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Research Article

Investigation of Travelling Wave Solutions for the (2+1)-Dimensional Ablowitz-Kaup-Newell-Segur Equation

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In this study, modified sub equation method was applied to the (2+1)-Ablowitz-Kaup-Newell-Segur (AKNS) equation. This analytical method, trigonometric, hyperbolic and rational type solutions have been produced. Contour, 3D and 2D graphs representing stationary wave are drawn by giving random values to the constants in these solutions. Using symbolic computation, this method is shown to be an effective, powerful and reliable tool for generating nonlinear evolution equations (NEDEs).

Keywords: Modified sub equation method; nonlinear evolution equation; exact solution.

Araştırma Makalesi

(2+1)-Boyutlu Ablowitz-Kaup-Newell-Segur Denklemi için Gezici Dalga Çözümlerinin İncelenmesi

Bu çalışmada, geliştirilmiş alt denklem yöntemi (2+1)-Ablowitz-Kaup-Newell-Segur (AKNS) denklemine uygulanmıştır. Bu analitik yöntemle trigonometrik, hiperbolik ve rasyonel tipte çözümleri üretilmiştir. Durağan dalgayı temsil eden kontur, 3 boyutlu ve 2 boyutlu grafikleri bu çözümlerdeki sabitlere rastgele değerler verilerek çizilir. Sembolik hesaplama kullanılarak, bu yöntemin doğrusal olmayan evrim denklemlerinin çözümlerini üretmek için etkili, güçlü ve güvenilir bir araç olduğu gösterilmiştir.

Anahtar Kelimeler: Geliştirilmiş alt denklem metodu; doğrusal olmayan evrim denklemi; tam çözüm.

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1. Introduction

Differential equations are used to describe complicated processes and systems in nature in many areas of mathematical modeling and applied sciences. Among these equations, NEDEs have an important place due to both their theoretical difficulties and practical importance. These equations, which are encountered in many disciplines from fluid dynamics to quantum physics and financial models, play an effective role in the analysis of nonlinear interactions and multivariable systems.

Many effective methods have been described to attain exact solutions of NEDEs. When the literature is reviewed, some of methods are: sech-tanh method [1], the functional variable method [2], Exp-function method [3], (G'/G)-expansion method [4], $(1/G')$- expansion method [5], [6], ansatz method [7], the sine-Gordon expansion method [8], extended Fan sub-equation method [9] and so on [10], [11], [12], [13], [14], [15], [16].

Traveling wave solutions obtained with these methods can be used in various application areas depending on the characteristics of the NEDEs studied.

We reviewed the travelling wave solutions of AKNS equation [17].

$$4u_{xt}+u_{xxxz}+8u_{xz}u_{x}+4u_{z}u_{xx}=0. \tag{1}$$

This study aims to obtain a travelling wave solution for the AKNS equation using the analytical method. The analytical method used in the work is the modified sub-equation method. This analytical technique has not been applied to solve the AKNS equation before, so our solutions are completely new. This research is important for the application of the AKNS equation in modeling physical phenomena such as plasma physics, nonlinear optics and water waves [18].

Scientists have presented many studies in the literature regarding the AKNS equation. Some of them are as follows: In their work, Issasfa and Lin constructed rational solutions made up of rogue wave and lumped soliton solutions using the Bilinear method and the Ansatz technique [19]. In the study, the analytical and numerical techniques Khater II method and extended cubic-B-spline scheme have been taken into consideration in order to find wave solutions for the AKNS equation [20]. Alfalqi and Khater obtained solitary wave solutions in AKNS equation using modified exp-$\phi(\zeta)$ expansion method [21].

The rest of the article is as follows. The methodology of the method used is given in chapter 2. The applications of the considered method are given in chapter 3. Conclusions are given in the final chapter 4.

