I})\right), g^{-1}\left(\sum_{j=1}^n \varpi_j g(\nu^u_{\tau_j,I})\right)\right]\right), \\ \left(\left[h^{-1}\left(\sum_{j=1}^n \varpi_j h(\mu^l_{\tau_j,F})\right), h^{-1}\left(\sum_{j=1}^n \varpi_j h(\mu^u_{\tau_j,F})\right)\right],\right. \\ \left.\left[g^{-1}\left(\sum_{j=1}^n \varpi_j g(\nu^l_{\tau_j,F})\right), g^{-1}\left(\sum_{j=1}^n \varpi_j g(\nu^u_{\tau_j,F})\right)\right]\right) \end{array} \right\rangle. \tag{4.2}$$

Remark 4.6. Consider the additive generator defined by $g = -\log(t)$. Then we obtain the Algebraic weighted geometric aggregation operator given by

$$WG_A(\tau_1,\ldots,\tau_n) = \left\langle \begin{array}{l} \left(\left[\prod_{j=1}^n(\mu^l_{\tau_j,T})^{\varpi_j}, \prod_{j=1}^n(\mu^u_{\tau_j,T})^{\varpi_j}\right], \left[1-\prod_{j=1}^n(1-\nu^l_{\tau_j,T})^{\varpi_j}, 1-\prod_{j=1}^n(1-\nu^u_{\tau_j,T})^{\varpi_j}\right]\right), \\ \left(\left[1-\prod_{j=1}^n(1-\mu^l_{\tau_j,I})^{\varpi_j}, 1-\prod_{j=1}^n(1-\mu^u_{\tau_j,I})^{\varpi_j}\right], \left[\prod_{j=1}^n(\nu^l_{\tau_j,I})^{\varpi_j}, \prod_{j=1}^n(\nu^u_{\tau_j,I})^{\varpi_j}\right]\right), \\ \left(\left[1-\prod_{j=1}^n(1-\mu^l_{\tau_j,F})^{\varpi_j}, 1-\prod_{j=1}^n(1-\mu^u_{\tau_j,F})^{\varpi_j}\right], \left[\prod_{j=1}^n(\nu^l_{\tau_j,F})^{\varpi_j}, \prod_{j=1}^n(\nu^u_{\tau_j,F})^{\varpi_j}\right]\right) \end{array} \right\rangle.$$

5. An Application of IVIFNVs to an MCDM Problem

In this section, we implement an MCDM process within the proposed fuzzy environment using the WASPAS method. Specifically, we apply this extended WASPAS methodology to an MCDM problem focused on evaluating income equality among selected OECD countries.

5.1. **An Extended WASPAS Methodology.** The proposed steps of the extended WASPAS approach are outlined and summarized below.

- **Step 1:** Formulate a MCDM problem, where there are m alternatives denoted as $A = \{A_1, \ldots, A_m\}$ and n criteria denoted as $C = \{C_1, \ldots, C_n\}$.
- **Step 2:** A decision matrix D is constructed using the non-zero numerical data, showing the performance of different alternatives under various criteria

$$D = \begin{bmatrix} x_{11} & \ldots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \ldots & x_{mn} \end{bmatrix}.$$

- **Step 3:** The decision matrix is normalized to make the criteria comparable. The normalization method depends on whether a criterion is a benefit criterion or a cost criterion. If C_j is a benefit criteria, then the normalization formula is given with

$$\tilde{x}_{ij} = \frac{x_{ij}}{\max_i x_{ij}}, \quad i = 1, \ldots, m,$$

and if C_j is cost criteria, then the normalization formula is given by

$$\tilde{x}_{ij} = \frac{\min_i x_{ij}}{x_{ij}}, \quad i = 1, \ldots, m.$$

- **Step 4:** To establish a comparative scale, the range for each alternative across all criteria is calculated by subtracting the minimum from the maximum value. This range is subsequently segmented into seven levels using percentiles, with each level assigned a rating from r_1 (weakest) to r_7 (strongest) according to Table 2. Based on these levels, IVIFNVs are assigned to each alternative for each criterion. This method is consistent with techniques found in Fei et al. [18], Aydoğan and Ünver [6], Bozyiğit et al.[11] and Ünver [48].

