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On Some Inequalities for Convex and s-Convex Functions

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Abstract
In this paper, we present some integral inequalities for convex and $s-$convex functions and we give some remarks and propositions.

Keywords: *Convex functions, Hermite-Hadamard Inequality, s-convex functions.*

1. *Introduction*

Let f be defined from I which is a subset of $[0,\infty)$ to $\mathbb{R}$ and a differantiable mapping on the interior of the interval I (I°) such that $f' \in L[a,b]$ where a and b belongs to I with a is less than b. If $|f'|$ is bounded and its upper bound is M, then the inequality given below holds:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{M}{b-a}\left[\frac{(x-a)^2+(b-x)^2}{2}\right]. \quad (1.1)$$

This inequality is Ostrowski inequality which is well known in the literature (see [5]). One can check [6] to see some new results related to Ostrowski inequality.

The function f defined from $[a,b]$ interval which is a subset of $\mathbb{R}$ to $\mathbb{R}$, is called to be convex if the inequality given below holds

$$f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$$

for all $x,y \in [a,b]$ and $\lambda \in [0,1]$.. If $(-f)$ is convex then f is called as concave.

In [3], Hudzik and Maligranda defined a new class of convex functions namely s-convex in the second sense..

This class of functions are defined from R_+ to R and it is said to be s-convex in the second sense providing that the inequality

$$f(\alpha x+\beta y) \leq \alpha^s f(x)+\beta^s f(y)$$

holds for $\alpha,\beta \geq 0$ with $\alpha+\beta=1$ with some fixed $s \in (0,1]$ and for all $x,y \in [0,\infty)$. K_s^2 is used to symbolise this class of functions.

For $s=1$ it is easy to see that, classical convexity is gathered from definition of s-convexity defined on $[0,\infty)$.

Alomari has handled the next equality to prove some theorems in [1].

Lemma 1 *[1] Let $f:[a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b). For $\lambda \in [0,1]$, $a+\lambda\frac{b-a}{2} \leq x \leq \frac{a+b}{2}$ and the mapping $K(x,t)$ defined as.*

$$K(x,t)=\begin{cases} t-\left(a+\lambda\dfrac{b-a}{2}\right), & t\in[a,x] \\ t-\dfrac{a+b}{2}, & t\in(x,a+b-x] \\ t-\left(b-\lambda\dfrac{b-a}{2}\right), & t\in(a+b-x,b] \end{cases}$$

we have

$$\int_a^b K(x,t)f'(t)dt=(b-a)\left[\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)\frac{f(x)+f(a+b-x)}{2}\right]-\int_a^b f(t)dt.$$

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Now we will give some theorems including Ostrowski type inequalities and also remarks and propositions by using previous lemma proved by Alomari .

2. Main Results

Theorem 1 *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable and absolutely continuous mapping on I° where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a,b]$ and $f' \in L[a,b]$ we have*

$$\left|(b-a)\left[\lambda \frac{f(a)+f(b)}{2} + (1-\lambda)\frac{f(x)+f(a+b-x)}{2}\right] - \int_a^b f(t)dt\right|$$

$$\leq \left[\frac{\lambda^3(b-a)^3}{24(x-a)} + \frac{(x-a)^2}{3} - \frac{\lambda(b-a)(x-a)}{4} + \frac{(a+b-2x)^2}{8}\right]$$

$$\times [|f'(x)| + |f'(a+b-2x)|]$$

$$+\left[\frac{\left(2x-2a-2\lambda(b-a)\right)^3}{24(x-a)} + \frac{(x-a)^2}{3} + \frac{\left(2a-2x+\lambda(b-a)\right)(x-a)}{4}\right]$$

$$\times [|f'(a)| + |f'(b)|]$$

for all $\lambda \in [0,1]$ and $a + \lambda \frac{b-a}{2} \leq x \leq \frac{a+b}{2}$.

