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Almost Hsu-Golden Structures on Manifolds

Büşra Ay[1], Mustafa Özkan[2,*]

[1]Graduate School of Natural and Applied Sciences, Gazi University, 06500 Ankara, Türkiye.
[2]Department of Mathematics, Faculty of Science, Gazi University, 06560 Ankara, Türkiye.

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Abstract. This study aims to define a new structure inspired by the Hsu and golden structures, termed the almost Hsu-golden structure. We investigate certain properties of this structure and define the almost Hsu-golden B-manifold. Finally, we also examine integrability and parallelism conditions and give some curvature relations for the almost Hsu-golden structure.

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1. Introduction

Hreţcanu [17] defined the golden structure, while Crasmareanu and Hreţcanu [9, 19, 20] examined its application to Riemannian manifolds. Notably, these studies created a suitable framework for subsequent research. Gezer, Salimov, and Cengiz [14] examined the integrability conditions of golden Riemannian structures. Özkan [26] studied the horizontal and complete lifts of the golden structure and its geometry on the tangent bundle. Akyol and Şahin [31] investigated the golden maps between golden Riemannian manifolds and the constancy of these maps. Özkan and Yılmaz [27] examined the r-lift and integrability conditions of the higher-order tangent bundle of the golden structure. Etayo, Santamaría, and Upadhyay [13] examined the first canonical and well-adapted connections to the golden Riemannian structure. Yaşar and Poyraz [28] defined the light-like hypersurfaces of a golden semi-Riemannian manifold and investigated their various properties. Hreţcanu and Blaga [7] defined golden warped product Riemannian manifolds and examined their curvature properties. Erdoğan and Yıldırım [11, 12] carried out studies on semi-invariant and totally umbilical semi-invariant submanifolds of a golden Riemannian manifold. Bahadır and Uddin [1] characterized golden Riemannian manifolds by examining their slant submanifolds. Hreţcanu and Blaga [18] examined warped product pointwise hemi-slant and semi-slant submanifolds in a locally golden Riemannian manifold. Şahin, Şahin, and Erdoğan [32] investigated Norden golden manifolds with a constant sectional curvature, proposing a new notion of Norden golden sectional curvature and examining semi-invariant submanifolds of the Norden golden space form. The existing literature shows that research in this area is constantly advancing and being enriched with new findings.

On the other hand, Hsu [21, 22] created the Hsu-structure and examined the integrability of this structure. Nivas and Verma [24] investigated the semi-symmetric non-metric connection on a manifold with a generalized Hsu-structure, examining the Nijenhuis tensor and integrability conditions of this manifold. Singh [30] introduced a general algebraic Hsu-structure and investigated its integrability conditions. Nivas and Verma [25] examined Hsu-structure manifolds,

*Corresponding Author
Email addresses: busra.ay1@gazi.edu.tr (B. Ay), ozkanm@gazi.edu.tr (M. Özkan)

submanifolds, and Hsu-metric structure manifolds. Bisht and Shanker [4] defined recurrence and different types of symmetry in the Hsu-structure manifold. Bisht [3] defined various forms of the Nijenhuis tensor according to the Hsu-structure and examined some of its properties. Bisht and Shanker [5] expressed the Nijenhius tensor in various forms in a hyperbolic Hsu-structure manifold. Bisht and Shanker [6] investigated the flatness of the Hsu-structure manifold for a variety of curvature tensors. Chand De, Gezer, and Karaman [10] defined the Hsu B-manifold and examined the curvature tensor field of the semi-symmetric F-connection on this manifold.

As possible connections and common applications between the golden structure and the Hsu-structure have not yet been sufficiently investigated, this study aims to fill this gap in the literature.

In this study, inspired by the Hsu and golden structures, we define a new structure called the almost Hsu-golden structure. This paper consists of six sections. The first section summarizes the studies conducted on the golden structure and the Hsu-structure from the past to the present, while the second section provides details on some fundamental ideas and notation. In the third section, we examine the mathematical properties and formulations of the considered structures, including definitions of the almost Hsu-golden structure and almost Hsu-golden B-manifold, along with an examination of certain properties. We investigate the integrability and parallelism conditions of this structure in the fourth section, and provide some curvature relations in the fifth section. We have provided a concise summary of the primary findings and contributions of the investigation in the final section.

2. Definitions and Notations

This section provides fundamental information regarding the Hsu-structure and the golden structure, in order to form a basis for development of the almost Hsu-golden structure.

Through this paper, $\mathfrak{M}$ refers to an n-dimensional C^∞-class differentiable manifold.

We assume that all tensor fields and connections on this manifold are of class C^∞. We denote by $\mathcal{X}(\mathfrak{M})$ the Lie algebra of the vector fields on $\mathfrak{M}$, and I denotes the identity operator on $\mathcal{X}(\mathfrak{M})$.

A polynomial structure with structure polynomial $\mathfrak{Q}(\mathfrak{X}) = \mathfrak{X}^2 - \mathfrak{X} - I$ is termed the golden structure. A golden structure on $\mathfrak{M}$ is defined as a tensor field Φ of type $(1,1)$ and of class C^∞, where $\Phi^2 - \Phi - I = 0$ is satisfied.

A golden Riemannian structure is defined as a pair $(\mathfrak{g},\Phi)$ on $\mathfrak{M}$, where $\mathfrak{g}$ represents a fixed Riemannian metric and $\mathfrak{g}(\Phi\mathfrak{X},\mathfrak{Y}) = \mathfrak{g}(\mathfrak{X},\Phi\mathfrak{Y})$ for $\mathfrak{X},\mathfrak{Y} \in \mathcal{X}(\mathfrak{M})$. The triple $(\mathfrak{M},\mathfrak{g},\Phi)$ is called the golden Riemannian manifold [9, 17, 19, 20].

Having established the foundational aspects of the golden structure, we now turn our attention to the Hsu-structure to further elucidate the comprehensive framework.

Hsu [21,22] constructed a new structure for any $(1,1)$-type tensor field $\mathfrak{F}$ and vector field $\mathfrak{X}$, such that $\mathfrak{F}^2(\mathfrak{X}) = \mathfrak{a}^{\mathfrak{r}} I(\mathfrak{X})$ on a differentiable manifold $\mathfrak{M}$ of class C^∞. This structure is called the Hsu-structure. Here $\mathfrak{a}$ is a complex number and $\mathfrak{r}$ is an integer. Under the given conditions, when $\mathfrak{a} = -1$ and $\mathfrak{r}$ is an odd number, the structure is classified as an almost complex structure. If $\mathfrak{a} = 1$ or $\mathfrak{r} = 0$ $(\mathfrak{a} \neq 0)$, it is categorized as an almost product structure. On the other hand, when $\mathfrak{a} = 0$ $(\mathfrak{r} \neq 0)$, it is classified as an almost tangent structure. Additionally if $\mathfrak{r} = 2$, then it is a $\mathfrak{G}\mathfrak{F}$-structure and for $\mathfrak{a} \neq 0$, it is a π-structure; for $\mathfrak{a} = \pm i$, it is an almost complex structure; for $\mathfrak{a} = 1$, it is an almost product structure; for $\mathfrak{a} = 0$, it is an almost tangent structure [10].

3. Almost Hsu-Golden Structures on Manifolds

In this section, we first introduce a new structure, the almost Hsu-golden structure, and discuss some aspects. Then, we define almost the Hsu-golden B-manifold.

Definition 3.1. Let $\mathfrak{M}$ be a differentiable manifold and $\mathfrak{J}$ be a $(1,1)$-type tensor field that satisfies the equation below

$$\mathfrak{J}^2 = \mathfrak{J} - \frac{1 - 5\mathfrak{a}^{\mathfrak{r}}}{4} I, \tag{3.1}$$

where $\mathfrak{r} \in \mathbb{Z}$, $\mathfrak{a} \in \mathbb{C} \setminus \{0\}$. Here, $\mathfrak{J}$ is called an almost Hsu-golden structure on $\mathfrak{M}$. Furthermore, we designate the pair $(\mathfrak{M},\mathfrak{J})$ as an almost Hsu-golden manifold.