Table 2. Linguistic terms and corresponding IVIFNS values

Linguistic term	IVIFNVs
Absolutely weak (AW)	$\langle([0.01, 0.02], [0.90, 0.95]), ([0.01, 0.02], [0.90, 0.95]), ([0.90, 0.95], [0.01, 0.02])\rangle$
Very weak (VW)	$\langle([0.10, 0.15], [0.75, 0.85]), ([0.10, 0.15], [0.75, 0.85]), ([0.75, 0.85], [0.10, 0.15])\rangle$
Fairly weak (FW)	$\langle([0.20, 0.25], [0.55, 0.75]), ([0.20, 0.25], [0.55, 0.75]), ([0.55, 0.75], [0.20, 0.25])\rangle$
Exactly equal (EE)	$\langle([0.45, 0.55], [0.40, 0.45]), ([0.45, 0.55], [0.40, 0.45]), ([0.45, 0.55], [0.40, 0.45])\rangle$
Fairly strong (FS)	$\langle([0.55, 0.75], [0.20, 0.25]), ([0.20, 0.25], [0.55, 0.75]), ([0.20, 0.25], [0.55, 0.75])\rangle$
Very strong (VS)	$\langle([0.75, 0.85], [0.10, 0.15]), ([0.10, 0.15], [0.75, 0.85]), ([0.10, 0.15], [0.75, 0.85])\rangle$
Absolutely strong (AS)	$\langle([0.90, 0.95], [0.01, 0.02]), ([0.01, 0.02], [0.90, 0.95]), ([0.01, 0.02], [0.90, 0.95])\rangle$

- **Step 5:** A weight vector is constructed based on expert opinions. Then, the total relative importance of alternative A_i is calculated separately using WSM and WPM. According to WSM, the total relative importance of an alternative is determined as the weighted sum of criterion values, while in WPM, it is computed as the product of the performance values of the alternative for each criterion raised to the power of the corresponding criterion weight. The total relative importance of alternative i according to WSM is calculated by

$$Q_i^{(1)} = \bigoplus_{j=1}^{n} \varpi_j \tilde{x}_{ij},$$

and the total relative importance of alternative A_i according to WPM is calculated by

$$Q_i^{(2)} = \bigotimes_{j=1}^{n} (\tilde{x}_{ij})^{\varpi_j}.$$

- **Step 6:** The overall fuzzy relative importance of alternatives is calculated by

$$Q_i = \lambda Q_i^{(1)} \oplus (1 - \lambda) Q_i^{(2)}.$$

- **Step 7:** To rank the alternatives, the overall relative importance values Q_i are defuzzified via the score function S defined in Definition 3.3. In this ranking, higher score values signify superior alternatives.

5.2. **Application on Income Equality.** Income equity, which varies significantly between countries, is typically evaluated using income distribution inequalities. This study aims to assess income equity within selected OECD countries. This evaluation, involving multiple criteria and alternatives, utilizes an MCDM framework, specifically the proposed extended WASPAS method.

- **Step 1:** 25 OECD countries are examined to assess income equity. Four criteria are determined to measure income distribution fairness: the Gini coefficient (C_1), the Palma ratio (C_2), the P90/P10 ratio (C_3) and the P90/P50 ratio (C_4). The OECD database is used for the criteria and data from the year 2022 are considered in the analysis [37].
- **Step 2:** As shown in Table 3, the decision-making matrix is constructed based on the alternatives and criteria according to the real data.

Table 3. Decision matrix

Countries	C_1	C_2	C_3	C_4	**Countries**	C_1	C_2	C_3	C_4
Austria	0.29	1.03	3.40	1.70	Luxembourg	0.30	1.08	3.70	1.90
Belgium	0.25	0.86	2.90	1.60	Mexico	0.40	1.86	5.80	2.40
Chile	0.45	2.40	7.30	2.90	Netherlands	0.29	1.05	3.20	1.80
Costa Rica	0.47	2.74	9.80	3.30	New Zealand	0.33	1.26	4.30	2.00
Czechia	0.25	0.85	3.10	1.70	Norway	0.26	0.90	3.10	1.70
Estonia	0.32	1.20	4.80	2.00	Poland	0.27	0.93	3.50	1.80
Finland	0.27	0.98	3.20	1.80	Slovak Republic	0.23	0.72	3.10	1.60
France	0.30	1.11	3.40	1.80	Slovenia	0.24	0.82	3.10	1.70
Hungary	0.29	1.06	3.60	1.80	Spain	0.32	1.17	4.50	2.00
Ireland	0.29	1.04	3.40	1.80	Sweden	0.29	1.07	3.30	1.70
Israel	0.34	1.33	5.52	2.07	United Kingdom	0.37	1.55	4.40	2.10
Korea	0.32	1.22	4.90	2.00	United States	0.40	1.82	6.30	2.30
Latvia	0.34	1.34	5.30	2.10					

- **Step 3:** In this problem, all criteria represent cost criteria. Therefore, normalization is performed according to the minimization criterion, and the normalized decision matrix is presented in Table 4.

Table 4. Normalized decision matrix

Countries	C_1	C_2	C_3	C_4	**Countries**	C_1	C_2	C_3	C_4
Austria	0.795	0.699	0.853	0.941	Luxembourg	0.765	0.667	0.784	0.842
Belgium	0.906	0.837	1.000	1.000	Mexico	0.566	0.387	0.500	0.667
Chile	0.505	0.300	0.397	0.552	Netherlands	0.786	0.686	0.906	0.889
Costa Rica	0.480	0.263	0.296	0.485	New Zealand	0.695	0.571	0.674	0.800
Czechia	0.909	0.847	0.935	0.941	Norway	0.864	0.800	0.935	0.941
Estonia	0.705	0.600	0.604	0.800	Poland	0.838	0.774	0.829	0.889
Finland	0.826	0.735	0.906	0.889	Slovak Republic	1.000	1.000	0.935	1.000
France	0.762	0.649	0.853	0.889	Slovenia	0.930	0.878	0.935	0.941
Hungary	0.770	0.679	0.806	0.889	Spain	0.717	0.615	0.644	0.800
Ireland	0.795	0.692	0.853	0.889	Sweden	0.781	0.673	0.879	0.941
Israel	0.657	0.541	0.525	0.774	United Kingdom	0.620	0.465	0.659	0.762
Korea	0.699	0.590	0.592	0.800	United States	0.572	0.396	0.460	0.696
Latvia	0.666	0.537	0.547	0.762					