Proof. From Lemma 1 and using the properties of the modulus, we have

$$|M| = \left|(b-a)\left[\lambda \frac{f(a)+f(b)}{2} + (1-\lambda)\frac{f(x)+f(a+b-x)}{2}\right] - \int_a^b f(t)dt\right|$$
$$\leq \int_a^b |K(x,t)||f'(t)|dt$$
$$= \int_a^x \left|t - \left(a + \lambda\frac{b-a}{2}\right)\right| |f'(t)|dt + \int_x^{a+b-x} \left|t - \frac{a+b}{2}\right| |f'(t)|dt$$
$$+ \int_{a+b-x}^b \left|t - \left(b - \lambda\frac{b-a}{2}\right)\right| |f'(t)|dt.$$

Since $|f'|$ is convex on $[a,b] = [a,x] \cup (x, a+b-x] \cup (a+b-x, b]$, we have

$$|f'(t)| \leq \frac{t-a}{x-a}|f'(x)| + \frac{x-t}{x-a}|f'(a)|, \qquad t \in a,x];$$

$$|f'(t)| \leq \frac{t-x}{a+b-2x}|f'(a+b-x)| + \frac{a+b-x-t}{a+b-2x}|f'(x)|, \qquad t \in (x, a+b-x];$$

and

$$|f'(t)| \leq \frac{t-a-b+x}{x-a}|f'(b)| + \frac{b-t}{x-a}|f'(a+b-x)|, \quad t \in (a+b-x, b];$$

which follows that

$$
\begin{aligned}
|M| &\leq \int_a^x \left|t-\left(a+\lambda\frac{b-a}{2}\right)\right|\left[\frac{t-a}{x-a}|f'(x)|+\frac{x-t}{x-a}|f'(a)|\right]dt \\
&+\int_x^{a+b-x}\left|t-\frac{a+b}{2}\right|\left[\frac{t-x}{a+b-2x}|f'(a+b-x)|+\frac{a+b-x-t}{a+b-2x}|f'(x)|\right]dt \\
&+\int_{a+b-x}^{b}\left|t-\left(b-\lambda\frac{b-a}{2}\right)\right|\left[\frac{t-a-b+x}{x-a}|f'(b)|+\frac{b-t}{x-a}|f'(a+b-x)|\right]dt \\
&=\int_a^{a+\lambda\frac{b-a}{2}}\left(a+\lambda\frac{b-a}{2}-t\right)\left[\frac{t-a}{x-a}|f'(x)|+\frac{x-t}{x-a}|f'(a)|\right]dt \\
&+\int_{a+\lambda\frac{b-a}{2}}^{x}\left(t-a-\lambda\frac{b-a}{2}\right)\left[\frac{t-a}{x-a}|f'(x)|+\frac{x-t}{x-a}|f'(a)|\right]dt \\
&+\int_x^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)\left[\frac{t-x}{a+b-2x}|f'(a+b-x)|+\frac{a+b-x-t}{a+b-2x}|f'(x)|\right]dt \\
&+\int_{\frac{a+b}{2}}^{a+b-x}\left(t-\frac{a+b}{2}\right)\left[\frac{t-x}{a+b-2x}|f'(a+b-x)|+\frac{a+b-x-t}{a+b-2x}|f'(x)|\right]dt \\
&+\int_{a+b-x}^{b-\lambda\frac{b-a}{2}}\left(b-\lambda\frac{b-a}{2}-t\right)\left[\frac{t-a-b+x}{x-a}|f'(b)|+\frac{b-t}{x-a}|f'(a+b-x)|\right]dt \\
&+\int_{b-\lambda\frac{b-a}{2}}^{b}\left(t-b+\lambda\frac{b-a}{2}\right)\left[\frac{t-a-b+x}{x-a}|f'(b)|+\frac{b-t}{x-a}|f'(a+b-x)|\right]dt \\
&=\frac{1}{24}\left[3(a+b-2x)^2+8(a-x)^2-6(a-b)(a-x)\lambda+\frac{(a-b)^3\lambda^3}{a-x}\right] \\
&\times(|f'(x)|+|f'(a+b-x)|) \\
&+\frac{1}{24}\left[4(a-x)^2-6(a-b)(a-x)\lambda+6(a-b)^2\lambda^2-\frac{(a-b)^3\lambda^3}{a-x}\right] \\
&\times(|f'(a)|+|f'(b)|)
\end{aligned}
$$

where we use the facts

$$
\begin{aligned}
&\int_a^{a+\lambda\frac{b-a}{2}}\left(a+\lambda\frac{b-a}{2}-t\right)(t-a)dt=\int_{b-\lambda\frac{b-a}{2}}^{b}\left(t-b+\lambda\frac{b-a}{2}\right)(b-t)dt \\
&=\frac{1}{48}\lambda^3(b-a)^3,
\end{aligned}
$$

