

Fixed-Point Approximation of Operators Satisfying (HRSC)-condition in $CAT(0)$ Spaces

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Abstract

In this paper, we have established strong and Δ -convergence results for the SP^* -iteration process applied to mappings satisfying the (HRSC)-condition in $CAT(0)$ spaces. Furthermore, a numerical example is provided to show the superiority of our results over existing ones and to illustrate the faster convergence of the SP^* -iteration process compared to several well-known iterative schemes.

Keywords: Fixed point, iteration process, mapping satisfying (HRSC)-condition, strong convergence, Δ -convergence, $CAT(0)$ space.

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1. Introduction

Let (X, d) be a metric space, $\mathcal{K}$ be a non-empty subset of X , and $\Phi : \mathcal{K} \rightarrow \mathcal{K}$ be a mapping. A point $w \in \Phi$ is called a fixed point of Φ if $\Phi w = w$ and we denote by $Fix(\Phi)$ the set of fixed points of Φ . A mapping $\Phi : \mathcal{K} \rightarrow \mathcal{K}$ is called contraction if there exists $\theta \in [0, 1)$ such that

$$d(\Phi w, \Phi v) \leq \theta d(w, v),$$

for all $w, v \in \mathcal{K}$. If $\theta = 1$ in inequality above, then Φ is said to be a nonexpansive mapping.

Definition 1.1. A mapping $\Phi : \mathcal{K} \rightarrow \mathcal{K}$ satisfies (C)-condition on $\mathcal{K}$ if for all $w, v \in \mathcal{K}$,

$$\frac{1}{2}d(w, \Phi w) \leq d(w, v) \Rightarrow d(\Phi w, \Phi v) \leq d(w, v).$$

Suzuki [33] showed that the mapping satisfying (C)-condition is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. Recently, Garcia-Falset et al. [12] studied the mapping satisfying (E)-condition that have a weaker property than (C)-condition.

Definition 1.2. A mapping $\Phi : \mathcal{K} \rightarrow X$ satisfies (E_μ) -condition on $\mathcal{K}$, if there exists $\mu \geq 1$ such that

$$d(w, \Phi v) \leq \mu d(w, \Phi w) + d(w, v)$$

for all $w, v \in \mathcal{K}$.

Moreover, it is said that Φ satisfies (E)-condition on $\mathcal{K}$, whenever Φ satisfies (E_μ) -condition, for some $\mu \geq 1$. It is clearly seen that if $\Phi : \mathcal{K} \rightarrow X$ is nonexpansive, then it satisfies (E_1) -condition and from Lemma 7 in [33] we know that if $\Phi : \mathcal{K} \rightarrow \mathcal{K}$ satisfies (C)-condition on $\mathcal{K}$, then Φ satisfies (E_3) -condition (see [12]). Proposition 1 in [12], we know also that if $\Phi : \mathcal{K} \rightarrow X$ a mapping which satisfies (E)-condition on $\mathcal{K}$ has some fixed point, then Φ is quasi-nonexpansive. Example 2 that is in [12] shows the converse is not true. In 2012, Karapınar [15] studied the (C)-condition and introduced a new condition called (HRSC)-condition (Hardy-Rogers-Suzuki-(C) condition).

Definition 1.3. [15] A mapping $\Phi : \mathcal{K} \rightarrow \mathcal{K}$ satisfies (HRSC)-condition on $\mathcal{K}$ if for all $w, v \in \mathcal{K}$,

$$\begin{aligned} \frac{1}{2}d(w, \Phi w) &\leq d(w, v) \Rightarrow \\ d(\Phi w, \Phi v) &\leq \frac{1}{5} \left[d(w, v) + d(w, \Phi w) + d(v, \Phi v) + d(w, \Phi v) + d(v, \Phi w) \right]. \end{aligned}$$

Moreover, Karapınar [15] gave some basic properties for a mapping satisfying (HRSC)-condition as follows.

Proposition 1.4. (i) If a mapping Φ satisfies (HRSC)-condition and has a fixed point, then it is a quasi-nonexpansive mapping.
(ii) If Φ is a mapping satisfying (HRSC)-condition, then

$$d(w, \Phi v) \leq 15 \left[d(w, \Phi w) \right] + d(w, v),$$

for all $w, v \in \mathcal{K}$

(iii) If Φ is a mapping satisfying (HRSC)-condition, then the set $\text{Fix}(\Phi)$ is closed.

It is clearly seen that if $\Phi : \mathcal{K} \rightarrow X$ is (HRSC)-condition, then Φ satisfies (E_μ) -condition, for $\mu = 15$.

Fixed points play a fundamental role in numerous fields, particularly in mathematics, where they often represent solutions to equations or equilibrium states in applied problems. The existence of fixed points is governed by a variety of fixed point theorems, which establish the conditions under which mappings admit such points. Over the years, fixed point methods have found wide-ranging applications across disciplines including informatics, biology, chemistry, economics, and engineering. While determining the exact value of a fixed point typically constitutes the final stage in problem-solving, proving its existence is a crucial and often nontrivial first step. Among the most effective techniques for approximating fixed points are iterative procedures. In recent years, this area has garnered increasing attention, with researchers proposing novel iterative schemes not only for classical nonexpansive mappings but also for more general classes of mappings. These studies have primarily focused on the convergence behavior of the proposed methods and have made substantial contributions to the development of fixed point theory.