- **Step 4:** The r_k values for each criterion is calculated, and the alternatives are given based on the linguistic terms presented in Table 5.
- **Step 5:** The weight vector is determined as $\varpi = (0.35, 0.25, 0.20, 0.20)$ based on expert opinions. The WA_A and WG_A-operators are used to aggregate the relative importance $Q_i^{(1)}$ and $Q_i^{(2)}$ of alternatives. Then, the

Table 5. Normalized fuzzy decision matrix with linguistic terms

Countries	C_1	C_2	C_3	C_4	**Countries**	C_1	C_2	C_3	C_4
Austria	FS	EE	VS	AS	Luxembourg	EE	FW	FS	FS
Belgium	VS	VS	AS	AS	Mexico	VW	VW	FW	FW
Chile	AW	AW	VW	AW	Netherlands	FS	EE	AS	VS
Costa Rica	AW	AW	AW	AW	New Zealand	FW	FW	EE	FS
Czechia	VS	VS	AS	AS	Norway	VS	VS	AS	AS
Estonia	EE	FW	EE	FS	Poland	FS	EE	VS	VS
Finland	FS	EE	AS	VS	Slovak Republic	AS	AS	AS	AS
France	EE	FW	VS	VS	Slovenia	AS	VS	AS	AS
Hungary	EE	FW	VS	VS	Spain	EE	FW	EE	FS
Ireland	FS	EE	VS	VS	Sweden	FS	FW	VS	AS
Israel	FW	FW	FW	EE	United Kingdom	VW	VW	EE	EE
Korea	FW	FW	FW	FS	United States	VW	VW	VW	FW
Latvia	FW	FW	FW	EE					

overall relative importance Q_i of the alternatives is calculated for $\lambda = 0.5$, and the results are presented in Table 6.

Table 6. Overall fuzzy relative importance of alternatives

Countries	$\mu^L_{T,i}$	$\mu^U_{T,i}$	$\upsilon^L_{T,i}$	$\upsilon^U_{T,i}$	$\mu^L_{I,i}$	$\mu^U_{I,i}$	$\upsilon^L_{I,i}$	$\upsilon^U_{I,i}$	$\mu^L_{F,i}$	$\mu^U_{F,i}$	$\upsilon^L_{F,i}$	$\upsilon^U_{F,i}$
Austria	0.6534	0.7806	0.1525	0.1983	0.1611	0.2178	0.6416	0.7595	0.1611	0.2178	0.6416	0.7595
Belgium	0.8170	0.8963	0.0509	0.0819	0.0509	0.0819	0.8170	0.8963	0.0509	0.0819	0.8170	0.8963
Chile	0.0223	0.0388	0.8738	0.9334	0.0306	0.0377	0.8740	0.9335	0.8738	0.9334	0.0223	0.0388
Costa Rica	0.0100	0.0200	0.9000	0.9500	0.0100	0.0200	0.9000	0.9500	0.9000	0.9500	0.0100	0.0200
Czechia	0.8170	0.8963	0.0509	0.0819	0.0509	0.0819	0.8170	0.8963	0.0509	0.0819	0.8170	0.8963
Estonia	0.5349	0.6698	0.2310	0.3013	0.2167	0.3036	0.5610	0.6906	0.2423	0.3235	0.5208	0.6454
Finland	0.5971	0.7423	0.2058	0.2553	0.2174	0.2804	0.5834	0.7174	0.2174	0.2804	0.5834	0.7174
France	0.5080	0.6096	0.2923	0.3904	0.2307	0.3705	0.5825	0.6976	0.3068	0.4199	0.4923	0.5801
Hungary	0.5080	0.6096	0.2923	0.3904	0.2307	0.3705	0.5825	0.6976	0.3068	0.4199	0.4923	0.5801
Ireland	0.6094	0.7474	0.1990	0.2526	0.2095	0.2756	0.5969	0.7244	0.2095	0.2756	0.5969	0.7244
Israel	0.2466	0.3079	0.5197	0.6921	0.2462	0.5500	0.5197	0.6926	0.5300	0.7118	0.2373	0.2882
Korea	0.2662	0.3561	0.4716	0.6439	0.2000	0.4854	0.5500	0.7500	0.4716	0.6439	0.2662	0.3561
Latvia	0.2466	0.3079	0.5197	0.6921	0.2462	0.5500	0.5197	0.6926	0.5300	0.7118	0.2373	0.2882
Luxembourg	0.4208	0.5555	0.3502	0.4445	0.2815	0.4240	0.4972	0.6496	0.3663	0.4764	0.4041	0.5236
Mexico	0.1367	0.1878	0.6730	0.8122	0.2000	0.2932	0.6733	0.8123	0.6730	0.8122	0.1367	0.1878
Netherlands	0.6534	0.7806	0.1525	0.1983	0.1611	0.2178	0.6416	0.7595	0.1611	0.2178	0.6416	0.7595
New Zealand	0.3137	0.4123	0.4428	0.5877	0.2462	0.4854	0.5197	0.6926	0.4525	0.6063	0.3044	0.3937
Norway	0.8170	0.8963	0.0509	0.0819	0.0509	0.0819	0.8170	0.8963	0.0509	0.0819	0.8170	0.8963
Poland	0.6094	0.7474	0.1990	0.2526	0.2095	0.2756	0.5969	0.7244	0.2095	0.2756	0.5969	0.7244
Slovak Republic	0.9000	0.9500	0.0100	0.0200	0.0100	0.0200	0.9000	0.9500	0.0100	0.0200	0.9000	0.9500
Slovenia	0.8673	0.9293	0.0243	0.0424	0.0243	0.0424	0.8673	0.9293	0.0243	0.0424	0.8673	0.9293
Spain	0.4014	0.5140	0.3925	0.4860	0.3302	0.4854	0.4673	0.5910	0.4189	0.5378	0.3751	0.4622
Sweden	0.5871	0.7112	0.1789	0.2622	0.1179	0.2178	0.6761	0.8220	0.1789	0.2622	0.5871	0.7112
United Kingdom	0.2227	0.2980	0.6134	0.7020	0.2935	0.2932	0.6155	0.7068	0.6339	0.7402	0.2050	0.2598
United States	0.1179	0.1686	0.7118	0.8314	0.2000	0.2212	0.7119	0.8314	0.7118	0.8314	0.1179	0.1686