2. Modified sub-equation method

Let us consider this method to generate solutions to NEDEs [22]. Regard the NEDEs as

$$K(u,u_t,u_x,u_z,u_{xx},...)=0. \tag{2}$$

Applying the wave transmutation

$$\xi=x+z-ct,\ u(x,z,t)=U(\xi)=u, \tag{3}$$

here c is speed of wave. Eq. (2) turns into ODE

$$H(U,U',-cU',U'',...)=0. \tag{4}$$

In the form, Eq. (4) is assumed to has

$$U(\xi)=a_0+\sum_{i=1}^{s}\left(a_i\Phi^i(\xi)+a_{-i}\Phi^{-i}(\xi)\right), \tag{5}$$

solution. Here at least one of the coefficients "a_s" is nonzero. In here a_i, $(0 \le i \le s)$ are constants to be determined, $s \in \{1,2,3,...\}$ which is going to be found in Eq. (4) by balancing term is found considering principle of balance and solution of Riccati equation is $\Phi(\xi)$

$$\Phi'(\xi) = \mu + (\Phi(\xi))^2, \tag{6}$$

where μ is any constant. Some special solutions of the Riccati equation in (6) are given as follows.

$$\Phi(\xi) = \begin{cases} -\sqrt{-\mu}\tanh\left(\sqrt{-\mu}\xi\right), & \mu < 0 \\ -\sqrt{-\mu}\coth\left(\sqrt{-\mu}\xi\right), & \mu < 0 \\ \sqrt{\mu}\tan\left(\sqrt{\mu}\xi\right), & \mu > 0 \\ -\sqrt{\mu}\cot\left(\sqrt{\mu}\xi\right), & \mu > 0 \\ -\dfrac{1}{\xi + R}, & \mu = 0\ (R \text{ is a const.}) \end{cases}. \tag{7}$$

When we apply Eq. (6) and Eq. (5) to Eq. (4), we obtain a new polynomial depending on $\Phi(\xi)$; by placing this polynomial in the nonlinear algebraic system in $a_i, (i = 0,1,...,s)$, setting all coefficients to zero, we obtain $\Phi^i(\xi)$, $(i = 0,1,...,s)$. In order to reach the solution in such nonlinear systems, we need to determine the constants c, μ, R, a_i, $(i = 0,1,...,s)$. By means of these constants, we substitute the solutions of Eq. (6) into Eq. (4) together with the formula (7), thus obtaining the analytical solutions for Eq. (2).

3. Application of the method

Consider Eq. (1). We can transform Eq. (1) into a nonlinear ODE by implementing the $\xi = x + z - ct$ transformation.

$$-4cU' + U''' + 6(U')^2 = 0. \tag{8}$$

From the definition of the term balancing in Eq. (8), $s = 1$ is obtained and is written as follows according to Eq. (5).

$$U(\xi) = a_0 + a_1\Phi(\xi) + a_2\frac{1}{\Phi(\xi)}. \tag{9}$$

If Eq. (9) is placed in Eq. (8) and the required editing are done, the following equation systems may be written:

$$\begin{aligned} &(\Phi(\xi))^0: && -4c\mu a_1 + 2\mu^2 a_1 + 6\mu^2 a_1^2 + 4ca_2 - 2\mu a_2 - 24\mu a_1 a_2 + 6a_2^2 = 0, \\ &(\Phi(\xi))^2: && -4ca_1 + 8\mu a_1 + 12\mu a_1^2 - 12a_1 a_2 = 0, \\ &(\Phi(\xi))^4: && 6a_1 + 6a_1^2 = 0, \\ &\frac{1}{(\Phi(\xi))^2}: && 4c\mu a_2 - 8\mu^2 a_2 - 12\mu^2 a_1 a_2 + 12\mu a_2^2 = 0, \\ &\frac{1}{(\Phi(\xi))^4}: && -6\mu^3 a_2 + 6\mu^2 a_2^2 = 0. \end{aligned} \tag{10}$$

a_1, a_2 and c, μ constants are attained from Eq. (10) the system employing a software program.