Numerous recent studies have significantly contributed to this development by introducing new classes of mappings, generalizing contraction principles, and constructing modified iterative schemes in various abstract settings, including Banach spaces, $CAT(0)$ spaces, b -metric spaces, S -metric spaces, and generalized metric spaces. These works have established strong and Δ -convergence results, extended classical theorems, explored applications to differential and integral equations, and revealed connections with operator theory and fractal geometry (see, for instance, [[6], [13], [16], [23] - [25], [26]-[29], [37], [38]]). This remains an ongoing area of mathematical research, with many prominent researchers continuing to explore the qualitative behavior of iterative methods and their applications in nonlinear analysis.

Let X be a real Banach space and $\mathcal{K}$ be a nonempty subset of X , and $\Phi : \mathcal{K} \rightarrow \mathcal{K}$ be a mapping. We have $\{\eta_n\}, \{\vartheta_n\}$ and $\{\tau_n\}$ real sequences in $[0, 1]$. Recently, Phuengrattana and Suantai ([22]) defined the SP-iteration as follows:

$$\begin{cases} z_n = (1 - \eta_n)w_n + \eta_n \Phi w_n, \\ v_n = (1 - \vartheta_n)z_n + \vartheta_n \Phi z_n, \\ w_{n+1} = (1 - \tau_n)v_n + \tau_n \Phi v_n, \quad \forall n \in \mathbb{N}. \end{cases} \quad (1.1)$$

where $w_1 \in \mathcal{K}$.

In 2014, Kadioglu and Yıldırım [14] introduced Picard Normal S-iteration process as follows:

$$\begin{cases} z_n = (1 - \vartheta_n)w_n + \vartheta_n \Phi w_n, \\ v_n = (1 - \tau_n)z_n + \tau_n \Phi z_n, \\ w_{n+1} = \Phi v_n, \quad \forall n \in \mathbb{N}. \end{cases} \quad (1.2)$$

where $w_1 \in \mathcal{K}$.

The following iterative scheme in the context of $CAT(0)$ spaces is known as the SP^* -iteration process, introduced by Temir and Korkut [34], and it has been employed to establish convergence results for certain generalized nonexpansive mappings in $CAT(0)$ spaces (see [34]-[36]). It was demonstrated that the SP^* -iteration process converges faster than both the SP-iteration scheme and the Picard Normal S-iteration process. Let X be a $CAT(0)$ and $\mathcal{K}$ be a nonempty subset of X , and $\Phi : \mathcal{K} \rightarrow \mathcal{K}$ be a mapping. We have $\{\eta_n\}, \{\vartheta_n\}$ and $\{\tau_n\}$ real sequences in $[0, 1]$. Now we give SP^* -iteration process: for arbitrary $w_1 \in \mathcal{K}$ construct a sequence $\{w_n\}$ by

$$\begin{cases} z_n = \Phi((1 - \eta_n)w_n \oplus \eta_n \Phi w_n), \\ v_n = \Phi((1 - \vartheta_n)z_n \oplus \vartheta_n \Phi z_n), \\ w_{n+1} = \Phi((1 - \tau_n)v_n \oplus \tau_n \Phi v_n), \quad \forall n \in \mathbb{N}. \end{cases} \quad (1.3)$$

Inspired and motivated by these facts, in this paper, we prove some convergence theorems of SP^* -iterative process generated by (1.3) to fixed point of mappings satisfying (HRSC)-condition in $CAT(0)$ spaces. In addition, we provide an example that satisfies (HRSC)-condition but the mapping does not satisfy (C)-condition.

2. Preliminaries

Fixed point theory in $CAT(0)$ spaces was first studied by Kirk (see [17], [18]). He showed that every nonexpansive (single-valued) mapping defined on a bounded, closed, convex subset of a complete $CAT(0)$ space has a fixed point. Since then, fixed point theory in $CAT(0)$ spaces has developed rapidly, and many related papers have appeared (see [7]-[11], [17]-[20]). On the other hand, it is well known that not every Banach space is a $CAT(0)$ space. In fact, the $CAT(0)$ condition requires the metric to satisfy certain triangle comparison properties, which impose strong restrictions on the curvature and geometric structure of the space. Among Banach spaces, only Hilbert spaces, whose norms are induced by an inner product, are known to be $CAT(0)$ spaces. This distinction reveals the profound geometric and topological differences between general Banach spaces and $CAT(0)$ spaces. Consequently, this fact encourages the extension of fixed point and convergence results from Hilbert and Banach spaces to the nonlinear framework of $CAT(0)$ spaces.

