Journal of New Theory
dergipark.org.tr/tr/pub/jnt
Open Access

ISSN: 2149-1402 (51) (2025) 52-64 doi.org/10.53570/jnt.1686180

Soft Set Extensions via Isotonic Spaces: Theory and Application to Risk-Based Decision Making

Mustafa Burç Kandemir[1]

Article Info
Received: 28 Apr 2025
Accepted: 14 Jun 2025
Published: 30 Jun 2025
Research Article

Abstract— This paper investigates the theoretical structure and practical implications of isotonic extensions of soft sets. It utilizes isotonic operators—functions satisfying groundedness and order-preserving properties—to derive new soft sets that reflect observed attributes and potential latent associations within a system. This study presents foundational results on preserving key soft set structures under isotonic extension and examines how internal approximation relations evolve under such operators. The study provides an application to infectious disease risk modeling in a hospital environment as a practical demonstration. Here, isotonic extensions enable the identification of asymptomatic but exposed individuals, offering a novel mathematical approach to decision-making under uncertainty.

Keywords — *Soft sets, closure operators, isotonic spaces, decision making, medical diagnosis*

Mathematics Subject Classification (2020) 03E47, 54A05

1. Introduction

The necessity of coping with uncertainty and incomplete information is a fundamental challenge in modern mathematical modeling. In this context, the soft set theory, introduced by Molodtsov in 1999 [1], presents a significant innovation, particularly in modeling uncertainty through parametric representations. By providing a more flexible structure compared to classical logic frameworks, soft sets have found applications in numerous fields, such as multi-criteria decision making, information systems, medicine, engineering, and economics. However, existing approaches in the literature predominantly focus on observable, direct data, living out the modeling of indirect, implicit, or potentially risky relationships. This limitation leads to the inadequacy of decision models, especially in areas where indirect interactions are decisive, such as epidemiology, security analysis, and network theory.

The definition of a topology on a set extends beyond the traditional axioms for open sets, encompassing collections of closed sets, neighborhood systems, closure operators, and interior operators, among other constructs. For instance, Day [2] and Hausdorff [3] have developed topological concepts by leveraging the notions of convergence, closure, and neighborhoods. Kuratowski [4] has pioneered a distinct approach to constructing a topological structure on a non-empty set U through the definition of a closure operator $\mu : P(U) \to P(U)$, where $P(U)$ is the power set of U. Utilizing this framework, the closure operator satisfying the established axioms enables the definition of the topological space

[1]mbkandemir@mu.edu.tr (Corresponding Author)
[1]Department of Mathematics, Faculty of Science, Muğla Sıtkı Koçman University, Muğla, Türkiye

(U, μ) by identifying closed sets as those satisfying $\mu(X) = X$. Furthermore, Kuratowski has broadened the scope of topological spaces by relaxing the axiom $\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y)$, thereby defining closure spaces. Conversely, Čech's approach [5] to the definition of closure spaces omits the idempotence axiom $\mu(\mu(X)) = \mu(X)$. The terms "Kuratowski closure space" and "Čech closure space" are employed in the literature to mitigate terminological ambiguity. Additionally, Gnilka [6–8] and Hammer [9, 10] have preferred the term "extended topological space" over "closure space". These studies have investigated fundamental concepts, such as compactness, quasi-metrizability, symmetry, and continuity through the lens of closure operators. More recently, Stadler and Stadler [11] and Stadler et al. [12, 13] have unveiled a topological approach to chemical organizations, evolutionary theory, and combinatorial chemistry, elucidating the relationships between topological concepts, such as similarity, neighborhood, connectedness, and continuity within chemical and biological contexts. In these interdisciplinary studies, the authors have considered the foundational concepts of closure and isotonic spaces, defining an isotonic space as a closure space (U, μ) that satisfies only the axioms of groundedness, i.e., the condition $\mu(\emptyset) = \emptyset$, and isotonicity, i.e., the condition $X \subseteq Y \Rightarrow \mu(X) \subseteq \mu(Y)$, for all $X, Y \in P(U)$. Moreover, Habil and Elzenati [14, 15] have explored the notions of connectedness and lower and upper separation axioms in isotonic spaces.

Current research concerning soft sets primarily concentrates on fundamental set operations, equivalence structures, decision-making algorithms, and generalized operators. For instance, Maji et al. [16] have defined basic operations on soft sets; Ali et al. [17] extended these operations; and Molodtsov [18] provided a theoretical foundation for correct operation definitions in soft sets. Rapid advancements in soft set theory have led to the definition of a multitude of novel operations, such as multiplication and complementation on soft sets, along with their various modifications. The theoretical properties of these operations have been extensively studied in [19–28]. Alongside these, numerous studies concern variations of soft sets and their applications to decision-making problems [29–43]. Nevertheless, the vast majority of these studies are based on models where only existing information is processed. A framework for systematically including implicit relationships, chains of contact, or potential impacts into the model remains absent within the classical structure. This gap can lead to serious consequences, particularly in decision-making problems involving high uncertainty, such as the detection asymptomatic infections.

This study proposes a novel mathematical approach by extending soft sets through isotonic operators in this context. These operators consider not only the observed information but also the potential relationships arising from the structural nature of the system. Thus, elements that are not directly observable but are systemically at risk can be incorporated into the model. For instance, in a hospital, an asymptomatic individual, while not exhibiting direct symptoms, may carry a risk due to past contact with symptomatic individuals. The inability to integrate such indirect information into the classical soft set structure leads to deficient decision-making processes; the isotonic operator-extended soft sets aim to bridge this gap.