- **Step 6:** The scores of Q_i are calculated, and the final ranking of alternatives is determined based on the descending order of Q_i values. As the value of the score function decreases, income inequality increases. The results are shown in Figure 5.
  Based on the score values, the five countries with the highest income equality are ranked as

$$\text{Slovak Republic} \succ \text{Slovenia} \succ \text{Belgium} \approx \text{Czechia} \approx \text{Norway}.$$

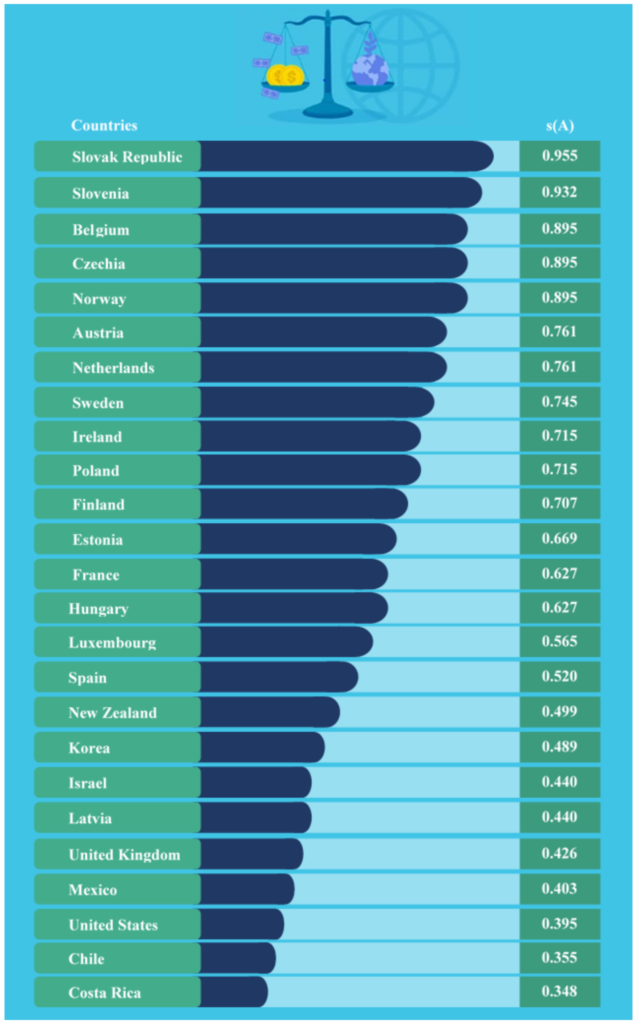

Figure 5. Scores of each country based on the extended WASPAS analysis

Conversely, the five countries with the highest income inequality are

$$\text{Costa Rica} \prec \text{Chile} \prec \text{United States} \prec \text{Mexico} \prec \text{United Kingdom}.$$

6. Comparative Analysis

In this section, a comparative analysis is performed. The results are compared with those obtained using the Complex Proportional Assessment (COPRAS) [63], Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [24], and Simple Additive Weighting (SAW) [22] methods. During the implementation of the methods, individual scores are calculated for each criterion; the resulting values are used to construct the decision matrix, which served as the basis for subsequent analyses.