Recently, Kirk and Panyanak [20] employed the concept of Δ -convergence, introduced by Lim [21], to establish analogs of certain weak convergence results originally proven in Banach spaces, within the framework of $CAT(0)$ spaces. Subsequently, Dhompongsa and Panyanak [7] extended these Δ -convergence results to the Picard, Mann, and Ishikawa iteration processes for nonexpansive mappings in $CAT(0)$ spaces.

Moreover, various convergence results for generalized nonexpansive mappings have been obtained in $CAT(0)$ spaces through different iterative methods (see [1], [2], [27], [30], [31], [34]-[36]).

We now collect some elementary facts about $CAT(0)$ spaces which will be used in the sequel.

If w, v_1, v_2 are points of a $CAT(0)$ spaces, and if v_0 is the midpoint of the segment $[v_1, v_2]$ then the $CAT(0)$ inequality implies

$$d^2(w, v_0) \leq \frac{1}{2}d^2(w, v_1) + \frac{1}{2}d^2(w, v_2) - \frac{1}{4}d^2(v_1, v_2).$$

This is the (CN) inequality of Bruhat and Tits [4]. In fact, a geodesic space is a $CAT(0)$ space if and only if it satisfies the (CN) inequality ([3], p. 163).

In the sequel, we need the following definitions and useful lemmas to prove our main results of this paper.

Lemma 2.1. ([7]) Let X be a $CAT(0)$ space.

(i) For $u, v \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [w, v]$ such that $d(w, z) = td(w, v)$ and $d(v, z) = (1-t)d(w, v)$.

(ii) For $w, v \in X$ and $t \in [0, 1]$, we have

$$d((1-t)w \oplus tv, z) \leq (1-t)d(w, z) + td(v, z).$$

Let $\{w_n\}$ be a bounded sequence in a closed convex subset $\mathcal{K}$ of a $CAT(0)$ space X . For $w \in X$, set $r(w, \{w_n\}) = \limsup_{n \rightarrow \infty} d(w, \{w_n\})$. The asymptotic radius $r(\{w_n\})$ of $\{w_n\}$ is given by $r(\mathcal{K}, \{w_n\}) = \inf_n \{r(w, \{w_n\}) : w \in \mathcal{K}\}$ and the asymptotic center of $\{w_n\}$ relative to $\mathcal{K}$ is the set $A(\mathcal{K}, \{w_n\}) = \{w \in \mathcal{K} : r(w, \{w_n\}) = r(\mathcal{K}, \{w_n\})\}$. It is known that, in a $CAT(0)$ space, $A(\mathcal{K}, \{w_n\})$ consists of exactly one point; see [10], Proposition 7.

We now recall the definition of Δ -convergence and weak convergence in $CAT(0)$ space.

Definition 2.2. ([20],[21]) A sequence $\{w_n\}$ in a $CAT(0)$ space X is said to Δ -converge to $w \in X$ if w is the unique asymptotic center of every subsequence $\{w_n\}$. In this case we write $\Delta\text{-}\lim_{n \rightarrow \infty} w_n = w$ and call w is the Δ -limit of $\{w_n\}$.

Lemma 2.3. ([20]) Given $\{w_n\} \in X$ such that $\{w_n\}$, Δ -converges to w and given $v \in X$ with $v \neq w$, then

$$\limsup_{n \rightarrow \infty} d(w_n, w) < \limsup_{n \rightarrow \infty} d(w_n, v).$$

Lemma 2.4. ([20]) Every bounded sequence in a complete $CAT(0)$ space always has a Δ -convergent subsequence.

Lemma 2.5. ([9]) Let $\mathcal{K}$ be closed convex subset of a complete $CAT(0)$ space and $\{w_n\}$ be a bounded sequence in $\mathcal{K}$. Then asymptotic center of $\{w_n\}$ is in $\mathcal{K}$.

3. Convergence of SP*-iteration process for mappings satisfying (HRCS)-condition

Lemma 3.1. Let $\mathcal{K}$ be a nonempty closed convex subset of a complete $CAT(0)$ space X , Φ be a mapping satisfying (HRCS)-condition with $\text{Fix}(\Phi) \neq \emptyset$. For arbitrary chosen $w_1 \in \mathcal{K}$, let $\{w_n\}$ be a sequence generated by (1.3) with $\{\tau_n\}$, $\{\vartheta_n\}$ and $\{\eta_n\}$ real sequences in $[0, 1]$. Assume that $\liminf_{n \rightarrow \infty} (1 - \eta_n)\eta_n > 0$, $\liminf_{n \rightarrow \infty} (1 - \vartheta_n)\vartheta_n > 0$ and $\liminf_{n \rightarrow \infty} (1 - \tau_n)\tau_n > 0$. Then $\text{Fix}(\Phi) \neq \emptyset$ if and only if $\{w_n\}$ is bounded and $\lim_{n \rightarrow \infty} d(w_n, \Phi w_n) = 0$.