The core problem of this study is the inability of classical soft sets to systematically model indirect and potential information; the central hypothesis, on the other hand, posits that "soft sets extended with isotonic operators will be an effective tool in incorporate implicit relationships into decision systems by enhancing their sensitivity". The studies conducted in this direction in the literature are quite limited and mostly confined to specific examples. This study aims to reveal the structural properties of the isotonic extension at a theoretical level and demonstrate the model's functionality through a real-world application scenario.

Within this framework, the structure of the study is organized as follows: The second section presents the necessary preliminary information and conceptual foundation. The third section provides definitions of soft sets defined on isotonic spaces and the fundamental definitions for the extension of these structures. The fourth section examines the structural properties of the isotonic extension in detail and presents various theoretical results. The fifth section, on the other hand, conducts an exemplary application on infectious disease risk in a hospital setting, discussing the advantages of the isotonic extension compared to classical models. Finally, the conclusion section provides a general evaluation based on the findings obtained and offered by suggestions for future research.

2. Preliminaries

In this section, we lay the groundwork by introducing essential definitions and concepts from soft set theory and topology, fundamental to understanding the proposed methodology.

Definition 2.1. [1] Let U be a universe of discourse and E be the set of all parameters associated with the elements of U. The ordered pair (F, E) is called a soft set over U, where $F : E \to P(U)$ is a set-valued function.

Definition 2.2. [18] Let (F, E) be a soft set over U. Then, the family

$$APP(F, E) = \{F(p) \mid p \in E\}$$

is designated as a family of approximate descriptions, contingent upon the selection of E.

Definition 2.3. [18] Let (F, E_1) and (G, E_2) be two soft sets over a universe U.

i. (F, E_1) and (G, E_2) are termed equal soft sets, denoted by $(F, E_1) = (G, E_2)$, if $E_1 = E_2$ and $F = G$.

ii. (F, E_1) and (G, E_2) are termed equivalent soft sets, denoted by $(F, E_1) \cong (G, E_2)$, if $APP(F, E_1) = APP(G, E_2)$.

Note that this equivalence holds if and only if for every $p \in E_1$, there exists a $q \in E_2$ such that $F(p) = G(q)$, and for every $q \in E_2$, there exists a $p \in E_1$ such that $G(q) = F(p)$.

Definition 2.4. [18] Let U be a universe of discourse.

i. A unary operation Φ on soft sets is a mapping over U that associates a soft set (F, E_1) with another soft set (G, E_2), i.e., $\Phi(F, E_1) = (G, E_2)$. Moreover, Φ is deemed correct if $(F, E_1) \cong (G, E_2) \Rightarrow \Phi(F, E_1) \cong \Phi(G, E_2)$.

ii. A binary operation Θ on soft sets is a mapping that assigns to any two soft sets (F, E_1) and (G, E_2) over U, a novel soft set (H, E_3), i.e., $\Theta((F, E_1), (G, E_2)) = (H, E_3)$. Moreover, Θ is considered correct if $(F_1, E_1) \cong (F_2, E_2) \wedge (G_1, E_3) \cong (G_2, E_4) \Rightarrow \Theta((F_1, E_1), (G_1, E_3)) \cong \Theta((F_2, E_2), (G_2, E_4))$.

iii. A relationship Ω between two soft sets (F, E_1) and (G, E_2) is a mapping assigning the values 0 or 1 to $\Omega((F, E_1), (G, E_2))$. If $\Omega((F, E_1), (G, E_2)) = 1$, then it is denoted by $(F, E_1)\Omega(G, E_2)$. Moreover, Ω is correct if $(F_1, E_1) \cong (F_2, E_2) \wedge (G_1, E_3) \cong (G_2, E_4) \Rightarrow \Omega((F_1, E_1), (G_1, E_3)) = \Omega((F_2, E_2), (G_2, E_4))$.

Definition 2.5. [18] The complement of a soft set (F, E) is defined as a unary operation, denoted by $(F, E)^c = (F^c, E)$, where $F^c(p) = U \setminus F(p)$, for all $p \in E$.

Definition 2.6. [18] Let (F, E_1) and (G, E_2) be two soft sets over a universe U.

i. The intersection of (F, E_1) and (G, E_2) is a binary operation denoted by $(F, E_1)\widetilde{\cap}(G, E_2) = (H, E_1 \times E_2)$, where $H(p, q) = F(p) \cap G(q)$, for all $(p, q) \in E_1 \times E_2$.

ii. The union of (F, E_1) and (G, E_2) is a binary operation denoted by $(F, E_1)\widetilde{\cup}(G, E_2) = (K, E_1 \times E_2)$, where $K(p, q) = F(p) \cup G(q)$, for all $(p, q) \in E_1 \times E_2$.