$$
\begin{aligned}
&\int_{a+\lambda\frac{b-a}{2}}^{x}\left(t-a-\lambda\frac{b-a}{2}\right)(x-t)dt=\int_{a+b-x}^{b-\lambda\frac{b-a}{2}}\left(b-\lambda\frac{b-a}{2}-t\right)(t-a-b+x)dt \\
&=\frac{1}{48}(2x-2a+a\lambda-b\lambda)^3,
\end{aligned}
$$

$$
\begin{aligned}
&\int_x^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)(t-x)dt=\int_{\frac{a+b}{2}}^{a+b-x}\left(t-\frac{a+b}{2}\right)(a+b-x-t)dt \\
&=\frac{1}{48}(a+b-2x)^3,
\end{aligned}
$$

$$
\begin{aligned}
&\int_x^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)(a+b-x-t)dt=\int_{\frac{a+b}{2}}^{a+b-x}\left(t-\frac{a+b}{2}\right)(t-x)dt \\
&=\frac{5}{48}(a+b-2x)^3,
\end{aligned}
$$

$$
\int_{a+b-x}^{b-\lambda\frac{b-a}{2}}\left(b-\lambda\frac{b-a}{2}-t\right)(b-t)dt=\int_{a+\lambda\frac{b-a}{2}}^{x}\left(t-a-\lambda\frac{b-a}{2}\right)(t-a)dt
$$

$$= -\frac{1}{48}(2x - 2a + a\lambda - b\lambda)^2(4a - 4x + \lambda a - \lambda b)$$

and

$$\int_a^{a+\lambda\frac{b-a}{2}} \left(a + \lambda\frac{b-a}{2} - t\right)(x - t)dt = \int_{b-\lambda\frac{b-a}{2}}^{b} \left(t - b + \lambda\frac{b-a}{2}\right)(t - a - b + x)dt$$

$$= \frac{1}{48}\lambda^2(b - a)^2(6x - 6a + \lambda a - \lambda b).$$

So, the proof is completed.

Remark 1 *In Theorem 1, if we choose just $\lambda = 0$ and $\lambda = 0$ with $x = \frac{a+b}{2}$ we have Theorem 5 and Corollary $2 - (2)$ respectively which were proved in [2].*

Theorem 2 *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable and absolutely continuous mapping on I° where $a, b \in I$ with $a < b$. If $|f'|$ is $s-$convex on $[a, b]$ and $f' \in L[a, b]$ we have*

$$\left|(b - a)\left[\lambda\frac{f(a)+f(b)}{2} + (1 - \lambda)\frac{f(x)+f(a+b-x)}{2}\right] - \int_a^b f(t)dt\right|$$

$$\leq \left[\frac{2}{(x-a)^s(s+1)(s+2)}\left(\lambda\frac{b-a}{2}\right)^{s+2} + \frac{(x-a)^2}{s+2} - \left(\lambda\frac{b-a}{2}\right)\frac{(x-a)}{s+1}\right.$$

$$\left. + \frac{s2^s+1}{2^{s+1}(s+1)(s+2)}(a + b - 2x)^2\right][|f'(x)| + |f'(a + b - 2x)|]$$

$$+ \left[\frac{2}{(x-a)^s(s+1)(s+2)}\left(x - a - \lambda\frac{b-a}{2}\right)^{s+2} + \frac{(x-a)^2}{s+2}\right.$$

$$\left. + \left(a + \lambda\frac{b-a}{2} - x\right)\frac{(x-a)}{s+1}\right][|f'(a)| + |f'(b)|]$$

for all $\lambda \in [0,1]$ and $a + \lambda\frac{b-a}{2} \leq x \leq \frac{a+b}{2}$.