According to this new structure, if we take

- $\mathfrak{a} = 1$ or $\mathfrak{r} = 0$ $(\mathfrak{a} \neq 0)$, then we have a golden structure defined in [9];
- $\mathfrak{a} = 0$ $(\mathfrak{r} \neq 0)$, we obtain a tangent golden structure defined in [9];
- $\mathfrak{a} = -1$ and r is an odd number, then we have a complex golden structure defined in [9];
- $\mathfrak{r} = 2$,

- for $\mathfrak{a} = 0$, we obtain a tangent golden structure defined in [9];
- for $\mathfrak{a} = 1$, we obtain a golden structure defined in [9];
- for $\mathfrak{a} = \mp i$, we obtain a complex golden structure defined in [9].

We highlight certain characteristics of these structures in the following:

Proposition 3.2.

i. *An almost Hsu-golden structure $\mathfrak{J}$ has two eigenvalues: $\sigma = \frac{1+\sqrt{5\mathfrak{a}^{\mathfrak{r}}}}{2}$ and $1-\sigma$.*

ii. *$\forall \mathfrak{p} \in \mathfrak{M}$, the almost Hsu-golden structure $\mathfrak{J}$ is an isomorphism with $1-5\mathfrak{a}^{\mathfrak{r}} \neq 0$ in the tangent space $\mathfrak{T}_{\mathfrak{p}}\mathfrak{M}$.*

iii. *It can be concluded that $\mathfrak{J}$ is invertible with $1-5\mathfrak{a}^{\mathfrak{r}} \neq 0$, and its inverse, written as $\hat{\mathfrak{J}} = \mathfrak{J}^{-1}$, satisfies the equation $\hat{\mathfrak{J}}^2 = \frac{4}{1-5\mathfrak{a}^{\mathfrak{r}}}(\hat{\mathfrak{J}} - I)$.*

Proof.

i. If σ is the eigenvalue of $\mathfrak{J}$ on $\mathcal{X}(\mathfrak{M})$, then $\mathfrak{J}(\mathfrak{X}) = \lambda\mathfrak{X}$ for all $\mathfrak{X} \in \mathcal{X}(\mathfrak{M})$. From (3.1), we have

$$\begin{aligned}\mathfrak{J}^2(\mathfrak{X}) = \mathfrak{J}(\mathfrak{X}) - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}I(\mathfrak{X}) &\Rightarrow \lambda^2\mathfrak{X} = \left(\lambda - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\right)\mathfrak{X}\\ &\Rightarrow \lambda^2 = \lambda - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\ .\end{aligned}$$

Thus, $\lambda_1 = \frac{1+\sqrt{5\mathfrak{a}^{\mathfrak{r}}}}{2} = \sigma$ and $\lambda_2 = \frac{1-\sqrt{5\mathfrak{a}^{\mathfrak{r}}}}{2} = 1-\sigma$ are the eigenvalues of the almost Hsu-golden structure $\mathfrak{J}$.

ii. As $Ker\mathfrak{J} = \{\mathfrak{X} \in \mathcal{X}(\mathfrak{M}) \mid \mathfrak{J}(\mathfrak{X}) = 0\}$ and $\mathfrak{J}$ is linear, from (3.1), we have

$$\begin{aligned}\mathfrak{J}(\mathfrak{J}(\mathfrak{X})) = \mathfrak{J}(\mathfrak{X}) - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}I(\mathfrak{X}) &\Rightarrow \mathfrak{J}(0) = 0 - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\mathfrak{X}\\ &\Rightarrow 0 = 0 - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\mathfrak{X}\\ &\Rightarrow \mathfrak{X} = 0\\ &\Rightarrow Ker\mathfrak{J} = \{0\}.\end{aligned}$$

Therefore, $\mathfrak{J}$ is injective. We now proceed to verify its surjectivity:

$$\begin{aligned}\dim \mathcal{X}(\mathfrak{M}) = rank\mathfrak{J} + \dim(Ker\mathfrak{J}) &\Rightarrow \dim \mathcal{X}(\mathfrak{M}) = \dim \mathfrak{J}(\mathcal{X}(\mathfrak{M}))\\ &\Rightarrow \mathcal{X}(\mathfrak{M}) = \mathfrak{J}(\mathcal{X}(\mathfrak{M})).\end{aligned}$$

Hence, we establish that $\mathfrak{J}$ is surjective.

iii. Given that $\mathfrak{J}$ is an isomorphism, $\mathfrak{J}$ it is invertible. Therefore, from (3.1),

$$\begin{aligned}\mathfrak{J}^2 = \mathfrak{J} - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}I &\Rightarrow \mathfrak{J}^2\mathfrak{J}^{-1} = \mathfrak{J}\mathfrak{J}^{-1} - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\mathfrak{J}^{-1}\\ &\Rightarrow \mathfrak{J} = I - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\mathfrak{J}^{-1}\\ &\Rightarrow \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\mathfrak{J}^{-1}\mathfrak{J}^{-1} = \mathfrak{J}^{-1} - I\\ &\Rightarrow \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\hat{\mathfrak{J}}^2 - \hat{\mathfrak{J}} + I = 0.\end{aligned}$$

□

Proposition 3.3. *It follows that $\tilde{\mathfrak{J}} = I - \mathfrak{J}$ is an almost Hsu-golden structure if $\mathfrak{J}$ is an almost Hsu-golden structure.*

Proof. This can be directly derived from (3.1). □

Theorem 3.4. *An almost Hsu-golden structure is induced by a Hsu-structure $\mathfrak{F}$ in the following way:*

$$\mathfrak{J} = \frac{1}{2}\left(I + \sqrt{5}\mathfrak{F}\right).$$

On the other hand, any almost Hsu-golden structure $\mathfrak{J}$ produces a Hsu-structure:

$$\mathfrak{F} = \sqrt{\mathfrak{a}^{\mathfrak{r}}}\frac{2\mathfrak{J} - I}{2\sigma - 1}.$$

Proof. For a Hsu-structure $\mathfrak{F}$, we find that

$$\begin{aligned}\mathfrak{J}^2 &= (\mathfrak{J} \circ \mathfrak{J})\\ &= \frac{1}{4}\left[\left(I + \sqrt{5}\mathfrak{F}\right) \circ \left(I + \sqrt{5}\mathfrak{F}\right)\right]\\ &= \frac{1}{4}\left(I + \sqrt{5}\mathfrak{F}\right) - \frac{1}{4}\left(1 - 5\mathfrak{a}^{\mathfrak{r}}\right) I\\ &= \mathfrak{J} - \frac{1 - 5\mathfrak{a}^{\mathfrak{r}}}{4} I.\end{aligned}$$

Thus, $\mathfrak{J}$ is an almost Hsu-golden structure. We now proceed to examine the almost Hsu-golden structure $\mathfrak{J}$. This yields

$$\begin{aligned}\mathfrak{F}^2 &= \mathfrak{F} \circ \mathfrak{F}\\ &= \frac{\mathfrak{a}^{\mathfrak{r}}}{(2\sigma - 1)^2}\left[(2\mathfrak{J} - I) \circ (2\mathfrak{J} - I)\right]\\ &= \frac{5(\mathfrak{a}^{\mathfrak{r}})^2}{(2\sigma - 1)^2} I\\ &= \mathfrak{a}^{\mathfrak{r}} I,\end{aligned}$$

establishing that $\mathfrak{F}$ is a Hsu-structure. □

Definition 3.5. An almost Hsu-golden B-structure is defined by a pair $(\mathfrak{g}, \mathfrak{J})$, where $\mathfrak{g}$ is a B-metric, such that

$$\mathfrak{g}(\mathfrak{J}\mathfrak{X}, \mathfrak{Y}) = \mathfrak{g}(\mathfrak{X}, \mathfrak{J}\mathfrak{Y}) \tag{3.2}$$

or, equivalently,

$$\mathfrak{g}(\mathfrak{J}\mathfrak{X}, \mathfrak{J}\mathfrak{Y}) = \mathfrak{g}(\mathfrak{X}, \mathfrak{J}\mathfrak{Y}) - \frac{1 - 5\mathfrak{a}^{\mathfrak{r}}}{4}\mathfrak{g}(\mathfrak{X}, \mathfrak{Y}). \tag{3.3}$$

The triple $(\mathfrak{M}, \mathfrak{g}, \mathfrak{J})$ is referred to as an almost Hsu-golden B-manifold.