Definition 2.7. [18] Let (F, E_1) and (G, E_2) be two soft sets over a universe U.

i. (F, E_1) is termed an internal approximation of (G, E_2), denoted by $(F, E_1)\widetilde{\subseteq}(G, E_2)$, if, for all $q \in E_2$ such that $G(q) \neq \emptyset$, there exists a $p \in E_1$ that satisfies $\emptyset \neq F(p) \subseteq G(q)$.

ii. (F, E_1) is termed an external approximation for (G, E_2), denoted by $(F, E_1)\widetilde{\supseteq}(G, E_2)$, if, for all $q \in E_2$ such that $G(q) \neq \emptyset$, there exists a $p \in E_1$ which satisfies $U \neq F(p) \supseteq G(q)$.

Definition 2.8. [18] Let (F, E) be a soft set over a universe U.

i. (F, E) is classified as a null soft set, denoted by $\widetilde{\emptyset}$, if and only if $APP(F, E) = \{\emptyset\}$.

ii. (F, E) is classified as an absolute soft set, denoted by $\widetilde{\mathbb{U}}$, if and only if $APP(F, E) = \{U\}$.

Definition 2.9. [4] Let $U \neq \emptyset$. Then, a function $\mu : P(U) \to P(U)$ is called a Kuratowski closure operator if it satisfies the following properties, for all $X, Y \in P(U)$:

(K0) $\mu(\emptyset) = \emptyset$ (groundedness)

(K1) $X \subseteq Y \Rightarrow \mu(X) \subseteq \mu(Y)$ (isotonicity)

(K2) $X \subseteq \mu(X)$ (expansiveness)

(K3) $\mu(X \cup Y) \subseteq \mu(X) \cup \mu(Y)$ (sub-additivity)

(K4) $\mu(\mu(X)) = \mu(X)$ (idempotence)

Definition 2.10. [4,5] Let $U \neq \emptyset$ and $\mu : P(U) \to P(U)$ be a function.

i. A topological space (U, μ) can be defined by a Kuratowski closure operator μ, where closed sets are the sets $X \subseteq U$ satisfying the condition $\mu(X) = X$.

ii. An ordered pair (U, μ) is called a closure space such that μ satisfies the conditions (K0)-(K3), where μ is called a closure operator.

Lemma 2.11. [5] Let (U, μ) be a closure space. Then, the following hold:

i. $\mu(X) \cup \mu(Y) \subseteq \mu(X \cup Y)$, for all $X, Y \in P(U)$

ii. $\mu(X \cap Y) \subseteq \mu(X) \cap \mu(Y)$, for all $X, Y \in P(U)$

Definition 2.12. [11–13] Let $U \neq \emptyset$ and $\mu : P(U) \to P(U)$ be a function. Then, the ordered pair (U, μ) is called an isotonic space if μ satisfies the conditions (K0) and (K1).

Example 2.13. Let $U = \{a, b\}$ and consider the function $\mu : P(U) \to P(U)$ defined by $\mu(\emptyset) = \emptyset$, $\mu(\{a\}) = \{b\}$, $\mu(\{b\}) = \{b\}$, and $\mu(U) = U$. It can be observed that μ is grounded. Furthermore, it is isotonic because $\mu(A) \subseteq \mu(B)$, for all $A \subseteq B$. Therefore, (U, μ) is an isotonic space. It must be noted that μ is not a Kuratowski closure operator and not a closure operator because $\{a\} \nsubseteq \mu(\{a\})$.

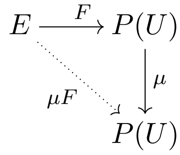
3. Soft Sets over Isotonic Spaces

In this section, we provide some results based on the relationship between soft sets and isotonic and closure spaces. Unless otherwise claimed, we consider the parameter set E for all soft sets.

Definition 3.1. Let (U, μ) be an isotonic space and (F, E) and (G, E) be soft sets over U. If $\mu \circ F = G$, i.e., $(\mu \circ F)(p) = G(p)$, for all $p \in E$, then (G, E) is called an isotonic extension of (F, E) and denoted by $(\mu F, E)$.

This definition allows for to extend a soft set by incorporating external information, such as indirect contact or inferred proximity, through the isotonic operator. It can be observed that Definition 3.1

yields the following commutative diagram:



Furthermore, since the operator μ defines a new soft set, it functions as a unary operation among soft sets. Consequently, the following proposition is derived.

Proposition 3.2. Let (U, μ) be an isotonic space and (F, E) and (G, E) be soft sets over U. If $(F, E) \cong (G, E)$, then $(\mu F, E) \cong (\mu G, E)$.

Proof. Let (U, μ) be an isotonic space, (F, E) and (G, E) be soft sets over U, and $(F, E) \cong (G, E)$. Then, there exist $p, q \in E$ such that $F(p) = G(q)$. Thus, $\mu(F(p)) = \mu(G(q))$. Hence, there exist $p, q \in E$ such that $(\mu \circ F)(p) = (\mu \circ G)(q)$. Therefore, $APP(F, E) = APP(G, E)$ and thus $(\mu F, E) \cong (\mu G, E)$. □

It should be noted that the converse of this proposition is not always true. For example, let $U = \{1, 2, 3\}$ and $E = \{p\}$ and define the soft sets (F, E) and (G, E) such that $F(p) = \{1\}$ and $G(p) = \{1, 2\}$. Consider the isotonic operator $\mu : P(U) \to P(U)$ given by $\mu(X) = \begin{cases} X \cup \{2\}, & X = \{1\} \\ X, & \text{otherwise} \end{cases}$. Since $\mu(\{1\}) = \{1, 2\}$ and $\mu(\{1, 2\}) = \{1, 2\}$, then $(\mu \circ F)(p) = \mu(F(p)) = \mu(\{1\}) = \{1, 2\}$ and $(\mu \circ G)(p) = \mu(G(p)) = \mu(\{1, 2\}) = \{1, 2\}$ and thus $(\mu F, E) \cong (\mu G, E)$ and $(F, E) \ncong (G, E)$.