Proof. From Lemma 1 and using the properties of the modulus, we have

$$|M| = \left|(b - a)\left[\lambda\frac{f(a)+f(b)}{2} + (1 - \lambda)\frac{f(x)+f(a+b-x)}{2}\right] - \int_a^b f(t)dt\right|$$

$$\leq \int_a^b |K(x, t)||f'(t)|dt$$

$$= \int_a^x \left|t - \left(a + \lambda\frac{b-a}{2}\right)\right| |f'(t)|dt + \int_x^{a+b-x} \left|t - \frac{a+b}{2}\right| |f'(t)|dt$$

$$+ \int_{a+b-x}^{b} \left|t - \left(b - \lambda\frac{b-a}{2}\right)\right| |f'(t)|dt.$$

Since $|f'|$ is $s-$convex on $[a, b] = [a, x] \cup (x, a + b - x] \cup (a + b - x, b]$, therefore we have

$$|f'(t)| \leq \left(\frac{t-a}{x-a}\right)^s |f'(x)| + \left(\frac{x-t}{x-a}\right)^s |f'(a)|, \qquad t \in a, x];$$

$$|f'(t)| \leq \left(\frac{t-x}{a+b-2x}\right)^s |f'(a+b-x)| + \left(\frac{a+b-x-t}{a+b-2x}\right)^s |f'(x)|, \qquad t \in (x, a+b-x]$$

and

$$|f'(t)| \leq \left(\frac{t-a-b+x}{x-a}\right)^s |f'(b)| + \left(\frac{b-t}{x-a}\right)^s |f'(a+b-x)|, \quad t \in (a+b-x, b]$$

which follows that

$$|M| \leq \int_a^x \left|t - \left(a + \lambda\frac{b-a}{2}\right)\right| \left[\left(\frac{t-a}{x-a}\right)^s |f'(x)| + \left(\frac{x-t}{x-a}\right)^s |f'(a)|\right] dt$$
$$+ \int_x^{a+b-x} \left|t - \frac{a+b}{2}\right| \left[\left(\frac{t-x}{a+b-2x}\right)^s |f'(a+b-x)| + \left(\frac{a+b-x-t}{a+b-2x}\right)^s |f'(x)|\right] dt$$
$$+ \int_{a+b-x}^b \left|t - \left(b - \lambda\frac{b-a}{2}\right)\right| \left[\left(\frac{t-a-b+x}{x-a}\right)^s |f'(b)| + \left(\frac{b-t}{x-a}\right)^s |f'(a+b-x)|\right] dt$$
$$= \int_a^{a+\lambda\frac{b-a}{2}} \left(a + \lambda\frac{b-a}{2} - t\right) \left[\left(\frac{t-a}{x-a}\right)^s |f'(x)| + \left(\frac{x-t}{x-a}\right)^s |f'(a)|\right] dt$$
$$+ \int_{a+\lambda\frac{b-a}{2}}^x \left(t - a - \lambda\frac{b-a}{2}\right) \left[\left(\frac{t-a}{x-a}\right)^s |f'(x)| + \left(\frac{x-t}{x-a}\right)^s |f'(a)|\right] dt$$
$$+ \int_x^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) \left[\left(\frac{t-x}{a+b-2x}\right)^s |f'(a+b-x)| + \left(\frac{a+b-x-t}{a+b-2x}\right)^s |f'(x)|\right] dt$$
$$+ \int_{\frac{a+b}{2}}^{a+b-x} \left(t - \frac{a+b}{2}\right) \left[\left(\frac{t-x}{a+b-2x}\right)^s |f'(a+b-x)| + \left(\frac{a+b-x-t}{a+b-2x}\right)^s |f'(x)|\right] dt$$
$$+ \int_{a+b-x}^{b-\lambda\frac{b-a}{2}} \left(b - \lambda\frac{b-a}{2} - t\right) \left[\left(\frac{t-a-b+x}{x-a}\right)^s |f'(b)| + \left(\frac{b-t}{x-a}\right)^s |f'(a+b-x)|\right] dt$$
$$+ \int_{b-\lambda\frac{b-a}{2}}^b \left(t - b + \lambda\frac{b-a}{2}\right) \left[\left(\frac{t-a-b+x}{x-a}\right)^s |f'(b)| + \left(\frac{b-t}{x-a}\right)^s |f'(a+b-x)|\right] dt$$
$$= \left[\frac{2}{(x-a)^s(s+1)(s+2)}\left(\lambda\frac{b-a}{2}\right)^{s+2} + \frac{(x-a)^2}{s+2} - \left(\lambda\frac{b-a}{2}\right)\frac{(x-a)}{s+1}\right.$$
$$\left. + \frac{s2^{s+1}+1}{2^{s+1}(s+1)(s+2)}(a+b-2x)^2\right] \times [|f'(x)| + |f'(a+b-2x)|]$$
$$+ \left[\frac{2}{(x-a)^s(s+1)(s+2)}\left(x - a - \lambda\frac{b-a}{2}\right)^{s+2} + \frac{(x-a)^2}{s+2}\right.$$
$$\left. \times + \left(a + \lambda\frac{b-a}{2} - x\right)\frac{(x-a)}{s+1}\right] \times [|f'(a)| + |f'(b)|]$$