4. Integrability and Parallelism of Almost Hsu-Golden Structure

In this section, we investigate the structure's integrability and parallelism conditions.

We introduce some tensor fields characterizing the properties of the Hsu-golden distributions defined by $\mathfrak{J}$ for an almost Hsu-golden B-structure $(\mathfrak{J}, \mathfrak{g})$ on the smooth manifold $\mathfrak{M}$, where ∇ is the Levi–Civita connection of $\mathfrak{g}$, as follows:

- The Nijenhuis tensor of $\mathfrak{J}$
$$\mathfrak{N}_{\mathfrak{J}}(\mathfrak{X}, \mathfrak{Y}) = \mathfrak{J}([\mathfrak{J}\mathfrak{X}, \mathfrak{Y}] + [\mathfrak{X}, \mathfrak{J}\mathfrak{Y}] - \mathfrak{J}([\mathfrak{X}, \mathfrak{Y}])) - [\mathfrak{J}\mathfrak{X}, \mathfrak{J}\mathfrak{Y}], \tag{4.1}$$
- The Jordan tensor associated with $\mathfrak{J}$
$$\mathfrak{M}_{\mathfrak{J}}(\mathfrak{X}, \mathfrak{Y}) = \mathfrak{J}(\{\mathfrak{J}\mathfrak{X}, \mathfrak{Y}\} + \{\mathfrak{X}, \mathfrak{J}\mathfrak{Y}\} - \mathfrak{J}(\{\mathfrak{X}, \mathfrak{Y}\})) - \{\mathfrak{J}\mathfrak{X}, \mathfrak{J}\mathfrak{Y}\}, \tag{4.2}$$
- The deformation tensor associated with $\mathfrak{J}$
$$\mathfrak{H}_{\mathfrak{J}}(\mathfrak{X}, \mathfrak{Y}) = (\mathfrak{J} \circ \nabla_{\mathfrak{X}}\mathfrak{J} - \nabla_{\mathfrak{J}\mathfrak{X}}\mathfrak{J})(\mathfrak{Y}), \tag{4.3}$$
which satisfies $2\mathfrak{H}_{\mathfrak{J}} = \mathfrak{N}_{\mathfrak{J}} + \mathfrak{M}_{\mathfrak{J}}$ [8, 16].

The complementary distributions on $\mathfrak{M}$ are represented by $\mathfrak{V}$ and $\mathfrak{S}$, corresponding to σ and $1 - \sigma$, respectively. Given the corresponding projections, $\mathfrak{v}$ and $\mathfrak{s}$, we have $\mathfrak{v}^2 = \mathfrak{v}$, $\mathfrak{s}^2 = \mathfrak{s}$, $\mathfrak{v} + \mathfrak{s} = I$, and $\mathfrak{v}\mathfrak{s} = \mathfrak{s}\mathfrak{v} = 0$. Then, it is clear that

$$\mathfrak{v} = \frac{\sigma - 1}{2\sigma - 1} I + \frac{1}{2\sigma - 1}\mathfrak{J} \tag{4.4}$$

and

$$\mathfrak{s} = \frac{\sigma}{2\sigma - 1} I - \frac{1}{2\sigma - 1}\mathfrak{J}. \tag{4.5}$$

Proposition 4.1. *Let $(\mathfrak{M}, \mathfrak{g}, \mathfrak{J})$ be an almost Hsu-golden B-manifold. For arbitrary vector fields $\mathfrak{X}$ and $\mathfrak{Y}$ on $\mathfrak{M}$,*

$$\mathfrak{N}_{\mathfrak{F}}(\mathfrak{X}, \mathfrak{Y}) = \frac{4}{5}\mathfrak{N}_{\mathfrak{J}}(\mathfrak{X}, \mathfrak{Y}). \tag{4.6}$$

Proof. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be arbitrary vector fields on $\mathfrak{M}$. From (3.1), (4.1), and Theorem 3.4, we have

$$\begin{aligned}
\mathfrak{N}_{\mathfrak{F}}(\mathfrak{X},\mathfrak{Y}) &= \mathfrak{a}^{\mathfrak{r}}[\mathfrak{X},\mathfrak{Y}] + \frac{1}{5}[(2\mathfrak{J}-I)\mathfrak{X},(2\mathfrak{J}-I)\mathfrak{Y}] - \frac{1}{5}(2\mathfrak{J}-I)[(2\mathfrak{J}-I)\mathfrak{X},\mathfrak{Y}] \\
&\quad - \frac{1}{5}(2\mathfrak{J}-I)[\mathfrak{X},(2\mathfrak{J}-I)\mathfrak{Y}] \\
&= \mathfrak{a}^{\mathfrak{r}}[\mathfrak{X},\mathfrak{Y}] + \frac{2}{5}[\mathfrak{J}\mathfrak{X},(2\mathfrak{J}-I)\mathfrak{Y}] - \frac{1}{5}[\mathfrak{X},(2\mathfrak{J}-I)\mathfrak{Y}] - \frac{2}{5}(2\mathfrak{J}-I)[\mathfrak{J}\mathfrak{X},\mathfrak{Y}] \\
&\quad + \frac{1}{5}(2\mathfrak{J}-I)[\mathfrak{X},\mathfrak{Y}] - \frac{2}{5}(2\mathfrak{J}-I)[\mathfrak{X},\mathfrak{J}\mathfrak{Y}] + \frac{1}{5}(2\mathfrak{J}-I)[\mathfrak{X},\mathfrak{Y}] \\
&= \left(\frac{5\mathfrak{a}^{\mathfrak{r}}-1}{5} + \frac{4}{5}\mathfrak{J}\right)[\mathfrak{X},\mathfrak{Y}] - \frac{4}{5}\mathfrak{J}[\mathfrak{J}\mathfrak{X},\mathfrak{Y}] - \frac{4}{5}\mathfrak{J}[\mathfrak{X},\mathfrak{J}\mathfrak{Y}] + \frac{4}{5}[\mathfrak{J}\mathfrak{X},\mathfrak{J}\mathfrak{Y}] \\
&= \frac{4}{5}\left(\mathfrak{J} - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\right)[\mathfrak{X},\mathfrak{Y}] - \frac{4}{5}\mathfrak{J}[\mathfrak{J}\mathfrak{X},\mathfrak{Y}] - \frac{4}{5}\mathfrak{J}[\mathfrak{X},\mathfrak{J}\mathfrak{Y}] + \frac{4}{5}[\mathfrak{J}\mathfrak{X},\mathfrak{J}\mathfrak{Y}] \\
&= \frac{4}{5}\mathfrak{N}_{\mathfrak{J}}(\mathfrak{X},\mathfrak{Y}).
\end{aligned}$$

□

Remark 4.2.

- **i.** If $\mathfrak{N}_{\mathfrak{J}} = 0$, then $\mathfrak{J}$ is integrable.
- **ii.** For all vector fields $\mathfrak{X}$ and $\mathfrak{Y}$ on $\mathfrak{M}$, the distribution $\mathfrak{V}$ is integrable if $\mathfrak{s}[\mathfrak{v}\mathfrak{X},\mathfrak{v}\mathfrak{Y}] = 0$, and $\mathfrak{S}$ is integrable if $\mathfrak{v}[\mathfrak{s}\mathfrak{X},\mathfrak{s}\mathfrak{Y}] = 0$.