Figure 6 presents the performance scores of various alternatives evaluated using four different MCDM methods: WASPAS, COPRAS, TOPSIS, and SAW. The results highlight a general agreement among methods, particularly in the ranking of the top-performing alternatives. The Slovak Republic consistently achieves the highest score across all methods (WASPAS: 0.955, COPRAS: 1.000, TOPSIS: 1.000, SAW: 1.000), indicating its strong overall performance regardless of the evaluation technique used. Similarly, Slovenia, Belgium, Czechia, and Norway rank among the top alternatives, with relatively high scores across all four methods. However, notable differences arise in the mid and lower rankings. For instance, Austria and the Netherlands obtain similar scores in WASPAS (0.761), while their

COPRAS scores differ (0.861 and 0.744, respectively). A more pronounced discrepancy is observed in Estonia, where its WASPAS score (0.669) is significantly higher than its COPRAS (0.544) and TOPSIS (0.447) scores, highlighting the variation in how methods handle decision criteria and weighting.At the lower end of the rankings, Costa Rica is consistently ranked the lowest in all methods. Similarly, Chile, Mexico, and the United States receive low scores across all methods, though their rankings vary slightly.

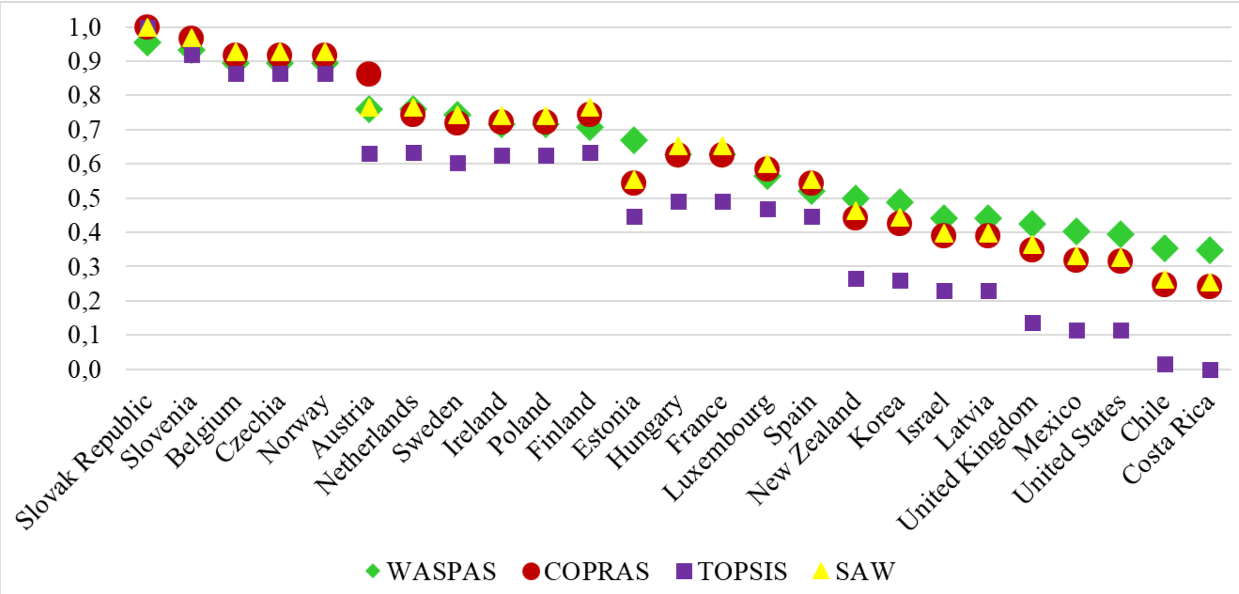

Figure 6. Country scores according to different applied methods

Figure 7 presents the ranking results of countries in terms of income equality according to the WASPAS, COPRAS, TOPSIS, and SAW methods.

7. Conclusion

Income inequality is a significant economic and social issue that arises from income disparities among individuals or households, leading to inequalities within society. This inequality directly impacts economic growth, social cohesion, and equal opportunities, resulting in variations in overall welfare levels. Various methods have been developed to measure income inequality. This study introduces IVIFNSs and presents their theoretical framework. IVIFNSs can be effectively used in the analysis of data containing uncertainty and vagueness. In this study, the income justice performance of countries is addressed as a MCDM problem within the IVIFNS environment. Income justice criteria and their corresponding weights are determined through expert opinions. The WASPAS method, a combination of the WSM and WPM, is used to solve the problem. The performance of the evaluated countries regarding income justice is ranked. In future research, performance evaluations of countries in various areas, such as economic growth, the effectiveness of education systems, healthcare systems, and integration into technology and digitalization, could be considered. Comparisons in these areas will allow for a more comprehensive analysis of the development levels of countries. Moreover, studies conducted in different fuzzy environments can address uncertainty and complexity more accurately, enabling more precise analyses. This approach can provide new perspectives and deepen cross-domain comparisons, leading to more reliable results.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