Corollary 3.3. Let (U, μ) be an isotonic space. Then, the isotonic extension operation Φ on soft sets over U defined by $\Phi(F, E) = (\mu F, E)$ is correct.

Proposition 3.4. Let (U, μ) be an isotonic space. Then, the null soft set $\widetilde{\emptyset}$ is preserved the under isotonic extension in Corollary 3.3.

Proof. Let (U, μ) be an isotonic space and $\widetilde{\emptyset} = (F, E)$. Then, $APP\,(F, E) = \{\emptyset\}$. By the property of groundedness, $\mu(\emptyset) = \emptyset$. Thus, $APP(\mu F, E) = \{\emptyset\}$. Hence, $\Phi\left(\widetilde{\emptyset}\right) = \widetilde{\emptyset}$. □

This result implies that the isotonic operator preserves the structure of complete absence (null soft set), ensuring no unintended elements are added during extension.

Proposition 3.5. Let (U, μ) be an isotonic space. Then, the absolute soft set $\widetilde{\mathbb{U}}$ is preserved under isotonic extension if and only if $\mu(U) = U$.

Proof. Let (U, μ) be an isotonic space and $(F, E) = \widetilde{\mathbb{U}}$.

($\Rightarrow$): Suppose that the absolute soft set (F, E) is preserved under isotonic extension. Then, $APP(F, E) = \{U\} = \{\mu(U)\} = APP(\mu F, E)$. Thus, $\mu(U) = U$.

($\Leftarrow$): Suppose that $\mu(U) = U$. Then, $\mu(F(p)) = \mu(U) = U$, for all $p \in E$. Thus, $APP(\mu F, E) = \{U\}$. Hence, $\Phi(F, E) = (F, E)$. Therefore, the absolute soft set (F, E) is preserved under isotonic extension. □

3.1. Structural Properties of Isotonic Extensions

In this subsection, we provide structural properties of isotonic extensions. These properties collectively show that isotonic extension behaves structurally consistently, preserving logical relations among soft sets and enabling risk propagation mechanisms.

Proposition 3.6. Let (U, μ) be an isotonic space, (F, E) and (G, E) be two soft sets over U, and $(H, E \times E) = (F, E)\widetilde{\cup}(G, E)$. Then, the following hold:

i. $(\mu F, E)\widetilde{\cup}(\mu G, E)\widetilde{\subseteq}(\mu H, E \times E)$

ii. If μ is preserved under the union operation, then $(\mu H, E \times E) = (\mu F, E)\widetilde{\cup}(\mu G, E)$

The proof can be observed from Lemma 2.11 (*i*).

Proposition 3.7. Let (U, μ) be an isotonic space, (F, E) and (G, E) be two soft sets over U, and $(H, E \times E) = (F, E)\widetilde{\cap}(G, E)$. Then,

i. $(\mu H, E \times E)\widetilde{\subseteq}(\mu F, E)\widetilde{\cap}(\mu G, E)$

ii. If μ is preserved under the intersection operation, then $(\mu H, E \times E) = (\mu F, E)\widetilde{\cap}(\mu G, E)$

The proof can be observed from Lemma 2.11 (*ii*).

It is important to note that the complement of the isotonic extension of a soft set may not be equal to the isotonic extension of the complement of the soft set. Indeed, consider the soft set (F, E), where $U = \{1, 2, 3\}$, $E = \{p\}$, and $F(p) = \{1\}$. Let the isotonic operator $\mu : \mathcal{P}(U) \to \mathcal{P}(U)$ be defined by

$$\mu(X) = \begin{cases} X \cup \{2\}, & X = \{1\} \\ X, & \text{otherwise} \end{cases}$$

Since $F^c(p) = U \setminus F(p) = \{2, 3\}$, then $(\mu \circ F^c)(p) = \mu(\{2, 3\}) = \{2, 3\}$. Furthermore, $(\mu \circ F)(p) = \mu(\{1\}) = \{1, 2\}$ and $(\mu \circ F)^c(p) = U \setminus \{1, 2\} = \{3\}$. Therefore, $(\mu \circ F^c)(p) \neq (\mu \circ F)^c(p)$.

Proposition 3.8. Let (U, μ) be an isotonic space. If $(F, E)\widetilde{\subseteq}(G, E)$, then $(\mu F, E)\widetilde{\subseteq}(\mu G, E)$.

Proof. Let (U, μ) be an isotonic space and $(F, E)\widetilde{\subseteq}(G, E)$. According to the definition of internal approximation, for all $q \in E$, there exists a $p \in E$ such that $\emptyset \neq F(p) \subseteq H(q)$. Since μ is grounded and isotonic, $\mu(F(p)) \subseteq \mu(H(q))$. Thus, $(\mu F, E)\widetilde{\subseteq}(\mu H, E)$. □

Proposition 3.9. Let (U, μ) be an isotonic space. If the closure operator is extensive, then $(F, E)\widetilde{\subseteq}(\mu F, E)$.