Proof. Assume that $\text{Fix}(\Phi) \neq \emptyset$. Φ is a quasi-nonexpansive because $\Phi : \mathcal{K} \rightarrow \mathcal{K}$ is a mapping satisfying (HRCS)-condition(see,[15],Proposition 3). Using (1.3), for any $p \in \text{Fix}(\Phi)$, because of Φ quasi-nonexpansive mapping, then we have

$$\begin{aligned} d^2(z_n, p) &= d^2(\Phi((1 - \eta_n)w_n \oplus \eta_n \Phi w_n), p) \\ &\leq d^2((1 - \eta_n)w_n \oplus \eta_n \Phi w_n, p) \\ &\leq (1 - \eta_n)d^2(w_n, p) + \eta_n d^2(\Phi w_n, p) \\ &\quad - (1 - \eta_n)\eta_n d^2(\Phi w_n, w_n) \\ &\leq d^2(w_n, p) - (1 - \eta_n)\eta_n d^2(\Phi w_n, w_n) \leq d^2(w_n, p). \end{aligned} \quad (3.1)$$

Using (1.3) and (3.1), we get

$$\begin{aligned} d^2(v_n, p) &= d^2(\Phi((1 - \vartheta_n)z_n \oplus \vartheta_n \Phi z_n), p) \\ &\leq d^2((1 - \vartheta_n)z_n \oplus \vartheta_n \Phi z_n, p) \\ &\leq (1 - \vartheta_n)d^2(z_n, p) + \vartheta_n d^2(\Phi z_n, p) \\ &\quad - (1 - \vartheta_n)\vartheta_n d^2(\Phi z_n, z_n) \\ &\leq d^2(z_n, p) - (1 - \vartheta_n)\vartheta_n d^2(\Phi z_n, z_n) \\ &\leq d^2(z_n, p) \leq d^2(w_n, p). \end{aligned} \quad (3.2)$$

By using (1.3) and (3.2), we get

$$\begin{aligned} d^2(w_{n+1}, p) &= d^2(\Phi((1 - \tau_n)v_n \oplus \tau_n \Phi v_n), p) \\ &\leq d^2((1 - \tau_n)v_n \oplus \tau_n \Phi v_n, p) \\ &\leq (1 - \tau_n)d^2(v_n, p) + \tau_n d^2(\Phi v_n, p) \\ &\quad - (1 - \tau_n)\tau_n d^2(\Phi v_n, v_n) \\ &\leq d^2(v_n, p) - (1 - \tau_n)\tau_n d^2(\Phi v_n, v_n) \\ &\leq d^2(v_n, p) \leq d^2(w_n, p). \end{aligned} \quad (3.3)$$

This implies that $\{d(w_n, p)\}$ is bounded and non-increasing for all $p \in \text{Fix}(\Phi)$. Put $\lim_{n \rightarrow \infty} d(w_n, p) = \gamma$. From (3.1) and (3.2), we have

$$\limsup_{n \rightarrow \infty} d(z_n, p) \leq \limsup_{n \rightarrow \infty} d(w_n, p) = \gamma$$

and

$$\limsup_{n \rightarrow \infty} d(v_n, p) \leq \limsup_{n \rightarrow \infty} d(z_n, p) \leq \limsup_{n \rightarrow \infty} d(w_n, p) = \gamma.$$

From (3.3), we can get $d(w_{n+1}, p) \leq d(v_n, p)$. Therefore $\gamma \leq \liminf_{n \rightarrow \infty} d(v_n, p)$. Thus we have $\gamma = \lim_{n \rightarrow \infty} d(v_n, p)$. Next

$$\gamma = \lim_{n \rightarrow \infty} d(v_n, p) \leq \lim_{n \rightarrow \infty} d(z_n, p) \leq \lim_{n \rightarrow \infty} d(w_n, p) = \gamma.$$

Now, using (3.1), we know that

$$d^2(z_n, p) \leq d^2(w_n, p) - (1 - \eta_n)\eta_n d^2(\Phi w_n, w_n).$$

Thus

$$(1 - \eta_n)\eta_n d^2(\Phi w_n, w_n) \leq d^2(w_n, p) - d^2(z_n, p)$$

so that

$$d^2(\Phi w_n, w_n) \leq \frac{1}{(1 - \eta_n)\eta_n} [d^2(w_n, p) - d^2(z_n, p)].$$

We have

$$\lim_{n \rightarrow \infty} d^2(\Phi w_n, w_n) \leq 0.$$

Hence $\lim_{n \rightarrow \infty} d(\Phi w_n, w_n) = 0$.