where we use the facts

$$\int_a^{a+\lambda\frac{b-a}{2}} \left(a + \lambda\frac{b-a}{2} - t\right)\left(\frac{t-a}{x-a}\right)^s dt = \int_{b-\lambda\frac{b-a}{2}}^b \left(t - b + \lambda\frac{b-a}{2}\right)\left(\frac{b-t}{x-a}\right)^s dt$$
$$= \frac{1}{(x-a)^s(s+1)(s+2)}\left(\lambda\frac{b-a}{2}\right)^{s+2},$$

$$\int_{a+\lambda\frac{b-a}{2}}^x \left(t - a - \lambda\frac{b-a}{2}\right)\left(\frac{x-t}{x-a}\right)^s dt = \int_{a+b-x}^{b-\lambda\frac{b-a}{2}} \left(b - \lambda\frac{b-a}{2} - t\right)\left(\frac{t-a-b+x}{x-a}\right)^s dt$$
$$= \frac{1}{(x-a)^s(s+1)(s+2)}\left(x - a - \lambda\frac{b-a}{2}\right)^{s+2},$$

$$\int_x^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)\left(\frac{t-x}{a+b-2x}\right)^s dt=\int_{\frac{a+b}{2}}^{a+b-x}\left(t-\frac{a+b}{2}\right)\left(\frac{a+b-x-t}{a+b-2x}\right)^s dt$$
$$=\frac{1}{2^{s+2}(s+1)(s+2)}(a+b-2x)^2,$$

$$\int_x^{\frac{a+b}{2}}\left(\frac{a+b}{2}-t\right)\left(\frac{a+b-x-t}{a+b-2x}\right)^s dt=\int_{\frac{a+b}{2}}^{a+b-x}\left(t-\frac{a+b}{2}\right)\left(\frac{t-x}{a+b-2x}\right)^s dt$$
$$=\frac{s2^{s+1}+1}{2^{s+2}(s+1)(s+2)}(a+b-2x)^2,$$

$$\int_{a+b-x}^{b-\lambda\frac{b-a}{2}}\left(b-\lambda\frac{b-a}{2}-t\right)\left(\frac{b-t}{x-a}\right)^s dt$$
$$=\int_{a+\lambda\frac{b-a}{2}}^{x}\left(t-a-\lambda\frac{b-a}{2}\right)\left(\frac{t-a}{x-a}\right)^s dt$$
$$=\frac{1}{(x-a)^s}\left[\frac{(x-a)^{s+2}}{s+2}-\lambda\frac{b-a}{2}\frac{(x-a)^{s+1}}{s+1}+\frac{1}{(s+1)(s+2)}\left(\lambda\frac{b-a}{2}\right)^{s+2}\right]$$

and

$$\int_a^{a+\lambda\frac{b-a}{2}}\left(a+\lambda\frac{b-a}{2}-t\right)\left(\frac{x-t}{x-a}\right)^s dt$$
$$=\int_{b-\lambda\frac{b-a}{2}}^{b}\left(t-b+\lambda\frac{b-a}{2}\right)\left(\frac{t-a-b+x}{x-a}\right)^s dt$$
$$=\frac{1}{(x-a)^s}\left[\frac{(x-a)^{s+2}}{s+2}+\left(a+\lambda\frac{b-a}{2}-x\right)\frac{(x-a)^{s+1}}{s+1}+\frac{1}{(s+1)(s+2)}\left(x-a-\lambda\frac{b-a}{2}\right)^{s+2}\right].$$

Remark 2 *In Theorem 2, if we choose $s=1$, Theorem 2 reduces to Theorem 1.*

Remark 3 *In Theorem 2, if we choose $\lambda=0$, we obtain Theorem 4.10.1 which is proved in [4].*