Proof. Let (U, μ) be an isotonic space and μ be extensive. Then, for all $p \in E$ such that $(\mu F)(p) \neq \emptyset$, $\emptyset \neq F(p) \subseteq (\mu F)(p)$. Thus, (F, E) is an internal approximation of $(\mu F, E)$, i.e., $(F, E)\widetilde{\subseteq}(\mu F, E)$. □

Definition 3.10. Let (U, μ) be an isotonic space and (F, E) be a soft set over U. If $(\mu F, E) = (F, E)$, then (F, E) is called a μ-closed soft set over U.

Proposition 3.11. Let (U, μ) be an isotonic space and (F, E) be a soft set over U. Then, (F, E) is a μ-closed soft set over U if and only if for all $p \in E$, $F(p)$ is closed in the isotonic space (U, μ).

Proof. Let (U, μ) be an isotonic space and (F, E) be a soft set over U.

($\Rightarrow$): Suppose that (F, E) is a μ-closed soft set over U. Then, $(\mu F, E) = (F, E)$. Thus, $\mu(F(p)) = F(p)$, for all $p \in E$. Hence, $F(p)$ is closed in the isotonic space (U, μ), for all $p \in E$.

($\Leftarrow$): Suppose that $F(p)$ is closed in the isotonic space (U, μ), for all $p \in E$. Then, $\mu(F(p)) = F(p)$, for all $p \in E$. Hence, $(\mu F, E) = (F, E)$. Therefore, (F, E) is a μ-closed soft set over U. □

Proposition 3.12. Let (U, μ) be an isotonic space and (F, E) and (G, E) be two soft sets over U. If (F, E) and (G, E) are μ-closed soft sets over U and μ is extensive, then $(F, E)\widetilde{\cap}(G, E)$ is a μ-closed soft set over U.

Proof. Let (U, μ) be an isotonic space, (F, E) and (G, E) be two soft sets over U, (F, E) and (G, E) be μ-closed soft sets over U, μ be extensive, and $(F, E)\tilde{\cap}(G, E) = (H, E \times E)$. By Proposition 3.7 (i), $(\mu H, E \times E)\tilde{\subseteq}(\mu F, E)\tilde{\cap}(\mu G, E)$. Since (F, E) and (G, E) are μ-closed soft sets over U, $(\mu F, E) = (F, E)$ and $(\mu G, E) = (G, E)$, which implies $(\mu H, E \times E)\tilde{\subseteq}(H, E \times E)$. Moreover, since μ is extensive, $H(p, q) \subseteq \mu(H(p, q))$, for all $(p, q) \in E \times E$. Thus, $(\mu H, E \times E) = (H, E \times E)$. Hence, $(F, E)\tilde{\cap}(G, E)$ is a μ-closed soft set over U. □

Proposition 3.13. Let (U, μ) be an isotonic space and (F, E) and (G, E) be two soft sets over U. If (F, E) and (G, E) are μ-closed soft sets over U and μ is sub-additive, then $(F, E)\tilde{\cup}(G, E)$ is a μ-closed soft sets over U.

The proof can be observed from Proposition 3.6 and the property of sub-additivity.

4. An Application of Isotonic Extensions of Soft Sets to Medical Diagnosis

To demonstrate the practical utility of isotonic extensions of soft sets, we consider an example involving infectious disease surveillance in a hospital setting. The goal is to detect symptomatic patients and those with potential exposure risks. This section details a novel approach to infectious disease surveillance through soft set-based risk modeling.

Algorithm 1 Core Algorithm Implemented in the Application

Input

1. The universal set U is defined as the collection of all patients or individuals.

2. The parameter set E whose elements represent symptoms or risk indicators.

3. The initial soft set (F, E), where the observed individuals are considered for each parameter.

4. The exposure rule $\mathcal{R} = \{(C_i, a_i) \mid C_i \subseteq U, a_i \in U\}$.

Output

1. The isotonic extension $(\mu F, E)$.

2. Risk frequencies obtained by the function $risk : U \to \mathbb{N}$ defined by $risk(h) = |\{p \in E \mid h \in (\mu F)(p)\}|$.

3. Priority ranking descending by risk scores.

Steps

Step 1. Define the isotonic operator $\mu(X) = X \cup \{a_i \mid (C_i, a_i) \in \mathcal{R} \text{ and } X \cap C_i \neq \emptyset\}$.

Step 2. Obtain the isotonic extension $(\mu F, E)$.

Step 3. Calculete risk frequencies, for each individual.

Step 4. Apply the prioritization rule, i.e., rank individuals in descending order according to their risk frequencies. Determine arbitrary priority for equal risk scores and consider additional criteria.

Following a respiratory disease outbreak, a metropolitan hospital faces a subtle yet significant challenge: identifying not only patients who exhibit overt symptoms but also asymptomatically infected individuals who may be silently transmitting the disease within the hospital environment. Conventional diagnostic methodologies predominantly focus on symptomatic individuals; however, the transmission dynamics of infectious diseases often transcend such clinical presentations.

In this paper, we introduce an innovative soft set-based decision support model that extends beyond the analysis of observable symptoms by incorporating exposure-based information through the application of an isotonic operator. The central objective is to develop a mathematical framework, utilizing soft sets enriched with contact-tracing semantics, to effectively model latent infection risk within a hospital milieu.