Conversely, suppose that the sequence $\{w_n\}$ is bounded and satisfies $\lim_{n \rightarrow \infty} d(w_n, \Phi w_n) = 0$. Let $A(\{w_n\}) = \{p\}$. Then, by Lemma 2.5, $p \in \mathcal{K}$. Since Φ satisfies the (HRSC)-condition on $\mathcal{K}$, there exists $\mu > 1$ (e.g., $\mu = 15$ in Proposition 1.4) such that

$$d(w_n, \Phi p) \leq 15 [d(\Phi w_n, w_n)] + d(w_n, p)$$

which implies that

$$\begin{aligned} \limsup_{n \rightarrow \infty} d(w_n, \Phi p) &\leq \limsup_{n \rightarrow \infty} \{15[d(\Phi w_n, w_n)] + d(w_n, p)\} \\ &\leq \limsup_{n \rightarrow \infty} d(w_n, p). \end{aligned}$$

As a consequence of the uniqueness of the asymptotic center, hence $\Phi p = p$. This completes the proof. $\square$

Now, we prove the Δ -convergence theorem of an iterative process generated by (1.3) in $CAT(0)$ spaces.

Theorem 3.2. Let $X, \mathcal{K}, \Phi$ and $\{w_n\}$ be as in Lemma 3.1 with $\text{Fix}(\Phi) \neq \emptyset$. Then $\{w_n\}$, Δ -converges to a fixed point of Φ .

Proof. Lemma 3.1 ensures that the sequence $\{w_n\}$ is bounded and satisfies $\lim_{n \rightarrow \infty} d(\Phi w_n, w_n) = 0$. Let us define $W_\Delta(w_n) = \bigcup A(\{\xi_n\})$, where the union is taken over all subsequences $\{\xi_n\}$ of $\{w_n\}$. We claim that $W_\Delta(w_n) \subseteq \text{Fix}(\Phi)$. Let $\xi \in W_\Delta(w_n)$. Then, there exists a subsequence $\{\xi_n\}$ of $\{w_n\}$ such that $A(\{\xi_n\}) = \{\xi\}$. By Lemmas 2.4 and 2.5, there exists a subsequence $\{\zeta_n\}$ of $\{\xi_n\}$ such that $\Delta\text{-}\lim_{n \rightarrow \infty} \zeta_n = \zeta \in \mathcal{K}$. Since $\lim_{n \rightarrow \infty} d(\zeta_n, \Phi \zeta_n) = 0$, and Φ satisfies the (HRSC)-condition, it follows that

$$d(\zeta_n, \Phi \zeta) \leq 15d(\Phi \zeta_n, \zeta_n) + d(\zeta_n, \zeta).$$

Taking the limit superior on both sides and applying the Opial property, we deduce that $\zeta \in \text{Fix}(\Phi)$.

Next, we claim that $\xi = \zeta$. Suppose, for contradiction, that $\xi \neq \zeta$. Since, by Lemma 3.1, the limit $\lim_{n \rightarrow \infty} d(w_n, \zeta)$ exists, and due to the uniqueness of asymptotic centers, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} d(\zeta_n, \zeta) &< \lim_{n \rightarrow \infty} d(\zeta_n, \xi) \leq \lim_{n \rightarrow \infty} d(\xi_n, \xi) \\ &< \lim_{n \rightarrow \infty} d(\xi_n, \zeta) = \lim_{n \rightarrow \infty} d(w_n, \zeta) \\ &= \lim_{n \rightarrow \infty} d(\zeta_n, \zeta), \end{aligned}$$

which is a contradiction. Thus, $\xi = \zeta \in \text{Fix}(\Phi)$, and hence $W_\Delta(w_n) \subseteq \text{Fix}(\Phi)$.

To prove that $\{w_n\}$, Δ -converges to a fixed point of Φ , it suffices to show that $W_\Delta(w_n)$ consists of exactly one point. By Lemmas 2.4 and 2.5, there exists a subsequence $\{\zeta_n\}$ of $\{\xi_n\}$ such that $\Delta\text{-}\lim_{n \rightarrow \infty} \zeta_n = \zeta \in \mathcal{K}$. Let $A(\{\xi_n\}) = \{\xi\}$ and $A(\{w_n\}) = \{p\}$. As shown earlier, $\xi = \zeta$ and $\zeta \in \text{Fix}(\Phi)$. We now claim that $p = \zeta$. Assume the contrary. Then the existence of $\lim_{n \rightarrow \infty} d(w_n, \zeta)$ and the uniqueness of asymptotic centers imply:

$$\begin{aligned} \lim_{n \rightarrow \infty} d(\zeta_n, \zeta) &< \lim_{n \rightarrow \infty} d(\zeta_n, p) \leq \lim_{n \rightarrow \infty} d(w_n, p) \\ &< \lim_{n \rightarrow \infty} d(w_n, \zeta) = \lim_{n \rightarrow \infty} d(\zeta_n, \zeta), \end{aligned}$$

which is again a contradiction. Hence, $p = \zeta \in \text{Fix}(\Phi)$. Therefore, $W_\Delta(w_n) = \{p\}$. In conclusion, $W_\Delta(w_n)$ is a singleton whose unique element is a fixed point of Φ . This completes the proof of the Δ -convergence of the sequence $\{w_n\}$. $\square$

In the next result, we prove the strong convergence theorem as follows.

Theorem 3.3. Let $X, \mathcal{K}, \Phi$ and $\{w_n\}$ be as in Lemma 3.1 with $\text{Fix}(\Phi) \neq \emptyset$ such that $\mathcal{K}$ is compact subset of X . Then $\{w_n\}$ converges strongly to a fixed point of Φ .