3. Applications to Some Special Means

We now consider the applications of our theorems to the following special means

a) The arithmetic mean:

$$A=A(a,b):=\frac{a+b}{2},\quad a,b\geq 0,$$

b) The harmonic mean:

$$H=H(a,b):=\frac{2ab}{a+b},\ a,b\geq 0,$$

c) The logarithmic mean:

$$L = L(a,b) := \begin{cases} a & \text{if} a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if} a \neq b \end{cases}, \quad a,b \geq 0$$

d) The p −logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} a & \text{if} a = b \\ \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if} a \neq b \end{cases}, p \in \mathbb{R}\backslash\{-1,0\}; \quad a,b \geq 0$$

We now derive some sophisticated bounds of the above means.

Proposition 1 *For all* $a,b,x > 0$, $\lambda \in [0,1]$ *and* $a + \lambda\frac{b-a}{2} \leq x \leq \frac{a+b}{2}$. *we have*

$$|(b-a)[\lambda A(a^n,b^n) + (1-\lambda)A(x^n,(a+b-x)^n)] - (b-a)L_n^n(a,b)|$$
$$\leq K \times A(x^{n-1},(a+b-x)^{n-1}) + L \times A(a^{n-1},b^{n-1})$$

where

$$K = \frac{n}{12}\left[3(a+b-2x)^2 + 8(a-x)^2 - 6(a-b)(a-x)\lambda + \frac{(a-b)^3\lambda^3}{a-x}\right]$$
$$L = \frac{n}{12}\left[4(a-x)^2 - 6(a-b)(a-x)\lambda + 6(a-b)^2\lambda^2 - \frac{(a-b)^3\lambda^3}{a-x}\right].$$

Proof. By applying $f(x) = x^n$ for $n \geq 2$ to Theorem 1, we get the result.

Proposition 2 *For all* $a,b,x > 0$, $\lambda \in [0,1]$ *and* $a + \lambda\frac{b-a}{2} \leq x \leq \frac{a+b}{2}$. *we have*

$$|(b-a)[\lambda H^{-1}(a,b) + (1-\lambda)H^{-1}(x,a+b-x)] - (b-a)L^{-1}(a,b)|$$
$$\leq M \times H^{-1}(x^2,(a+b-x)^2) + N \times H^{-1}(a^2,b^2)$$

where

$$M = \frac{1}{12}\left[3(a+b-2x)^2 + 8(a-x)^2 - 6(a-b)(a-x)\lambda + \frac{(a-b)^3\lambda^3}{a-x}\right]$$
$$N = \frac{1}{12}\left[4(a-x)^2 - 6(a-b)(a-x)\lambda + 6(a-b)^2\lambda^2 - \frac{(a-b)^3\lambda^3}{a-x}\right].$$

Proof. By applying $f(x) = \frac{1}{x}$ to Theorem 1, we get the result.

Proposition 3 *For all* $a, b > 0$, $x,\lambda \in [0,1]$, $s \in (0,1)$ *and* $a + \lambda\frac{b-a}{2} \leq x \leq \frac{a+b}{2}$*. we have*

$$|(b-a)[\lambda A(a^s, b^s) + (1-\lambda)A(x^s, (a+b-x)^s)] - (b-a)L_s^s(a,b)|$$
$$\leq P \times A(x^{s-1}, (a+b-2x)^{s-1}) + R \times A(a^{s-1}, b^{s-1})$$

where

$$P = \frac{2}{(x-a)^s(s+1)(s+2)}\left(\lambda\frac{b-a}{2}\right)^{s+2} + \frac{(x-a)^2}{s+2} - \left(\lambda\frac{b-a}{2}\right)\frac{(x-a)}{s+1}$$
$$+\frac{s2^s+1}{2^{s+1}(s+1)(s+2)}(a+b-2x)^2$$
$$R = \frac{2}{(x-a)^s(s+1)(s+2)}\left(x-a-\lambda\frac{b-a}{2}\right)^{s+2} + \frac{(x-a)^2}{s+2} + \left(a+\lambda\frac{b-a}{2}-x\right)\frac{(x-a)}{s+1}$$

Proof. The assertion follows from Theorem 2 applied for $f(x) = x^s$.

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