Case 1: If

$$a_1 = -1, \quad a_2 = \mu, \quad c = -4\mu, \tag{11}$$

then substituting these values from (11) into (9), we obtain a hyperbolic solution to Eq. (1)

$$u_1(x,z,t) = -\frac{\mu \, coth\left[\sqrt{-\mu}\,(x+z+4t\mu)\right]}{\sqrt{-\mu}} + a_0 + \sqrt{-\mu}\, tanh\left[\sqrt{-\mu}\,(x+z+4t\mu)\right]. \tag{12}$$

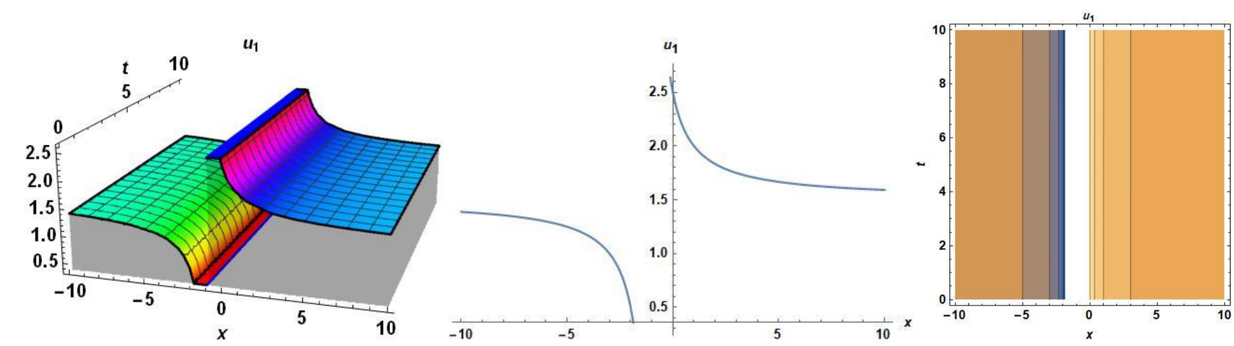

Figure 1. Graphics for $\mu = -0.0001, \quad a_0 = 1.5, \quad z = 1$ values of Eq. (12).

Case 2: If

$$a_1 = -1, \quad a_2 = \mu, \quad c = -4\mu, \tag{13}$$

then substituting these values from (13) into (9), we obtain a hyperbolic solution to Eq. (1)

$$u_2(x,z,t) = \sqrt{-\mu}\, coth\left[\sqrt{-\mu}\,(x+z+4t\mu)\right] + a_0 - \frac{\mu \, tanh\left[\sqrt{-\mu}\,(x+z+4t\mu)\right]}{\sqrt{-\mu}}. \tag{14}$$

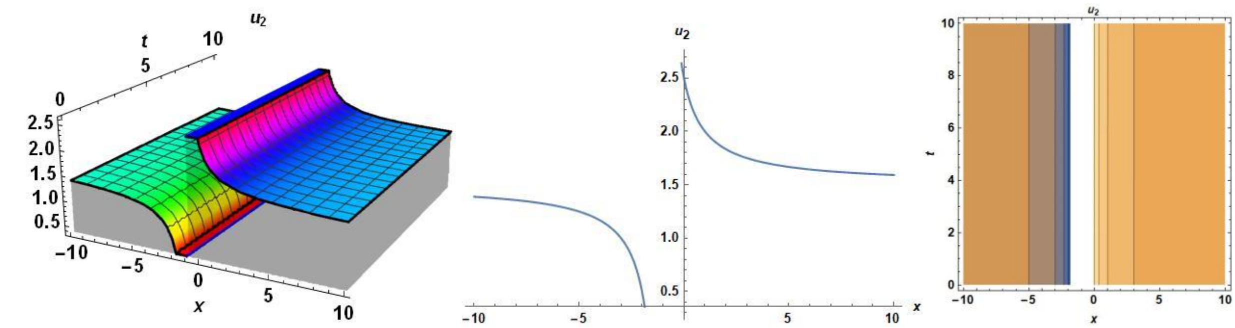

Figure 2. Graphics for $\mu = -0.0001, \quad a_0 = 1.5, \quad z = 1$ values of Eq. (14).