Consider patients who present without symptoms yet have a documented history of sharing rooms or interacting with confirmed cases. Their seemingly benign status raises a critical question: Are they truly risk-free?

4.1. Soft Set Modeling of Symptom Data

We define the universal set of patients as follows:

$$U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$$

Let the set of pertinent symptoms be represented by $E = \{p_1, p_2, p_3\}$, where p_1: Fever, p_2: Cough, and p_3: Sore throat. Consider the soft set (F, E) defined by

$$F(p_1) = \{h_1, h_2\}, \quad F(p_2) = \{h_1, h_3\}, \quad \text{and} \quad F(p_3) = \{h_5\}$$

Moreover, (F, E) can also be represented as shown in Table 1:

Table 1. A representation of (F, E)

Pertinent symptoms	**Patients exhibiting the symptom**
Fever (p_1)	$\{h_1, h_2\}$
Cough (p_2)	$\{h_1, h_3\}$
Sore throat (p_3)	$\{h_5\}$

4.2. Integration of Related Contact Data (Exposure (Semantics) Rules)

To enhance the model's granularity, we incorporate hospital contact data:

i. Patient h_4 shared a room with patients h_1 and h_2.

ii. Patient h_6 had close contact with patient h_5.

These documented connections serve as the basis for the subsequent application of the considered isotonic operator. This relationship is illustrated in Figure 1:

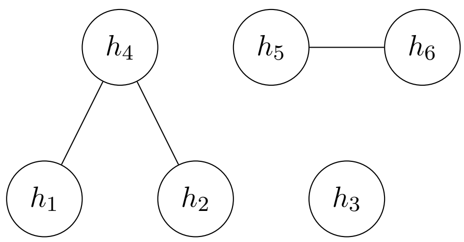


Figure 1. Contact network among patients

Thus, the exposure rule $\mathcal{R}$ is obtained as follows:

$$\mathcal{R} = \{(\{h_1, h_2\}, h_4), (\{h_5\}, h_6)\}$$

where $C_1 = \{h_1, h_2\}$ and $C_2 = \{h_5\}$; $a_1 = h_4$ and $a_2 = h_6$.

4.3. Isotonic Extension Operator: Capturing Exposure Risk

Consider the function $\mu : P(U) \to P(U)$ defined by

$$\mu(X) = X \cup \{a_i \mid (C_i, a_i) \in \mathcal{R} \text{ and } X \cap C_i \neq \emptyset\}$$

Then, μ is an isotonic operator on U. This operator is designed to capture the risk associated with indirect exposure and embodies a proactive infection control strategy by identifying individuals at elevated risk due to their proximity to confirmed cases. The isotonic extension effectively models the amplification of infection risk based on spatial and social proximity within the hospital environment. The isotonic extension of the soft set can be illustrated in Figure 2.

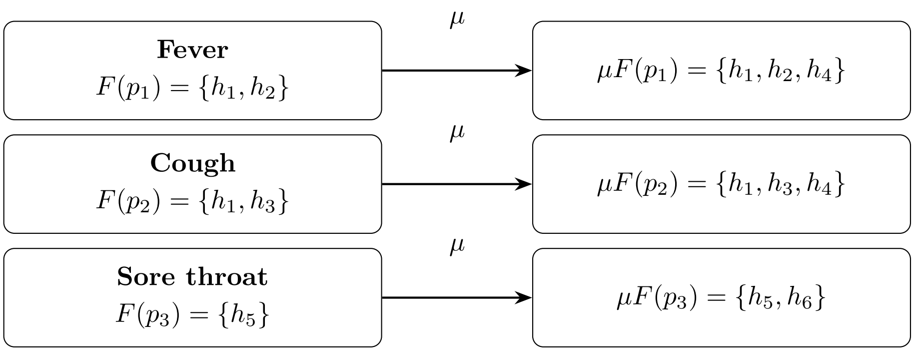


Figure 2. Diagram of the isotonic extension of (F, E)

4.4. Isotonic Extension of the Soft Set and Risk Frequencies

Applying the isotonic operator μ to each symptom-based patient set yields the following sets:

$$\mu F(p_1) = \{h_1, h_2, h_4\}, \quad \mu F(p_2) = \{h_1, h_3, h_4\}, \quad \text{and} \quad \mu F(p_3) = \{h_5, h_6\}$$

From these sets, we compute risk frequencies for each patient (see Table 2):

Table 2. Risk Frequencies of Patients

Patients	Frequencies
h_1	2
h_2	1
h_3	1
h_4	2
h_5	1
h_6	1

The visual representation illustrating the risk frequencies for each patient is provided in Figure 3:

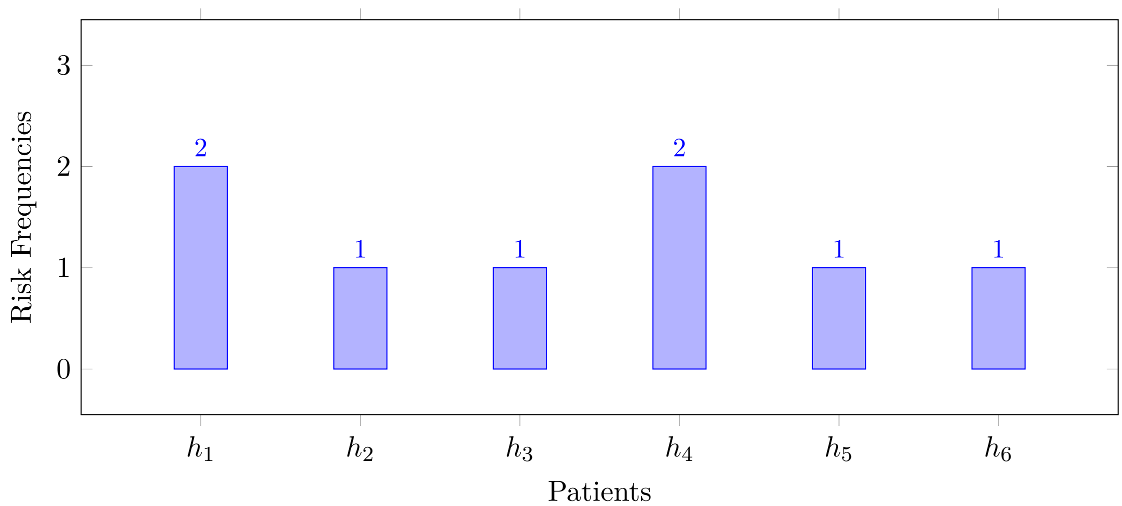


Figure 3. Risk frequencies for each patient computed from $(\mu F, E)$

4.5. Decision Outcome: Prioritization for Intervention

Employing a straightforward decision rule, prioritize the patient who appears in the highest number of extended symptom sets, the analysis reveals:

i. Patients h_1 and h_4 appear in two symptom sets of the isotonic extension.

ii. Patient h_1 is symptomatic.

iii. Patient h_4 is asymptomatic but identified as high-risk due to exposure.

Consequently, the proposed system recommends that both patients should be prioritized for isolation and further diagnostic testing. This simple yet powerful rule highlights how isotonic extension enhances the soft set model: it successfully identifies asymptomatic individuals (e.g., h_4) who, despite not presenting symptoms, pose a risk due to documented exposure. Traditional soft set models would fail to flag such individuals.

5. Conclusion

This paper presents a theoretical and applied framework for extending soft sets via isotonic operators. The proposed approach addresses a key limitation in classical soft set theory: the inability to represent indirect or latent information such as exposure risk. The study establishes a consistent and correct unary operation that preserves equivalence on soft sets, the null soft set, and the absolute soft set under certain conditions by introducing and formalizing the isotonic extension of a soft set. From a practical standpoint, the application to hospital-based infection surveillance demonstrates the real-world relevance of isotonic extensions of soft sets. The model identifies high-risk individuals not based solely on observed symptoms but also on indirect contact information, an essential advancement in decision-making under uncertainty. Future research may explore further generalizations using parameter-dependent isotonic operators or the integration of temporal dynamics, enabling real-time risk modeling in evolving systems.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

References

[1] D. A. Molodtsov, *Soft set theory-First results*, Computers and Mathematics with Application 37 (4-5) (1999) 19–31.

[2] M. M. Day, *Convergence, closure, and neighborhoods*, Duke Mathematical Journal 11 (1944) 181–199.

[3] F. Hausdorff, *Gestufte raume*, Fundamenta Mathematicae 25 (1) (1935) 486–502.

[4] K. Kuratowski, Topology, Academic Press, 1966.

[5] E. Ćech, Z. Frolik, M. Katétov, *Topological spaces*, Publishing House of the Czechoslovak Academy of Sciences, 1966.

[6] S. Gnilka, *On extended topologies I: Closure operators*, Commentationes Mathematicae 34 (1994) 81–94.

[7] S. Gnilka, *On extended topologies II: Compactness, quasi-metrizability, symmetry*, Commentationes Mathematicae 35 (1995) 147–162.

[8] S. Gnilka, *On continuity in extended topologies*, Commentationes Mathematicae 37 (1997) 99–108.

[9] P. C. Hammer, *Extended topology: Set-valued set functions*, Nieuw Archief voor Wiskunde 10 (1962) 55–77.

[10] P.C.Hammer, *Extended topology: Continuity I*, Portugaliae Mathematica 25 (2) (1964) 77–93.

[11] B. M. R. Stadler, P. F. Stadler, *Generalized topological spaces in evolutionary theory and combinatorial chemistry*, Journal of Chemical Information and Computer Sciences 42 (3) (2002) 577–585.

[12] B. M. R. Stadler, P. F. Stadler, M. Shpak, G. Wagner, *Recombination spaces, metrics, and pretopologies*, Zeitschrift für Physikalische Chemie 216 (2) (2002) 217–234.

[13] B. M. R. Stadler, P. F. Stadler, G. Wagner, W. Fontana, *The topology of the possible: Formal spaces underlying patterns of evolutionary change*, Journal of Theoretical Biology 213 (2) (2001) 241–274.

[14] E. D. Habil, K. A. Elzenati, *Topological properties in isotonic spaces*, Islamic University Journal 16 (2) (2008) 1–14.

[15] E. D. Habil, K. A. Elzenati, *Connectedness in isotonic spaces*, Turkish Journal of Mathematics 30 (2006) 247–262.

[16] P. K. Maji, R. Biswas, A. R. Roy, *Soft set theory*, Computers and Mathematics with Applications 45 (2003) 555–562.