WASPAS	COPRAS	TOPSIS	SAW
1.Slovak Republic	1.Slovak Republic	1.Slovak Republic	1.Slovak Republic
2.Slovenia	2.Slovenia	2.Slovenia	2.Slovenia
3.Belgium, Czechia, Norway	3.Belgium, Czechia, Norway	3.Belgium, Czechia, Norway	3.Belgium, Czechia, Norway
4.Austria, Netherlands	4.Austria	4.Finland, Netherlands, Austria	4.Finland, Netherlands, Austria
5.Sweden	5.Finland, Netherlands	5.Ireland, Poland	5.Sweden
6.Ireland, Poland	6.Ireland, Poland	6.Sweden	6.Ireland, Poland
7.Finland	7.Sweden	7.France, Hungary	7.France, Hungary
8.Estonia	8.France, Hungary	8.Luxembourg	8.Luxembourg
9.France, Hungary	9.Luxembourg	9.Estonia, Spain	9.Estonia, Spain
10.Luxembourg	10.Estonia, Spain	10.New Zealand	10.New Zealand
11. Spain	11.New Zealand	11.Korea	11.Korea
12. New Zealand	12.Korea	12.Israel, Latvia	12.Israel, Latvia
13.Korea	13.Israel, Latvia	13.United Kingdom	13.United Kingdom
14.Israel,Latvia	14.United Kingdom	14.Mexico	14.Mexico
15.United Kingdom	15.Mexico	15.United States	15.United States
16.Mexico	16.United States	16.Chile	16.Chile
17.Unites States	16.Chile	17.Costa Rica	17.Costa Rica
18.Chile	17.Costa Rica		
19.Costa Rica			

Figure 7. Ranking results of countries in terms of income equality according to the applied methods

Authors Contribution Statement

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

References

[1] Alvaredo, F., Chancel, L., Piketty, T., Saez, E., Zucman, G., *Global Inequality Dynamics: New Findings from WID.world*, American Economic Review, **107**(5)(2017), 404–409.

[2] Ashraf, S., Abdullah, S., Mahmood, T., Ghani, F., Mahmood, T., *Spherical fuzzy sets and their applications in multi-attribute decision-making problems*, J. Intell. Fuzzy Syst., **36**(3)(2019), 2829–2844.

[3] Ashraf Ganjoei, R., Akbarifard, H., Mashinchi, M., Jalaee Esfandabadi, S.A., *Applying of fuzzy Nonlinear Regression to Investigate the Effect of Information and Communication Technology (ICT) on Income Distribution*, vol.2021(2021), 1–11.

[4] Atanassov, K.T., *Intuitionistic fuzzy sets*, In Intuitionistic fuzzy sets, Physica, Heidelberg, 1999.

[5] Atanassov, K., Gargov, G., *Interval-valued fuzzy set*, Fuzzy Sets and Systems, **31**, 343–349 (1989).

[6] Aydoğan, B., Ünver, M., *A modified similarity measure for continuous function valued intuitionistic fuzzy sets and an application on classification*, Springer Proceeding in Mathematics and Statistics (accepted).

[7] Atkinson, A.B., *Income inequality in OECD countries: Data and Explanations*, CESifo Economic Studies, **49**(4)(2003), 479–513.

[8] Bai, Z., *An interval-vValued iIntuitionistic fuzzy TOPSIS method based on an improved score function*, The Scientific World Journal, 2013(2013), 1–6.

[9] Bedregal, B., Lima, L., Rocha, M., Dimuro, G., Bustince, H., *Interval-valued Atanassov intuitionistic t-norms and t-conorms endowed with the usual or admissible orders*, Comput. Appl. Math., **42**(1)(2023), 49.

[10] Beliakov, G., Bustince, H., Goswami, D.P., Mukherjee, U.K., Pal, N.R., *On averaging operators for Atanassov's intuitionistic fuzzy sets*, Information Sciences, **181**(6)(2011), 1116–1124.