Proposition 4.3. *Let $(\mathfrak{M},\mathfrak{g},\mathfrak{J})$ be an almost Hsu-golden B-manifold. The following relations hold:*

$$\begin{aligned}
\mathfrak{s}[\mathfrak{v}\mathfrak{X},\mathfrak{v}\mathfrak{Y}] &= \frac{1}{5\mathfrak{a}^{\mathfrak{r}}}\mathfrak{s}\mathfrak{N}_{\mathfrak{J}}(\mathfrak{v}\mathfrak{X},\mathfrak{v}\mathfrak{Y}), \\
\mathfrak{v}[\mathfrak{s}\mathfrak{X},\mathfrak{s}\mathfrak{Y}] &= \frac{1}{5\mathfrak{a}^{\mathfrak{r}}}\mathfrak{v}\mathfrak{N}_{\mathfrak{J}}(\mathfrak{s}\mathfrak{X},\mathfrak{s}\mathfrak{Y}), \\
\mathfrak{J}\mathfrak{v} &= \mathfrak{v}\mathfrak{J} = \sigma\mathfrak{v}, \\
\mathfrak{J}\mathfrak{s} &= \mathfrak{s}\mathfrak{J} = (1-\sigma)\mathfrak{s}.
\end{aligned}$$

Based on Remark 4.2 and Proposition 4.3, the following proposition can be established.

Proposition 4.4. *$\mathfrak{V}$ is integrable if and only if $\mathfrak{s}\mathfrak{N}_{\mathfrak{J}}(\mathfrak{v}\mathfrak{X},\mathfrak{v}\mathfrak{Y}) = 0$ and $\mathfrak{S}$ is integrable if and only if $\mathfrak{v}\mathfrak{N}_{\mathfrak{J}}(\mathfrak{s}\mathfrak{X},\mathfrak{s}\mathfrak{Y}) = 0$. The distributions $\mathfrak{V}$ and $\mathfrak{S}$ are both integrable if $\mathfrak{J}$ is integrable.*

Proposition 4.5. *It is evident that $\mathfrak{N}_{\mathfrak{v}} = \mathfrak{N}_{\mathfrak{s}} = \frac{1}{5\mathfrak{a}^{\mathfrak{r}}}\mathfrak{N}_{\mathfrak{J}} = \mathfrak{M}_{\mathfrak{v}} = \mathfrak{M}_{\mathfrak{s}}$ and $\mathfrak{H}_{\mathfrak{v}} = \mathfrak{H}_{\mathfrak{s}} = \frac{1}{5\mathfrak{a}^{\mathfrak{r}}}\mathfrak{H}_{\mathfrak{J}}$ apply to the two projection operators, $\mathfrak{v}$ and $\mathfrak{s}$.*

Proof. For an almost Hsu-golden structure $\mathfrak{J}$, Equations (4.1)–(4.3), Remark 4.2, and Equations (4.4)–(4.6) yield that

$$\mathfrak{N}_{\mathfrak{v}}[\mathfrak{X},\mathfrak{Y}] = \frac{1}{5\mathfrak{a}^{\mathfrak{r}}}\mathfrak{N}_{\mathfrak{J}}[\mathfrak{X},\mathfrak{Y}] = \mathfrak{N}_{\mathfrak{s}}[\mathfrak{X},\mathfrak{Y}]$$

and

$$\mathfrak{M}_{\mathfrak{v}}[\mathfrak{X},\mathfrak{Y}] = \frac{1}{5\mathfrak{a}^{\mathfrak{r}}}\mathfrak{M}_{\mathfrak{J}}[\mathfrak{X},\mathfrak{Y}] = \mathfrak{M}_{\mathfrak{s}}[\mathfrak{X},\mathfrak{Y}].$$

Furthermore, we obtain

$$\begin{aligned}
\mathfrak{H}_{\mathfrak{J}}(\mathfrak{X},\mathfrak{Y}) &= \frac{\mathfrak{N}_{\mathfrak{J}} + \mathfrak{M}_{\mathfrak{J}}}{2} \\
&= \frac{5\mathfrak{a}^{\mathfrak{r}}}{2}(\mathfrak{N}_{\mathfrak{v}}[\mathfrak{X},\mathfrak{Y}] + \mathfrak{M}_{\mathfrak{v}}[\mathfrak{X},\mathfrak{Y}]) \\
\Rightarrow \mathfrak{H}_{\mathfrak{J}}(\mathfrak{X},\mathfrak{Y}) &= 5\mathfrak{a}^{\mathfrak{r}}\mathfrak{H}_{\mathfrak{v}}(\mathfrak{X},\mathfrak{Y})
\end{aligned}$$

and

$$\begin{aligned}\mathfrak{H}_{\mathfrak{J}}(\mathfrak{X},\mathfrak{Y}) &= \frac{5\mathfrak{a}^{\mathfrak{r}}}{2}(\mathfrak{N}_{\mathfrak{s}}[\mathfrak{X},\mathfrak{Y}] + \mathfrak{M}_{\mathfrak{s}}[\mathfrak{X},\mathfrak{Y}])\\ \Rightarrow \mathfrak{H}_{\mathfrak{J}}(\mathfrak{X},\mathfrak{Y}) &= 5\mathfrak{a}^{\mathfrak{r}}\mathfrak{H}_{\mathfrak{s}}(\mathfrak{X},\mathfrak{Y}),\end{aligned}$$

from which we obtain that

$$\mathfrak{H}_{\mathfrak{v}}(\mathfrak{X},\mathfrak{Y}) = \mathfrak{H}_{\mathfrak{s}}(\mathfrak{X},\mathfrak{Y}) = \frac{1}{5\mathfrak{a}^{\mathfrak{r}}}\mathfrak{H}_{\mathfrak{J}}(\mathfrak{X},\mathfrak{Y}).$$ □

Corollary 4.6. *On a B-manifold with an almost Hsu-golden structure, the projectors $\mathfrak{v}$ and $\mathfrak{s}$ exhibit $\mathfrak{g}$-symmetry, while the distributions $\mathfrak{V}$ and $\mathfrak{S}$ are $\mathfrak{g}$-orthogonal. Additionally, the almost Hsu-golden structure is $\mathfrak{N}_{\mathfrak{J}}$-symmetric.*

Proof. The proof of this proposition is a direct consequence of Remark 4.2. □

Lemma 4.7. *Consider $(\mathfrak{M},\mathfrak{g},\mathfrak{J})$ as an almost Hsu-golden B-manifold. Then,*

$$\mathfrak{N}_{\mathfrak{J}}(\mathfrak{X},\mathfrak{Y}) = (\nabla_{\mathfrak{J}\mathfrak{X}}\mathfrak{J})\mathfrak{Y} - (\nabla_{\mathfrak{J}\mathfrak{Y}}\mathfrak{J})\mathfrak{X} + \mathfrak{J}(\nabla_{\mathfrak{Y}}\mathfrak{J})\mathfrak{X} - \mathfrak{J}(\nabla_{\mathfrak{X}}\mathfrak{J})\mathfrak{Y} \tag{4.7}$$

holds for all vector fields $\mathfrak{X}$ and $\mathfrak{Y}$ on $\mathfrak{M}$.