[17] M. I. Ali, F. Feng, X. Y. Liu, W. K. Min, M. Shabir, *On some new operations in soft set theory*, Computers and Mathematics with Applications 57 (2009) 1547–1553.

[18] D. A. Molodtsov, *Equivalence and correct operations for soft sets*, International Robotics & Automation Journal 4 (1) (2018) 18–21.

[19] N. Çağman, *Conditional complements of sets and their application to group theory*, Journal of New Results in Science 10 (3) (2021) 67–74.

[20] N. S. Stojanovic, *A new operation on soft sets: Extended symmetric difference of soft sets*, Military Technical Courier 69 (4) (2021) 779–791.

[21] A. Sezgin, F. N. Aybek, A. O. Atagün, *A new soft set operation: Complementary soft binary piecewise intersection operation*, Black Sea Journal of Engineering and Science 6 (4) (2023) 330–346.

[22] A. Sezgin, F. N. Aybek, N. B. Güngör, *A new soft set operation: Complementary soft binary piecewise union operation*, Acta Informatica Malaysia 7 (1) (2023) 38–53.

[23] A. Sezgin, E. Yavuz, *A new soft set operation: Soft binary piecewise symmetric difference operation*, Necmettin Erbakan Üniversitesi Fen ve Mühendislik Bilimleri Dergisi 5 (2) (2023) 189–208.

[24] A. Sezgin, A. M. Demirci, *A new soft set operation: Complementary soft binary piecewise star operation*, Ikonion Journal of Mathematics 5 (2) (2003) 24-52.

[25] A. Sezgin, N. H. Çam, *Soft difference-product: A new product for soft sets with its decision-making*, Journal of Amasya University, the Institute of Sciences and Technology 5 (2) (2024) 144–137.

[26] A. Sezgin, M. Sarıalioğlu, *A new soft set operation: Complementary soft binary piecewise theta (θ) operation*, Journal of Kadirli Faculty of Applied Sciences 4 (2) (2024) 325–357.

[27] A. Sezgin, N. Çağman, *A new soft set operation: Complementary soft binary piecewise difference operation*, Osmaniye Korkut Ata University Journal of the Institute of Science and Technology 7 (1) (2024) 58–94.

[28] A. Sezgin, E. Şenyiğit, M. Luzum, *A new product for soft sets with its decision-making: Soft gamma-product*, Earthline Journal of Mathematical Sciences 15 (2) (2025) 211–234.

[29] P. K. Maji, A. R. Roy, R. Biswas, *An application of soft sets in a decision making problem*, Computers and Mathematics with Applications 44 (8) (2002) 1077–1083.

[30] N. Çağman, S. Enginoğlu, *Soft matrix theory and its decision making*, Computers and Mathematics with Applications 59 (2010) 3308–3314.

[31] N. Çağman, S. Karataş, *Intuitionistic fuzzy soft set theory and its decision making*, Journal of Intelligence and Fuzzy Systems 24 (2013) 829–836.

[32] W. Xu, Z. Xiao, X. Dang, D. Yang, X. Yang, *Financial ratio selection for business failure prediction using soft set theory*, Knowledge-Based Systems 63 (2014) 59–67.

[33] M. B. Kandemir, *Monotonic soft sets and its applications*, Annals of Fuzzy Mathematics and Information 12 (2) (2016) 295–307.

[34] A. Z. Khameneh, A. Kılıçman, A. R. Salleh, *An adjustable approach to multi-criteria group decision-making based on a preference relationship under fuzzy soft information*, International Journal of Fuzzy System 19 (2017) 1840–1865.

[35] A. O. Atagün, H. Kamacı, O. Oktay, *Reduced soft matrices and generalized products with applications in decision making*, Neural Computing and Applications 29 (2018) 445–456.

[36] H. Kamacı, A. O. Atagün, A. Sönmezoğlu, *Row-products of soft matrices with applications in multiple-disjoint decision making*, Soft Computing 62 (2018) 892–914.

[37] M. B. Kandemir, *The concept of σ-algebraic soft set*, Soft Computing 22 (2018) 4353–4360.

[38] Z. Ali, T. Mahmood, M. Aslam, R. Chinram, *Another view of complex intuitionistic fuzzy soft sets based on prioritized aggregation operators and their applications to multiattribute decision making*, Mathematics 9 (2021) 30803–30816.

[39] M. Ali, A. Kılıçman, *On interval-valued fuzzy soft preordered sets and associated applications in decision-making*, Mathematics 9 (2021) 3142.

[40] G. Ali, H. Alolaiyan, D. Pamučar, M. Asif, N. Lateef, *A novel MADM framework under q-rung orthopair fuzzy bipolar soft sets*, Mathematics 9 (2021) 2163.

[41] S. Ali, M. Kousar, Q. Xin, D. Pamučar, M. S. Hameed, R. Fayyaz, *Belief and possibility belief interval-valued N-soft set and their applications in multi-attribute decision-making problems*, Entropy 23 (2021) 1498.

[42] O. Dalkılıç, *A novel approach to soft set theory in decision-making under uncertainty*, International Journal of Computer Mathematics 98 (10) (2021) 1935–1945.

[43] K. Taşköprü, E. Karaköse, *A soft set approach to relations and its application to decision making*, Mathematical Sciences and Applications E-Notes 11 (1) (2023) 1–13.