[11] Bozyiğit, M.C., Olgun, M., Ünver, M., Söylemez, D., *Parametric picture fuzzy cross-entropy measures based on d-Choquet integral for building material recognition*, Applied Soft Computing, **166**(2024), 112167.
[12] Bozyigit, M.C., Smarandache, F., Olgun, M., Ünver, M., *A new type of neutrosophic set in Pythagorean fuzzy environment and applications to multi-criteria decision-making*, International Journal of Neutrosophic Science, **20**(2)(2023), 107–134.
[13] Burtless, G., *Effects of growing wage disparities and changing family composition on the US income distribution*, European Economic Review, **43**(4–6)(1999), 853–865.
[14] Chakraborty, S., Zavadskas, E.K., *Applications of WASPAS method in manufacturing decision-making*, Informatica, **25**(1)(2014), 1–20.
[15] Cowell, F., Measuring Inequality, Oxford University Press, 2011.
[16] Dabla-Norris, E., Kochhar, K., Suphaphiphat, N., Ricka, F., Tsounta, E., *Causes and consequences of income inequality: A global perspective*, IMF Staff discussion note No. 15/13, Washington, DC: International Monetary Fund (2015).
[17] Deschrijver, G., Cornelis, C., *Representabılıty In Interval-Valued Fuzzy Set Theory*, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, **15**(03)(2007), 345–361.
[18] Fei, L., Wang, H., Chen, L., Deng, Y., *A new vector valued similarity measure for intuitionistic fuzzy sets based on OWA operators*, Iranian Journal of Fuzzy Systems, **16**(3)(2019), 113–126.
[19] Keshavarz Ghorabaee, M., Zavadskas, E.K., Amiri, M., Esmaeili, A., *Multi-criteria evaluation of green suppliers using an extended WASPAS method with interval type-2 fuzzy sets*, Journal of Cleaner Production, **137**(2016), 213–229.
[20] Giordani, P., Giorgi, G.M., *A fuzzy logic approach to poverty analysis based on the Gini and Bonferroni inequality indices*, Statistical Methods and Applications, **19**(4)(2010), 587–607.
[21] Gini, C., *Measurement of inequality of incomes*, The economic journal, **31**(121)(1921), 124–125.
[22] Harsanyi, J.C., *Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility*, Journal of political economy, **63**(4)(1955), 309–321.
[23] Hasnoi, L., Belhadj, B., *Fuzzy multidimensional inequality measurement. Policies to reduce inequality in Tunisia*, EuroEconomica, Danubius University of Galati, **2**(34)(2015), 21–28.
[24] Hwang, C.L., Yoon, K., Hwang, C.L., Yoon, K., *Methods for multiple attribute decision making*, Multiple attribute decision making: methods and applications a state-of-the-art survey, (1981), 58–191.
[25] Hussain, A., Majeed, S., Shoaib, M., Ali, H., Ullah, K., *WASPAS Method for Healthcare Systems based on Intuitionistic Fuzzy Information*, (2024).
[26] Juhn, C., Murphy, K. M., Pierce, B., *Wage inequality and the rise in returns to skill*, Journal of political Economy, **101**(3)(1993), 410–442.
[27] Kaasa, A., *Factors of Income Inequality and Their Influence Mechanisms: A Theoretical Overview*, University of Tartu Faculty of Economics and Business Administration Working Paper No. 40 , (2005).
[28] Kannan, J., Jayakumar, V., *Sustainable method for tender selection using linear Diophantine multi-fuzzy soft set*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, **72**(4)(2023), 976–991.
[29] Keeley, B., Income Inequality: The Gap between Rich and Poor, OECD Insights, OECD Publishing, Paris, 2015.
[30] Klement, E.P., Mesiar, R., Pap, E., Triangular norms, Kluwer Academic Publishers, Dordrecht, 2002.
[31] Klement, E.P., Mesiar, R., Pap, E., *Triangular norms. Position paper III: continuous t-norms*, Fuzzy Sets and Systems, **145**(3)(2004), 439–454.
[32] Liu, Y., Xie, N., *Amelioration operators of fuzzy number intuitionistic fuzzy geometric and their application to Multi-criteria decision-making*, 2009 Chinese Control and Decision Conference (2009).
[33] Mishra, A.R., Singh, R.K., Motwani, D., *Multi-criteria assessment of cellular mobile telephone service providers using intuitionistic fuzzy WASPAS method with similarity measures*, Granular Computing, **4**(2018), 511–529.
[34] Mishra, A.R., Rani, P., *Multi-criteria healthcare waste disposal location selection based on Fermatean fuzzy WASPAS method*, Complex and Intelligent Systems, **7**(2021), 2469–2484.
[35] Morelli, S., Smeeding, T., Thompson, J., Post-1970 Trends in Within-Country Inequality and Poverty, In: Handbook of Income Distribution, 2015.
[36] Lakshmana Gomathi Nayagam, V., Muralikrishnan, S., Sivaraman, G., *Multi-criteria decision-making method based on interval-valued intuitionistic fuzzy sets*, Expert Systems with Applications, **38**(3)(2011), 1464–1467.

[37] OECD, *Income distribution database*, Retrieved from https://data-explorer.oecd.org/vis?tm=DF_IDD&pg=0&snb=1&vw=tb&df[ds]=dsDisseminateFinalDMZ&df[id]=DSD_WISE_IDD%40DF_IDD&df[ag]=OECD.WISE.INE&df[vs]=&pd=2010%2C&dq=.A.INC_DISP..._T.METH2012.D_CUR.&ly[rw]=REF_AREA&ly[cl]=TIME_PERIOD&to[TIME_PERIOD]=false (2025).

[38] Ok, E.A., *Fuzzy measurement of income inequality: a class of fuzzy inequality measures*, Social Choice and Welfare, **12**(2)(1995), 111–116.