Proof. Considering the Lie bracket $[\mathfrak{X},\mathfrak{Y}] = \nabla_{\mathfrak{X}}\mathfrak{Y} - \nabla_{\mathfrak{Y}}\mathfrak{X}$ and the covariant derivative of $\mathfrak{J}$ with respect to $\mathfrak{X}$ and $\mathfrak{Y}$, $(\nabla_{\mathfrak{X}}\mathfrak{J})\mathfrak{Y} = \nabla_{\mathfrak{X}}\mathfrak{J}\mathfrak{Y} - \mathfrak{J}\nabla_{\mathfrak{X}}\mathfrak{Y}$, we obtain

$$\begin{aligned}\mathfrak{N}_{\mathfrak{J}}(\mathfrak{X},\mathfrak{Y}) =& \mathfrak{J}^2[\mathfrak{X},\mathfrak{Y}] + [\mathfrak{J}\mathfrak{X},\mathfrak{J}\mathfrak{Y}] - \mathfrak{J}[\mathfrak{J}\mathfrak{X},\mathfrak{Y}] - \mathfrak{J}[\mathfrak{X},\mathfrak{J}\mathfrak{Y}]\\ =& \mathfrak{J}(\nabla_{\mathfrak{X}}\mathfrak{Y}) - \mathfrak{J}(\nabla_{\mathfrak{Y}}\mathfrak{X}) - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\nabla_{\mathfrak{X}}\mathfrak{Y} + \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\nabla_{\mathfrak{Y}}\mathfrak{X}\\ &+ (\nabla\mathfrak{J}\mathfrak{X}\mathfrak{J})\mathfrak{Y} + \mathfrak{J}(\nabla_{\mathfrak{J}\mathfrak{X}}\mathfrak{Y}) - (\nabla_{\mathfrak{J}\mathfrak{Y}}\mathfrak{J})\mathfrak{X} - \mathfrak{J}(\nabla_{\mathfrak{J}\mathfrak{Y}}\mathfrak{X})\\ &- \mathfrak{J}(\nabla_{\mathfrak{J}\mathfrak{X}}\mathfrak{Y}) + \mathfrak{J}(\nabla_{\mathfrak{Y}}\mathfrak{J})\mathfrak{X} + \mathfrak{J}(\nabla_{\mathfrak{Y}}\mathfrak{X}) - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\nabla_{\mathfrak{X}}\mathfrak{Y}\\ &- \mathfrak{J}(\nabla_{\mathfrak{X}}\mathfrak{J})\mathfrak{Y} - \mathfrak{J}(\nabla_{\mathfrak{X}}\mathfrak{Y}) + \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\nabla_{\mathfrak{X}}\mathfrak{Y} + \mathfrak{J}(\nabla_{\mathfrak{J}\mathfrak{Y}}\mathfrak{X})\\ =& (\nabla_{\mathfrak{J}\mathfrak{X}}\mathfrak{J})\mathfrak{Y} - (\nabla_{\mathfrak{J}\mathfrak{Y}}\mathfrak{J})\mathfrak{X} + \mathfrak{J}(\nabla_{\mathfrak{Y}}\mathfrak{J})\mathfrak{X} - \mathfrak{J}(\nabla_{\mathfrak{X}}\mathfrak{J})\mathfrak{Y}.\end{aligned}$$ □

When $\mathfrak{N}_{\mathfrak{J}} = 0$, the almost Hsu-golden structure $\mathfrak{J}$ is termed integrable, and the triplet $(\mathfrak{M},\mathfrak{g},\mathfrak{J})$ is referred to as a Hsu-golden *B*-manifold. As a direct consequence of Lemma 4.7, we obtain the following result.

Corollary 4.8. *Consider an almost Hsu-golden B-manifold $(\mathfrak{M},\mathfrak{g},\mathfrak{J})$. When the condition $\nabla\mathfrak{J} = 0$ is satisfied, the Hsu-golden structure $\mathfrak{J}$ is integrable and the triplet $(\mathfrak{M},\mathfrak{g},\mathfrak{J})$ is termed a Hsu-golden B-manifold.*

Based on (4.7), we derive the following integrability condition expressed in terms of a Codazzi-like equation.

Theorem 4.9. *For a given almost Hsu-golden B-manifold $(\mathfrak{M},\mathfrak{g},\mathfrak{J})$, the integrability of $\mathfrak{J}$ is equivalent to the satisfaction of the following Codazzi-like equation:*

$$(\nabla_{\mathfrak{J}\mathfrak{X}}\mathfrak{J})\mathfrak{Y} - \mathfrak{J}(\nabla_{\mathfrak{X}}\mathfrak{J})\mathfrak{Y} = 0$$

for arbitrary vector fields $\mathfrak{X}$ and $\mathfrak{Y}$ on $\mathfrak{M}$.

For a pure tensor Ω of type $(0,a)$, the operator ϕ with respect to $\mathfrak{J}$ is explicitly given by the relation

$$\begin{aligned}(\phi_{\mathfrak{J}}\Omega)(\mathfrak{X},\mathfrak{Y}_1,\ldots,\mathfrak{Y}_a) =& (\mathfrak{J}\mathfrak{X})\Omega(\mathfrak{Y}_1,\ldots,\mathfrak{Y}_a) - \mathfrak{X}\Omega(\mathfrak{J}\mathfrak{Y}_1,\ldots,\mathfrak{Y}_a)\\ &+ \sum_{\lambda=1}^{a}\Omega(\mathfrak{Y}_1,\ldots,(L_{\mathfrak{Y}_\lambda}\mathfrak{J})\mathfrak{X},\ldots,\mathfrak{Y}_a)\end{aligned}$$

for $\mathfrak{X},\mathfrak{Y}_1,\ldots,\mathfrak{Y}_a \in \Gamma(\mathfrak{T}\mathfrak{M})$, where $L_{\mathfrak{Y}}$ denotes the Lie derivative with respect to $\mathfrak{Y}$.

The following theorem further relates to the integrable property of the Hsu-golden structure $\mathfrak{J}$.

Theorem 4.10. *Given an almost Hsu-golden B-manifold $(\mathfrak{M},\mathfrak{g},\mathfrak{J})$, the integrability of $\mathfrak{J}$ is characterized by the vanishing of $\phi_{\mathfrak{J}}\mathfrak{g}$; that is, $\phi_{\mathfrak{J}}\mathfrak{g} = 0$.*

Proof. As the proof's methodology closely resembles that of Theorem 2.1 in Gezer [14], we do not reproduce it here. □

In 1930, the Schouten–van Kampen connection was defined [29], while the Vrănceanu connection was introduced in 1931 [33]. Ianus provided coordinate-free formulas for these connections in [23], and Bejancu named them in [2].

Definition 4.11 ([15])**.** The Schouten–van Kampen and Vrănceanu connections are characterized as follows:

- The Schouten–van Kampen connection
  $\overset{\mathfrak{Sc}}{\nabla}_{\mathfrak{X}} \mathfrak{Y} = \mathfrak{v}(\nabla_{\mathfrak{X}}\mathfrak{v}\mathfrak{Y}) + \mathfrak{s}(\nabla_{\mathfrak{X}}\mathfrak{s}\mathfrak{Y})$;
- The Vrănceanu connection
  $\overset{\mathfrak{V}}{\nabla}_{\mathfrak{X}} \mathfrak{Y} = \mathfrak{v}(\nabla_{\mathfrak{v}\mathfrak{X}}\mathfrak{v}\mathfrak{Y}) + \mathfrak{s}(\nabla_{\mathfrak{s}\mathfrak{X}}\mathfrak{s}\mathfrak{Y}) + \mathfrak{v}[\mathfrak{s}\mathfrak{X}, \mathfrak{v}\mathfrak{Y}] + \mathfrak{s}[\mathfrak{v}\mathfrak{X}, \mathfrak{s}\mathfrak{Y}]$,

where $\mathfrak{X}, \mathfrak{Y} \in \mathcal{X}(\mathfrak{M})$.