Case 3: If

$$a_1 = -1, \quad a_2 = \mu, \quad c = -4\mu, \tag{15}$$

then substituting these values from (15) into (9), we obtain a trigonometric solution to Eq. (1)

$$u_3(x,z,t) = a_0 - \sqrt{\mu}\, tan\left[\sqrt{\mu}\,(x+z+4t\mu)\right] + \sqrt{\mu}\, cot\left[\sqrt{\mu}\,(x+z+4t\mu)\right]. \tag{16}$$

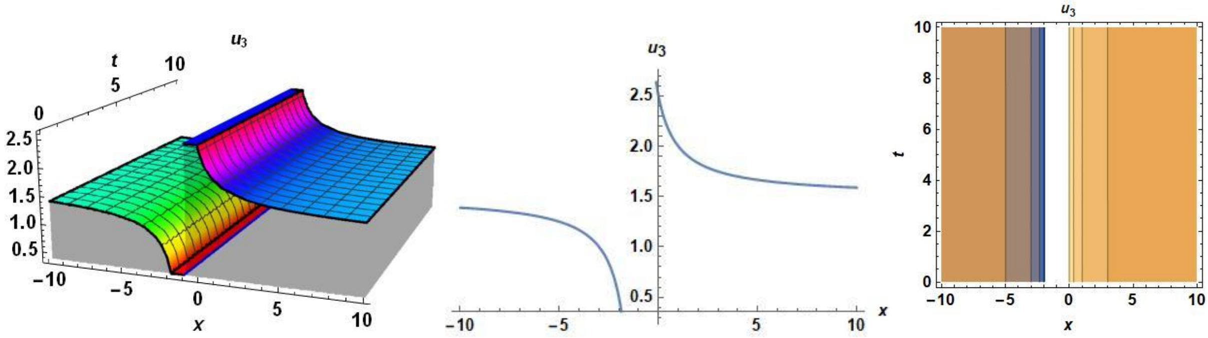

Figure 3. Graphics for $\mu = 0.0001, \quad a_0 = 1.5, \quad z = 1$ values of Eq. (16).

Case 4: If

$$\mu = 0, \quad a_1 = -1, \quad a_2 = \mu, \quad c = -4\mu, \tag{17}$$

then substituting these values from (17) into (9), we obtain a rational solution to Eq. (1)

$$u_4(x,z,t) = \frac{1}{r + x + z} + a_0. \tag{18}$$

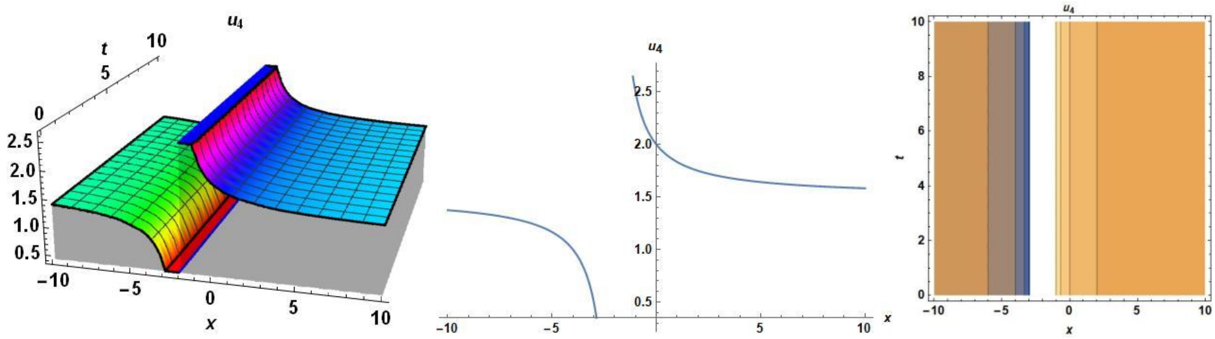

Figure 4. Graphics for $\mu = 0, \quad a_0 = 1.5, \quad r = 1, \quad z = 1$ values of Eq. (18).