[39] Palma, J.G., *Homogeneous middles vs. heterogeneous tails, and the end of the 'inverted-U': It's all about the share of the rich*, Development and Change, **42**(1), 87–153 (2011).

[40] Qureshi, Z., *Rising Inequality: A Major Issue of our Time*, Report, The Brookings Institution, Washington DC (2023).

[41] Raj, A.K, Bathusha, S.N.S., Hussain, S., *Self centered interval-valued intuitionistic fuzzy graph with an application*, Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, **72**(4)(2023), 1155–1172.

[42] Reiser, R.H.S., Bedregal, B., *Interval-valued intuitionistic fuzzy implications - Construction, properties and representability*, Information Sciences, **248**(2013), 68–88.

[43] Rutkowska, A., Kaczmarek-Majer, K., Bartkowiak, M., Hryniewicz, O., *Explaining the relation between life expectancy and income inequality with fuzzy linguistic summaries*, (2023).

[44] Sen, B., Growth and poverty reduction: macroeconomic experience, In: World Bank, Social Impact of Adjustment Operation, Washington DC: World Bank, Operations and Evaluation Department, 1995.

[45] Smarandache, F., A unifying field of logics. Neutrosophy: Neutrosophic probability, set and logic, American Research Press, Rehoboth, 1998.

[46] Stewart, F., Income Distribution and Development, In: John Toye ed. Trade and Development, Directions for the Twenty-first Century, Edward Elgar Publishing, 2003.

[47] Urosevic, S., Karabasevic, D., Stanujkic, D., Maksimovic, M., *An approach to personnel selection in the tourism industry based on the SWARA and the WASPAS methods*, Economic Computation and Economic Cybernetics Studies and Research, **51**(2017), 75–88.

[48] Ünver, M., *Gaussian Aggregation Operators and Applications to Intuitionistic Fuzzy Classification*, Journal of Classification, (2025).

[49] Ünver, M., Türkarslan, E., Çelik, N., Olgun, M., Ye, J., *Intuitionistic fuzzy-valued neutrosophic multi-sets and numerical applications to classification*, Complex Intell. Syst., **8**(2022), 1703–1721.

[50] Wang, S.F., *Interval-valued intuitionistic fuzzy Choquet integral operators based on Archimedean t-norm and their calculations*, Journal of Computational Analysis and Applications, **23**(4)(2017), 703–712.

[51] Wang, H., Smarandache, F., Zhang, Y., Sunderraman, R., *Interval Neutrosophic Sets and Logic: Theory and Applications in Computing*, Hexis, Phoenix (2005).

[52] Wang, Z., Li, K.W., Wang, W., *An approach to multiattribute decision-making with interval-valued intuitionistic fuzzy assessments and incomplete weights*, Information Sciences, **179**(17)(2009), 3026–3040.

[53] Wang, H., Smarandache, F., Zhang, Y. Q., Sunderraman, R., *Single valued neutrosophic sets*, Multispace Multistruct, **4**(2010), 410–413.

[54] Wang, W., Liu, X., Qin, Y., *Interval-valued intuitionistic fuzzy aggregation operators*, Journal of Systems Engineering and Electronics, **23**(4)(2012), 574–580.

[55] Xia, M., Xu, Z., Zhu, B., *Some issues on intuitionistic fuzzy aggregation operators based on Archimedean t-conorm and t-norm*, Knowledge-Based Systems, **31**(2012), 78–88.

[56] Xu, Z.S., *Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision-making*, Control and Decision, **22**(2007), 215–219.

[57] Xu, Z.S., Chen, J., *An Overview Of Distance And Similarity Measures Of Intuitioistic Fuzzy Sets*, International Journal Of Uncertainty, Fuzziness And Knowledge-Based Systems, **16**(04)(2008), 529–555.

[58] Xu, F., Yin, H., Wu, Q., *An axiomatic approch of interval-valued intuitionistic fuzzy rough sets based on interval-valued intuitionistic fuzzy approximation operators*, 2012 2nd International Conference on Consumer Electronics, Communications and Networks (CECNet) (2012).

[59] Yager, R.R., *Pythagorean Membership Grades in Multicriteria decision-making*, IEEE Transactions on Fuzzy Systems, **22**(4)(2014), 958–965.

[60] Ye, J., *A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets*, Journal of Intelligent and Fuzzy Systems, **26**(5)(2014), 2459–2466.
[61] Zadeh, L.A., *Fuzzy sets*, Information and Control, **8**(3)(1965), 338–353.
[62] Zadeh, L., *The concept of a linguistic variable and its application to approximate reasoning-I*, Inform. Sci., **8**(1975), 199–249.
[63] Zavadskas, E.K., Kaklauskas, A., Šarka, V., *The new method of multicriteria complex proportional assessment of projects*, (1994).
[64] Zavadskas, E.K., Turskis, Z., Antucheviciene, J., Zakarevicius, A., *Optimization of Weighted Aggregated Sum Product Assessment*, Elektronika Ir Elektrotechnika, **122**(6)(2012), 3–6.