Proposition 4.12. *Given any linear connection ∇ on $\mathfrak{M}$, the projectors $\mathfrak{v}$ and $\mathfrak{s}$ are parallel with respect to the Schouten–van Kampen and Vrănceanu connections. Additionally, $\mathfrak{J}$ is parallel with respect to the Schouten–van Kampen and Vrănceanu connections.*

Proof. According to Definition 4.11 and Remark 4.2, for all $\mathfrak{X}, \mathfrak{Y} \in \mathcal{X}(\mathfrak{M})$, the parallelism of projector $\mathfrak{v}$ with regard to the Schouten–van Kampen and Vrănceanu connections is characterized by

$$\begin{aligned}(\overset{\mathfrak{Sc}}{\nabla}_{\mathfrak{X}}\mathfrak{v})\mathfrak{Y} &= \overset{\mathfrak{Sc}}{\nabla}_{\mathfrak{X}}\mathfrak{v}\mathfrak{Y} - \mathfrak{v}(\overset{\mathfrak{Sc}}{\nabla}_{\mathfrak{X}}\mathfrak{Y})\\ &= \mathfrak{v}(\nabla_{\mathfrak{X}}\mathfrak{v}^{y}\mathfrak{Y}) + \mathfrak{s}(\nabla_{\mathfrak{X}}\mathfrak{s}(\mathfrak{v}\mathfrak{Y})) - \mathfrak{v}[\mathfrak{v}(\nabla_{\mathfrak{X}}\mathfrak{v}\mathfrak{Y}) + \mathfrak{s}(\nabla_{\mathfrak{X}}\mathfrak{s}\mathfrak{Y})]\\ &= 0\end{aligned}$$

and

$$\begin{aligned}(\overset{\mathfrak{V}}{\nabla}_{\mathfrak{X}}\mathfrak{v})\mathfrak{Y} &= \overset{\mathfrak{V}}{\nabla}_{\mathfrak{X}}\mathfrak{v}\mathfrak{Y} - \mathfrak{v}(\overset{\mathfrak{V}}{\nabla}_{\mathfrak{X}}\mathfrak{Y})\\ &= \mathfrak{v}(\nabla_{\mathfrak{v}\mathfrak{X}}\mathfrak{v}\mathfrak{Y}) + \mathfrak{v}[\mathfrak{s}\mathfrak{X}, \mathfrak{v}(\mathfrak{v}\mathfrak{Y})] - \mathfrak{v}(\nabla_{\mathfrak{v}\mathfrak{X}}\mathfrak{v}\mathfrak{Y}) - \mathfrak{v}[\mathfrak{s}\mathfrak{X}, \mathfrak{v}\mathfrak{Y}]\\ &= 0.\end{aligned}$$

Analogously, the parallelism of the projector $\mathfrak{s}$ with respect to these connections can be established. Furthermore, the almost Hsu-golden structure $\mathfrak{J}$ is parallel with regard to the Schouten–van Kampen and Vrănceanu connections precisely when

$$\begin{aligned}(\overset{\mathfrak{Sc}}{\nabla}_{\mathfrak{X}}\mathfrak{J})\mathfrak{Y} &= \overset{\mathfrak{Sc}}{\nabla}_{\mathfrak{X}}\mathfrak{J}\mathfrak{Y} - \mathfrak{J}(\overset{\mathfrak{Sc}}{\nabla}_{\mathfrak{X}}\mathfrak{Y})\\ &= \mathfrak{v}(\nabla_{\mathfrak{X}}\mathfrak{v}(\mathfrak{J}\mathfrak{Y}) + \mathfrak{s}(\nabla_{\mathfrak{X}}\mathfrak{s}(\mathfrak{J}\mathfrak{Y})) - \mathfrak{J}[\mathfrak{v}(\nabla_{\mathfrak{X}}\mathfrak{v}\mathfrak{Y}) + \mathfrak{s}(\nabla_{\mathfrak{X}}\mathfrak{s}\mathfrak{Y})]\\ &= 0\end{aligned}$$

and

$$\begin{aligned}(\overset{\mathfrak{V}}{\nabla}_{\mathfrak{X}}\mathfrak{J})\mathfrak{Y} &= \overset{\mathfrak{V}}{\nabla}_{\mathfrak{X}}\mathfrak{J}\mathfrak{Y} - \mathfrak{J}(\overset{\mathfrak{V}}{\nabla}_{\mathfrak{X}}\mathfrak{Y})\\ &= \mathfrak{v}(\nabla_{\mathfrak{v}\mathfrak{X}}\mathfrak{v}(\mathfrak{J}\mathfrak{Y}) + \mathfrak{s}(\nabla_{\mathfrak{s}\mathfrak{X}}\mathfrak{s}(\mathfrak{J}\mathfrak{Y}))) + \mathfrak{v}[\mathfrak{s}\mathfrak{X}, \mathfrak{v}(\mathfrak{J}\mathfrak{Y})] + \mathfrak{s}[\mathfrak{v}\mathfrak{X}, \mathfrak{s}(\mathfrak{J}\mathfrak{Y})]\\ &\quad -\mathfrak{J}\mathfrak{v}(\nabla_{\mathfrak{v}\mathfrak{X}}\mathfrak{v}\mathfrak{Y}) - \mathfrak{J}\mathfrak{s}(\nabla_{\mathfrak{s}\mathfrak{X}}\mathfrak{s}\mathfrak{Y}) - \mathfrak{J}\mathfrak{v}[\mathfrak{s}\mathfrak{X}, \mathfrak{v}\mathfrak{Y}] - \mathfrak{J}\mathfrak{v}[\mathfrak{v}\mathfrak{X}, \mathfrak{s}\mathfrak{Y}]\\ &= 0.\end{aligned}$$

□

Proposition 4.13. *For any linear connection ∇ on $\mathfrak{M}$, regarding both the Schouten–van Kampen and Vrănceanu connections, the distributions $\mathfrak{V}$ and $\mathfrak{S}$ satisfy the parallelism condition.*

Proof. Let $\mathfrak{X} \in \mathcal{X}(\mathfrak{M})$ and $\mathfrak{Y} \in \Gamma(\mathfrak{V})$. Given that $\mathfrak{Y} \in \Gamma(\mathfrak{V})$, it follows that $\mathfrak{s}(\mathfrak{Y}) = 0$ and $\mathfrak{v}(\mathfrak{Y}) = \mathfrak{Y}$, which yields

$$\begin{aligned}\overset{\mathfrak{Sc}}{\nabla}_{\mathfrak{X}}\mathfrak{Y} &= \mathfrak{v}(\nabla_{\mathfrak{X}}\mathfrak{v}\mathfrak{Y}) + \mathfrak{s}(\nabla_{\mathfrak{X}}\mathfrak{s}\mathfrak{Y})\\ &= \mathfrak{v}(\nabla_{\mathfrak{X}}\mathfrak{Y}) \in \mathfrak{V}\end{aligned}$$

and

$$
\begin{aligned}
\overset{\mathfrak{V}}{\nabla}_{\mathfrak{X}}\mathfrak{Y} &= \mathfrak{v}(\nabla_{\mathfrak{X}}\mathfrak{v}\mathfrak{Y}) + \mathfrak{s}(\nabla_{\mathfrak{X}}\mathfrak{s}\mathfrak{Y}) + \mathfrak{v}[\mathfrak{s}\mathfrak{X}, \mathfrak{v}\mathfrak{Y}] + \mathfrak{s}[\mathfrak{v}\mathfrak{X}, \mathfrak{s}\mathfrak{Y}] \\
&= (\mathfrak{v}(\nabla_{\mathfrak{X}}\mathfrak{Y}) + \mathfrak{v}[\mathfrak{s}\mathfrak{X}, \mathfrak{Y}]) \in \mathfrak{V}.
\end{aligned}
$$

Similarly, let $\mathfrak{X} \in \mathcal{X}(\mathfrak{M})$ and $\mathfrak{Y} \in \Gamma(\mathfrak{S})$. Given that $\mathfrak{Y} \in \Gamma(\mathfrak{S})$, it follows that $\mathfrak{v}(\mathfrak{Y}) = 0$ and $\mathfrak{s}(\mathfrak{Y}) = \mathfrak{Y}$, which yields

$$
\begin{aligned}
\overset{\mathfrak{S}c}{\nabla}_{\mathfrak{X}}\mathfrak{Y} &= \mathfrak{v}(\nabla_{\mathfrak{X}}\mathfrak{v}\mathfrak{Y}) + \mathfrak{s}(\nabla_{\mathfrak{X}}\mathfrak{s}\mathfrak{Y}) \\
&= \mathfrak{s}(\nabla_{\mathfrak{X}}\mathfrak{Y}) \in \mathfrak{S}
\end{aligned}
$$

and

$$
\begin{aligned}
\overset{\mathfrak{V}}{\nabla}_{\mathfrak{X}}\mathfrak{Y} &= \mathfrak{v}(\nabla_{\mathfrak{X}}\mathfrak{v}\mathfrak{Y}) + \mathfrak{s}(\nabla_{\mathfrak{X}}\mathfrak{s}\mathfrak{Y}) + \mathfrak{v}[\mathfrak{s}\mathfrak{X}, \mathfrak{v}\mathfrak{Y}] + \mathfrak{s}[\mathfrak{v}\mathfrak{X}, \mathfrak{s}\mathfrak{Y}] \\
&= (\mathfrak{s}(\nabla_{\mathfrak{X}}\mathfrak{Y}) + \mathfrak{s}[\mathfrak{v}\mathfrak{X}, \mathfrak{Y}]) \in \mathfrak{S}.
\end{aligned}
$$