Case 5: If

$$a_1 = -1, \quad a_2 = 0, \quad c = -\mu, \tag{19}$$

then substituting these values from (19) into (9), we obtain a hyperbolic solution to Eq. (1)

$$u_5(x,z,t) = a_0 + \sqrt{-\mu}\, tanh\left[\sqrt{-\mu}\left(x + z + t\mu\right)\right]. \tag{20}$$

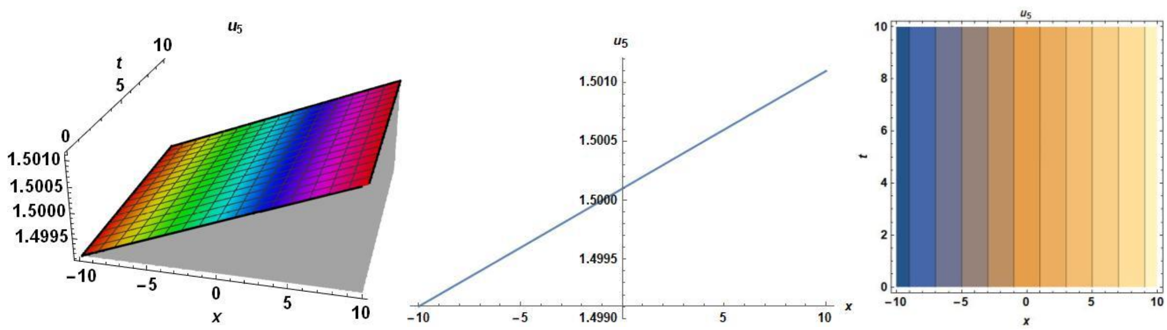

Figure 5. Graphics for $\mu = -0.0001, \quad a_0 = 1.5, \quad z = 1$ values of Eq. (20).

Case 6: If

$$a_1 = -1, \quad a_2 = 0, \quad c = -\mu, \tag{21}$$

then substituting these values from (21) into (9), we obtain a hyperbolic solution to Eq. (1)

$$u_6(x,z,t) = \sqrt{-\mu}\, coth\left[\sqrt{-\mu}\left(x + z + t\mu\right)\right] + a_0. \tag{22}$$

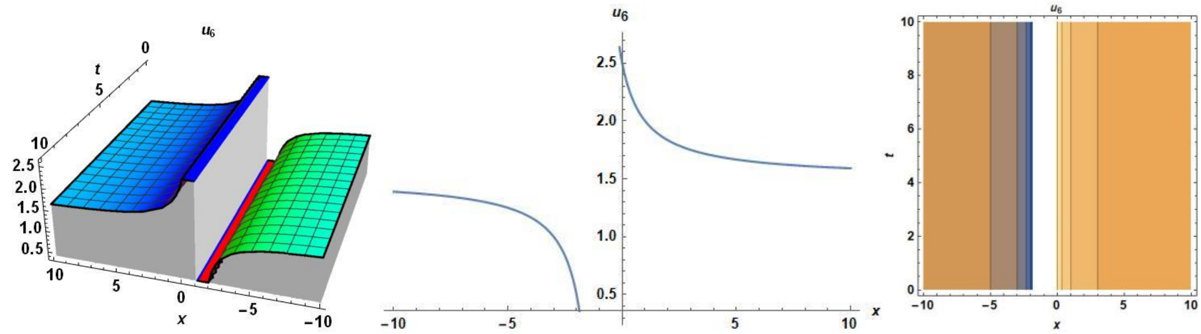

Figure 6. Graphics for $\mu = -0.0001, \quad a_0 = 1.5, \quad z = 1$ values of Eq. (22).