Proof. By Lemma 3.1, we have $\lim_{n \rightarrow \infty} d(w_n, \Phi w_n) = 0$. Since $\mathcal{K}$ is compact, by Lemma 2.4, there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and $p \in \mathcal{K}$ such that $\{w_{n_k}\}$ converges p . Then, by Proposition 1.4, for $\mu = 15$, we have

$$d(w_{n_k}, \Phi p) \leq 15 \left[d(\Phi w_{n_k}, w_{n_k}) \right] + d(w_{n_k}, p)$$

for all $k \geq 1$. So $\{w_{n_k}\}$ converges Φp . This implies $\Phi p = p$. Since Φ is quasi-nonexpansive, we have $d(w_{n+1}, p) \leq d(w_n, p)$ for all $n \in \mathbb{N}$. Therefore $\{w_n\}$ converges strongly to p . $\square$

Finally, we briefly discuss the strong convergence theorem using (I)-condition introduced by Senter and Dotson[32] in $CAT(0)$ space X as follows.

Theorem 3.4. Let Φ be a mapping satisfying the (HRCS)-condition on a nonempty closed convex subset $\mathcal{K}$ of a complete $CAT(0)$ space X . Let $\{w_n\}$ be the sequence defined as in Lemma 3.1, and suppose that $\text{Fix}(\Phi) \neq \emptyset$. If Φ also satisfies the (I)-condition, then the sequence $\{w_n\}$ defined by (3) converges strongly to a fixed point of Φ .

Proof. Let $\{w_n\}$ be a sequence in the fixed point set $\text{Fix}(\Phi)$ such that $w_n \rightarrow z$ for some $z \in \mathcal{K}$. Since $d(w_n, \Phi w_n) = 0 \leq d(w_n, z)$, by the (HRCS)-condition, we have $d(w_n, \Phi z) = d(\Phi w_n, \Phi z) \leq d(w_n, z)$.

Taking the limit on both sides yields

$$\lim_{n \rightarrow \infty} d(w_n, \Phi z) \leq \lim_{n \rightarrow \infty} d(w_n, z) = 0.$$

By the uniqueness of limits, it follows that $\Phi z = z$, hence $z \in \text{Fix}(\Phi)$. Therefore, $\text{Fix}(\Phi)$ is closed.

Next, by Lemma 3.1, we have $\lim_{n \rightarrow \infty} d(w_n, \Phi w_n) = 0$. Using condition (I), we obtain

$$\lim_{n \rightarrow \infty} f(d(w_n, \text{Fix}(\Phi))) \leq \lim_{n \rightarrow \infty} d(w_n, \Phi w_n) = 0,$$

which implies that $\lim_{n \rightarrow \infty} f(d(w_n, \text{Fix}(\Phi))) = 0$. Since $f: [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, we conclude that $\lim_{n \rightarrow \infty} d(w_n, \text{Fix}(\Phi)) = 0$. Hence, there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and a sequence $\{y_k\} \subset \text{Fix}(\Phi)$ such that $d(w_{n_k}, y_k) \leq \frac{1}{2^k}$, for all $k \geq 1$. By Lemma 3.1 again, we have

$$d(w_{n_{k+1}}, y_k) \leq d(w_{n_k}, y_k) \leq \frac{1}{2^k},$$

and thus,

$$\begin{aligned} d(p_{k+1}, y_k) &\leq d(p_{k+1}, w_{n_{k+1}}) + d(w_{n_{k+1}}, y_k) \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}. \end{aligned}$$

Therefore, $\{y_k\}$ is a Cauchy sequence in $\text{Fix}(\Phi)$. Since $\text{Fix}(\Phi)$ is closed, $\{y_k\}$ converges to some point $p \in \text{Fix}(\Phi)$.

To show that $\{w_n\}$ converges to p , observe that $d(w_{n_k}, p) \leq d(w_{n_k}, y_k) + d(y_k, p) \rightarrow 0$ as $k \rightarrow \infty$.

Hence, $\lim_{k \rightarrow \infty} d(w_{n_k}, p) = 0$. Since $\lim_{n \rightarrow \infty} d(w_n, p)$ exists, it follows that $w_n \rightarrow p$. $\square$

Next, we give the following example a mapping satisfying (HRCS)-condition, but it does not satisfy (C)-condition.

Example 3.5. Let $X = \mathbb{R}$ (a $CAT(0)$ space) and $\mathcal{K} = [0, 5]$. Define

$$\Phi: \mathcal{K} \rightarrow \mathcal{K}, \quad \Phi(w) = \begin{cases} \frac{w+2}{4}, & w \neq 5, \\ 3, & w = 5. \end{cases} \quad (1.3)$$

In order to see that Φ satisfies (HRCS)-condition on $[0, 5]$, we consider the following cases: Let the metric be the usual absolute value: $d(w, v) = |w - v|$. The mapping Φ satisfies the (HRSC)-condition if, for all $w, v \in [0, 5]$, the following implication holds:

$$\begin{aligned} \frac{1}{2}|w - \Phi(w)| &\leq |w - v| \Rightarrow \\ |\Phi w - \Phi v| &\leq \frac{1}{5}(|w - v| + |w - \Phi w| + |v - \Phi v| \\ &\quad + |w - \Phi v| + |v - \Phi w|). \end{aligned}$$

We will verify this inequality by considering all possible cases for $w, v \in [0, 5]$.