□

Theorem 4.14. *The set of linear connections ∇ such that $\nabla\mathfrak{J} = 0$ is*

$$
\nabla_{\mathfrak{X}}\mathfrak{Y} = \frac{1}{5\mathfrak{a}^{\mathfrak{r}}}\left[\frac{5\mathfrak{a}^{\mathfrak{r}}+1}{2}\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{Y} + 2\mathfrak{J}(\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{J}\mathfrak{Y}) - \mathfrak{J}(\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{Y}) - \tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{J}\mathfrak{Y}\right] + \mathfrak{O}_{\mathfrak{F}}\mathfrak{Q}(\mathfrak{X}, \mathfrak{Y}),
$$

where $\tilde{\nabla}$ represents an arbitrary fixed linear connection and $\mathfrak{Q}$ denotes a $(1,2)$-tensor field such that $\mathfrak{O}_{\mathfrak{F}}\mathfrak{Q}$ corresponds to an associated Obata operator

$$
\mathfrak{O}_{\mathfrak{F}}\mathfrak{Q}(\mathfrak{X}, \mathfrak{Y}) = \frac{1}{2}\left[\mathfrak{Q}(\mathfrak{X}, \mathfrak{J}\mathfrak{Y}) + \frac{\mathfrak{F}}{\sqrt{a^r}}\mathfrak{Q}\left(\mathfrak{X}, \frac{\mathfrak{F}}{\sqrt{a^r}}\mathfrak{J}\mathfrak{Y}\right)\right]
$$

for the corresponding almost Hsu-golden structure.

Proof. For the set of linear connections ∇, where $\tilde{\nabla}$ is an arbitrary fixed linear connection and $\mathfrak{Q}$ is a $(1,2)$-tensor field with $\mathfrak{O}_{\mathfrak{F}}\mathfrak{Q}$ as its associated Obata operator, the application of Equation (3.2) and Proposition 3.3 for all $\mathfrak{X}, \mathfrak{Y} \in \mathcal{X}(\mathfrak{M})$ yields

$$
\begin{aligned}
\nabla_{\mathfrak{X}}\mathfrak{J}\mathfrak{Y} &= \frac{1}{5\mathfrak{a}^{\mathfrak{r}}}\left[\frac{5\mathfrak{a}^{\mathfrak{r}}+1}{2}\left(\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{J}\mathfrak{Y}\right) + 2\mathfrak{J}\left(\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{J}^2\mathfrak{Y}\right) - \mathfrak{J}\left(\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{J}\mathfrak{Y}\right) - \tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{J}^2\mathfrak{Y}\right] \\
&\quad + \mathfrak{O}_{\mathfrak{F}}\mathfrak{Q}(\mathfrak{X}, \mathfrak{J}\mathfrak{Y}) \\
&= \frac{1}{5\mathfrak{a}^{\mathfrak{r}}}\left[\frac{5\mathfrak{a}^{\mathfrak{r}}-1}{2}\left(\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{J}\mathfrak{Y}\right) + \mathfrak{J}\left(\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{J}\mathfrak{Y}\right) - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\mathfrak{J}\left(\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{Y}\right)\right. \\
&\quad \left. + \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\,\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{Y}\right] + \mathfrak{O}_{\mathfrak{F}}\mathfrak{Q}(\mathfrak{X}, \mathfrak{J}\mathfrak{Y}),
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{J}(\nabla_{\mathfrak{X}}\mathfrak{Y}) &= \frac{1}{5\mathfrak{a}^{\mathfrak{r}}}\left[\frac{5\mathfrak{a}^{\mathfrak{r}}+1}{2}\mathfrak{J}\left(\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{Y}\right) + 2\mathfrak{J}^2\left(\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{J}\mathfrak{Y}\right) - \mathfrak{J}^2\left(\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{Y}\right) - \mathfrak{J}\left(\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{J}\mathfrak{Y}\right)\right] \\
&\quad + \mathfrak{J}(\mathfrak{O}_{\mathfrak{F}}\mathfrak{Q}(\mathfrak{X}, \mathfrak{Y})) \\
&= \frac{1}{5\mathfrak{a}^{\mathfrak{r}}}\left[\frac{5\mathfrak{a}^{\mathfrak{r}}-1}{2}\mathfrak{J}\left(\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{Y}\right) + \mathfrak{J}\left(\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{J}\mathfrak{Y}\right) - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{2}\,\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{J}\mathfrak{Y}\right. \\
&\quad \left. + \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\,\tilde{\nabla}_{\mathfrak{X}}\,\mathfrak{Y}\right] + \mathfrak{J}(\mathfrak{O}_{\mathfrak{F}}\mathfrak{Q}(\mathfrak{X}, \mathfrak{Y})),
\end{aligned}
$$

and

$$\mathfrak{O}_{\mathfrak{F}}\mathfrak{Q}(\mathfrak{X},\mathfrak{J}\mathfrak{Y}) = \frac{1}{2}\left[\mathfrak{Q}(\mathfrak{X},\mathfrak{J}\mathfrak{Y}) + \frac{\mathfrak{F}}{\sqrt{\mathfrak{a}^{\mathfrak{r}}}}\mathfrak{Q}\left(\mathfrak{X}, \frac{\mathfrak{F}}{2\sqrt{\mathfrak{a}^{\mathfrak{r}}}}\mathfrak{Y} + \frac{\sqrt{5}\mathfrak{a}^{\mathfrak{r}}}{2\sqrt{\mathfrak{a}^{\mathfrak{r}}}}\mathfrak{Y}\right)\right]$$
$$= \frac{1}{2}\left[(\mathfrak{J}\mathfrak{Q}(\mathfrak{X},\mathfrak{Y}) + \frac{1}{\mathfrak{a}^{\mathfrak{r}}}\mathfrak{F}\mathfrak{J}\mathfrak{Q}(\mathfrak{X},\mathfrak{F}\mathfrak{Y})\right].$$

Moreover,

$$\mathfrak{J}(\mathfrak{O}_{\mathfrak{F}}\mathfrak{Q}(\mathfrak{X},\mathfrak{Y})) = \frac{1}{2}\left[\mathfrak{J}\mathfrak{Q}(\mathfrak{X},\mathfrak{Y}) + \frac{\mathfrak{F}}{\sqrt{\mathfrak{a}^{\mathfrak{r}}}}\mathfrak{J}\mathfrak{Q}\left(\mathfrak{X}, \frac{\mathfrak{F}}{\sqrt{\mathfrak{a}^{\mathfrak{r}}}}\mathfrak{Y}\right)\right]$$

and $\mathfrak{O}_{\mathfrak{F}}\mathfrak{Q}(\mathfrak{X},\mathfrak{J}\mathfrak{Y}) = \mathfrak{J}(\mathfrak{O}_{\mathfrak{F}}\mathfrak{Q}(\mathfrak{X},\mathfrak{Y}))$. As a result, we obtain

$$(\nabla_{\mathfrak{X}}\mathfrak{J})\mathfrak{Y} = \nabla_{\mathfrak{X}}\mathfrak{J}\mathfrak{Y} - \mathfrak{J}(\nabla_{\mathfrak{X}}\mathfrak{Y}) = 0,$$

which implies that $\nabla\mathfrak{J} = 0$. □

5. Curvature Tensor Field of Almost Hsu-Golden *B*-Manifolds

This section provides some properties of the Riemannian curvature tensor of an almost Hsu-golden *B*-manifold $(\mathfrak{M},\mathfrak{J},\mathfrak{g})$.