Case 7: If

$$a_1 = -1, \quad a_2 = 0, \quad c = -\mu, \tag{23}$$

then substituting these values from (23) into (9), we obtain a trigonometric solution to Eq. (1)

$$u_7(x,z,t) = a_0 - \sqrt{\mu}\, tan\left[\sqrt{\mu}\left(x + z + t\mu\right)\right]. \tag{24}$$

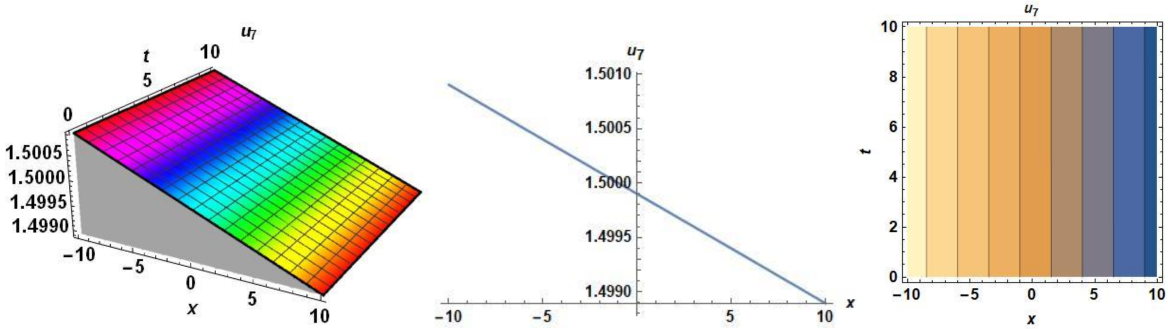

Figure 7. Graphics for $\mu = 0.0001, \quad a_0 = 1.5, \quad z = 1$ values of Eq. (24).

Case 8: If

$$a_1 = -1, \quad a_2 = 0, \quad c = -\mu, \tag{25}$$

then substituting these values from (25) into (9), we obtain a trigonometric solution to Eq. (1)

$$u_8(x,z,t) = a_0 + \sqrt{\mu}\, cot\left[\sqrt{\mu}\left(z + x + t\mu\right)\right]. \tag{26}$$

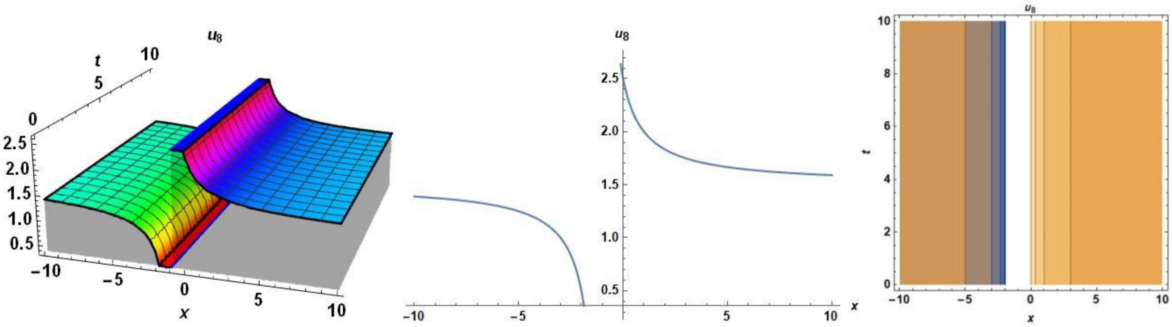

Figure 8. Graphics for $\mu = 0.0001, \quad a_0 = 1.5, \quad z = 1$ values of Eq. (26).

4. Conclusions

In this study, the modified sub equation method, which is the significant instrument used to attain the analytical solution of NEDEs, has been applied. As a consequence of this application, trigonometric, rational, hyperbolic solutions of AKNS equation were produced. The parameters in the solutions are given special values and visualized with contour, 3D and 2D graphics. When physical meanings are given to the constants, the scientific value of the results obtained increases even more. The prominent aspect of this method is that it can provide different and original solutions than other expansion methods. The applied approach has been found to be practical, reliable and suitable for future research. In addition, ready-made software packages have been used to reduce the complexity of the process.

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Conflict of Interest Statement

The author declares that she has no conflict of interest.

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