Case 1: $w, v = 5$

In this case, $\Phi w = \Phi v = 3$, so: $|\Phi w - \Phi v| = 0$, and the right-hand side is strictly positive. Hence, the inequality is satisfied.

Case 2: $w, v \in [0, 5]$

$$\begin{aligned}
|\Phi_w - \Phi_v| &= \left| \frac{w+2}{4} - \frac{v+2}{4} \right| = \left| \frac{w-v}{4} \right| \\
&\leq \frac{1}{5}|w-v| + \frac{1}{4}|w-v| + \frac{1}{20}|w-v| \\
&\leq \frac{1}{5}|w-v| + \frac{1}{5} \left| \frac{5w-5v}{4} \right| + \frac{1}{5} \left| \frac{3w-3v}{4} \right| \\
&\leq \frac{1}{5}|w-v| + \frac{1}{5} \left| \left(w - \left(\frac{v+2}{4} \right) \right) - \left(v - \left(\frac{w+2}{4} \right) \right) \right| \\
&\quad + \frac{1}{5} \left| \left(w - \left(\frac{w+2}{4} \right) \right) - \left(v - \left(\frac{v+2}{4} \right) \right) \right| \\
&\leq \frac{1}{5}|w-v| + \frac{1}{5} \left| \left(w - \left(\frac{v+2}{4} \right) \right) \right| + \frac{1}{5} \left| \left(v - \left(\frac{w+2}{4} \right) \right) \right| \\
&\quad + \frac{1}{5} \left| \left(w - \left(\frac{w+2}{4} \right) \right) \right| + \frac{1}{5} \left| \left(v - \left(\frac{v+2}{4} \right) \right) \right| \\
&= \frac{1}{5} (|w-v| + |w-\Phi_w| + |v-\Phi_v| + |w-\Phi_v| + |v-\Phi_w|).
\end{aligned}$$

Hence, the inequality is satisfied.

Case 3: $w = 5, v \in [0, 5)$

In this case: $\Phi_w = 3$ and $\Phi_v = \frac{v+2}{4}$. Then we have

$$\begin{aligned}
|\Phi_w - \Phi_v| &= \left| 3 - \frac{v+2}{4} \right| = \left| \frac{v-10}{4} \right| = \frac{1}{5} \left| \frac{5v-50}{4} \right| \\
&\leq \frac{1}{5} \left| \frac{4v-20}{4} \right| + \frac{1}{5} \left| \frac{v-30}{4} \right| \\
&\leq \frac{1}{5}|v-5| + \frac{1}{5}|2| + \frac{1}{5} \left| \frac{v-18}{4} \right| + \frac{1}{5} \\
&\leq \frac{1}{5}|v-5| + \frac{1}{5}|2| + \frac{1}{5} \left| \left(\frac{v+2}{4} \right) - 5 \right| \\
&\quad + \frac{1}{5}|3-v| + \frac{1}{5} \left| \frac{2-3v}{4} \right|
\end{aligned}$$

To see why this inequality holds, let us examine the following cases.

$$\frac{|2-3v|}{4} + |3-v| \geq 1 \text{ for } v \in [0, 5)$$

Define

$$\varphi(v) = \frac{|2-3v|}{4} + |3-v|,$$

and we wish to show $\varphi(v) \geq 1$ for all $v \in [0, 5)$.

The expressions inside the absolute values change sign at

$$2-3v=0 \implies v=\frac{2}{3}, \quad 3-v=0 \implies v=3.$$

Thus we split $[0, 5)$ into three subintervals:

$$[0, \frac{2}{3}], \quad [\frac{2}{3}, 3], \quad [3, 5).$$

Case 3.1: $0 \leq v \leq \frac{2}{3}$.

Here $2-3v \geq 0$ and $3-v \geq 0$, so

$$\varphi(v) = \frac{2-3v}{4} + (3-v) = \frac{2-3v+12-4v}{4} = \frac{14-7v}{4}.$$

We check

$$\frac{14-7v}{4} \geq 1 \iff 14-7v \geq 4 \iff 7v \leq 10 \iff v \leq \frac{10}{7}.$$

Since $\frac{2}{3} < \frac{10}{7}$, the inequality holds for all $v \in [0, \frac{2}{3}]$.

Case 3.2: $\frac{2}{3} \leq v \leq 3$.