Let $(\mathfrak{M},\mathfrak{J},\mathfrak{g})$ denote an almost Hsu-golden *B*-manifold, and let $\mathfrak{R}$ be the Riemannian curvature tensor of $\mathfrak{g}$. Within this framework, the curvature operator acting on vector fields $\mathfrak{X},\mathfrak{Y},\mathfrak{Z},\mathfrak{W}\in\mathcal{X}(\mathfrak{M})$ is characterized by

$$\mathfrak{R}(\mathfrak{X},\mathfrak{Y})\mathfrak{Z} = \nabla_{\mathfrak{Y}}\nabla_{\mathfrak{X}}\mathfrak{Z} - \nabla_{\mathfrak{X}}\nabla_{\mathfrak{Y}}\mathfrak{Z} + \nabla_{[\mathfrak{X},\mathfrak{Y}]}\mathfrak{Z}.$$

The corresponding curvature tensor, defined for smooth vector fields $\mathfrak{X},\mathfrak{Y},\mathfrak{Z},\mathfrak{W}\in\mathcal{X}(\mathfrak{M})$, takes the form

$$\mathfrak{R}(\mathfrak{X},\mathfrak{Y},\mathfrak{Z},\mathfrak{W}) = \mathfrak{g}(\mathfrak{R}(\mathfrak{X},\mathfrak{Y})\mathfrak{Z},\mathfrak{W}).$$

Lemma 5.1. *Let $(\mathfrak{M},\mathfrak{J},\mathfrak{g})$ denote an almost Hsu-golden B-manifold. We have that*

$$\mathfrak{R}(\mathfrak{J}\mathfrak{X},\mathfrak{J}\mathfrak{Y}) = \mathfrak{R}(\mathfrak{X},\mathfrak{J}\mathfrak{Y}) - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\mathfrak{R}(\mathfrak{X},\mathfrak{Y})$$

for any vector fields $\mathfrak{X}$ and $\mathfrak{Y}$ on $\mathfrak{M}$.

Proof. Through Equations (3.2) and (3.3), direct calculations, and the known properties of $\mathfrak{R}$, we establish that

$$\mathfrak{g}(\mathfrak{R}(\mathfrak{J}\mathfrak{X},\mathfrak{J}\mathfrak{Y})\mathfrak{Z},\mathfrak{W}) = \mathfrak{g}((\mathfrak{R}(\mathfrak{X},\mathfrak{J}\mathfrak{Y}) - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\mathfrak{R}(\mathfrak{X},\mathfrak{Y}))\mathfrak{Z},\mathfrak{W})$$

for all $\mathfrak{X},\mathfrak{Y},\mathfrak{Z},\mathfrak{W}\in\mathcal{X}(\mathfrak{M})$. Consequently,

$$\mathfrak{R}(\mathfrak{J}\mathfrak{X},\mathfrak{J}\mathfrak{Y}) = \mathfrak{R}(\mathfrak{X},\mathfrak{J}\mathfrak{Y}) - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\mathfrak{R}(\mathfrak{X},\mathfrak{Y}). \quad □$$

Proposition 5.2. *If $(\mathfrak{M},\mathfrak{J},\mathfrak{g})$ represents an almost Hsu-golden B-manifold, then the Riemannian curvature tensor $\mathfrak{R}$ of $\mathfrak{g}$ satisfies the following identities:*

- $\mathfrak{R}(\mathfrak{J}\mathfrak{X},\mathfrak{Y},\mathfrak{Z},\mathfrak{W}) = \mathfrak{R}(\mathfrak{X},\mathfrak{J}\mathfrak{Y},\mathfrak{Z},\mathfrak{W})$,
- $\mathfrak{R}(\mathfrak{X},\mathfrak{Y},\mathfrak{J}\mathfrak{Z},\mathfrak{W}) = \mathfrak{R}(\mathfrak{X},\mathfrak{Y},\mathfrak{Z},\mathfrak{J}\mathfrak{W})$,
- $\mathfrak{R}(\mathfrak{J}\mathfrak{X},\mathfrak{J}\mathfrak{Y},\mathfrak{Z},\mathfrak{W}) = \mathfrak{R}(\mathfrak{X},\mathfrak{J}\mathfrak{Y},\mathfrak{Z},\mathfrak{W}) - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\mathfrak{R}(\mathfrak{X},\mathfrak{Y},\mathfrak{Z},\mathfrak{W})$,
- $\mathfrak{R}(\mathfrak{X},\mathfrak{Y},\mathfrak{J}\mathfrak{Z},\mathfrak{J}\mathfrak{W}) = \mathfrak{R}(\mathfrak{X},\mathfrak{Y},\mathfrak{Z},\mathfrak{J}\mathfrak{W}) - \frac{1-5\mathfrak{a}^{\mathfrak{r}}}{4}\mathfrak{R}(\mathfrak{X},\mathfrak{Y},\mathfrak{Z},\mathfrak{W})$,
- $\mathfrak{R}(\mathfrak{X},\mathfrak{J}\mathfrak{Y},\mathfrak{J}\mathfrak{Z},\mathfrak{W}) = \mathfrak{R}(\mathfrak{J}\mathfrak{X},\mathfrak{Y},\mathfrak{J}\mathfrak{Z},\mathfrak{W}) = \mathfrak{R}(\mathfrak{J}\mathfrak{X},\mathfrak{Y},\mathfrak{Z},\mathfrak{J}\mathfrak{W}) = \mathfrak{R}(\mathfrak{X},\mathfrak{J}\mathfrak{Y},\mathfrak{Z},\mathfrak{J}\mathfrak{W})$

for any $\mathfrak{X},\mathfrak{Y},\mathfrak{Z},\mathfrak{W}\in\mathcal{X}(\mathfrak{M})$.

Proof. Through the application of Equations (3.2) and (3.3), Lemma 5.1, and the standard properties of $\mathfrak{R}$, one can verify that the proposition holds. □

Lemma 5.3. *Consider* $(\mathfrak{M}, \mathfrak{J}, \mathfrak{g})$ *as an almost Hsu-golden B-manifold, and let* $\mathfrak{R}$ *be its curvature tensor. We have the subsequent relation:*

$$\mathfrak{g}(\mathfrak{R}(\mathfrak{X}, \mathfrak{J}\mathfrak{X})\mathfrak{Y}, \mathfrak{Z}) = \mathfrak{g}(\mathfrak{R}(\mathfrak{X}, \mathfrak{J}\mathfrak{X})\mathfrak{Y}, \mathfrak{J}\mathfrak{Y}) = \mathfrak{g}(\mathfrak{R}(\mathfrak{X}, \mathfrak{J}\mathfrak{X})\mathfrak{J}\mathfrak{X}, \mathfrak{X}) = 0$$

for arbitrary vector fields $\mathfrak{X}$ *,* $\mathfrak{Y}$*, and* $\mathfrak{Z}$ *on* $\mathcal{X}(\mathfrak{M})$*.*

Proof. Using (3.2) and Proposition 5.2, the lemma follows immediately as a direct consequence of the algebraic properties involved. □

6. Conclusions

This paper presents a novel geometric structure termed the virtually Hsu-golden structure, derived from the recognised Hsu and golden structures. We commenced by synthesising the advancements and current findings pertaining to golden and Hsu-structures from historical to contemporary contexts. Subsequently, we established the definitions of the almost Hsu-golden structure and the almost Hsu-golden B-manifold, and analysed their fundamental aspects through theoretical examination and illustrative instances. Additionally, we examined the constraints for integrability and parallelism related to this novel structure. In the final part of the study, we established certain curvature relations that enhance understanding of its geometric properties. The findings of this study indicate that the almost Hsu-golden structure is a viable avenue for further investigation in differential geometry, with possible implications in both theoretical and applied mathematics.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

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