Here $2-3v \leq 0$ but $3-v \geq 0$, so

$$|2-3v| = 3v-2, \quad |3-v| = 3-v.$$

Thus

$$\varphi(v) = \frac{3v-2}{4} + (3-v) = \frac{3v-2+12-4v}{4} = \frac{10-v}{4}.$$

We check

$$\frac{10-v}{4} \geq 1 \iff 10-v \geq 4 \iff v \leq 6.$$

Since $v \leq 3 < 6$, the inequality holds for all $v \in [\frac{2}{3}, 3]$.

Case 3.3: $3 \leq v < 5$.

Here $2-3v \leq 0$ and $3-v \leq 0$, so $|2-3v| = 3v-2$, $|3-v| = v-3$. Hence

$$\varphi(v) = \frac{3v-2}{4} + (v-3) = \frac{3v-2+4v-12}{4} = \frac{7v-14}{4} = \frac{7(v-2)}{4}.$$

We check

$$\frac{7(v-2)}{4} \geq 1 \iff 7(v-2) \geq 4 \iff v-2 \geq \frac{4}{7} \iff v \geq \frac{18}{7}.$$

Since $v \geq 3 > \frac{18}{7}$, the inequality holds for all $v \in [3, 5)$.

In each subinterval of $[0, 5)$, we have shown $\varphi(v) \geq 1$. Therefore $\frac{|2-3v|}{4} + |3-v| \geq 1$ for all $v \in [0, 5)$. Each term is positive, and the total sum is strictly greater than $|\frac{v-10}{4}|$. Thus we have

$$\begin{aligned} |\Phi w - \Phi v| &\leq \frac{1}{5} \left(|w-v| + |w-\Phi w| + |v-\Phi v| \right. \\ &\quad \left. + |w-\Phi v| + |v-\Phi w| \right). \end{aligned}$$

Hence, by considering all possible cases for $w, v \in [0, 5]$, we conclude that the piecewise-defined mapping Φ satisfies the (HRSC)-condition. In order to we show that Φ does not satisfy (C)-condition, we take $w = 3.5, v = 5$, then we have

$$\frac{1}{2}|w-\Phi w| = \frac{1}{2} \left| 3.5 - \frac{(3.5)+2}{4} \right| = 1.0625 < 1.5 = |w-v|.$$

Next,

$$|\Phi w - \Phi v| = \left| \left(\frac{(3.5)+2}{4} \right) - 3 \right| = 1.625 > 1.5 = |w-v|.$$

Thus Φ does not satisfy (C)-condition.

Numerical result: We now compare convergence behavior of SP*-iteration process with other iteration processes using Example 3.5. From Figure 1, we see that the SP*-iteration process converges faster than SP-iteration and Picard Normal S-iteration processes. Let $\{a_n\} = \{b_n\} = \{c_n\} = 0.7$ and initial point be $w_1 = 0$. The fixed point of the mapping defined in Example 3.5 is 0.666667. These can be seen in Figure 1.

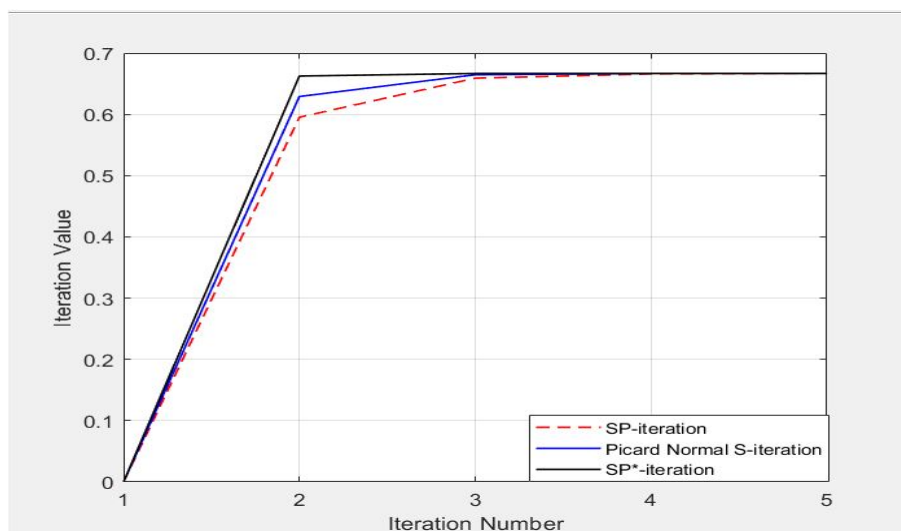


Figure 3.1: Convergence of SP-iteration, Picard Normal S-iteration and SP*-iteration processes to the fixed point 0.666667 of the mapping defined in Example 3.5.

4. Conclusions

We get some results on the strong and Δ -convergence of SP*-iteration process (1.3) for the mapping with (HRCS)-condition in nonlinear $CAT(0)$ spaces. In addition, we give an illustrative numerical example that satisfies (HRCS)-condition. As seen in Example 3.5, the mapping does not satisfy (C)-condition. Further, in future studies, iteration process can be developed and iteration that converges faster than prominent iterations can be presented.

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