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Research Article

On Dual Hyperbolic Narayana and Narayana-Lucas Hybrid Quaternions

Pankaj Kumar [1]* **and Shilpa Kapoor**[1]

[1] *Srinivasa Ramanujan Department of Mathematics, Central University of Himachal Pradesh, Dharamshala, Shahpur, India*
* *Corresponding author*

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Abstract

In this present communication, we introduce the novel concepts of dual hyperbolic Narayana quaternions and dual hyperbolic Narayana-Lucas quaternions within the framework of hybrid numbers. We also explore the connections between these newly defined quaternions and examine the mathematical properties they share. Additionally, we find the recurrence relations, Binet formulas, generating functions, exponential generating functions, and other meaningful identities. These newly introduced quaternions have significant applications in the fields of quantum physics, computer graphics, and robotics. Additionally, these identities and relationships we established also play an important role in the field of number theory and combinatorics.

1. Introduction

In the recent years, significant work has been done in the field of number sequences due to their wide range of applications in quantum physics, number theory, robotics, and image encryptions. Number sequences also plays an important role in the areas such as stock market, currency exchange and many others [1, 2]. An examination of specific number-theoretic properties in the context of algebra reveals a fascinating relationship between quaternions and number theory. Particularly, quaternions from classical number theory are useful tools for contemporary mathematics and engineering because they not only improve fundamental identities but also have uses in modular arithmetic, norm-based division tests, and matrix construction. Hamilton [3] was the first to introduce the concept of quaternions as:

$$H = \{p + iq + jr + ks \,|p, q, r, s \in \mathbb{R}, i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j\}.$$

where i, j, k are the quaternionic units and are non-commutative under the multiplication rule. Quaternions are important in many areas of applied mathematics, such as computer science, physics, differential geometry, quantum physics, engineering, and the calculation of rotational motions in three dimensions. Numerous studies have looked into the relationships between quaternions and the algebra of sequences. For more detailed insights, refer to the works cited in [3–12]. Dual hyperbolic quaternions can be considered an algebraic extension of quaternions, with the properties of hyperbolic and dual numbers. Quaternions form a basis in 3D rotation and algebra, while dual hyperbolic quaternions are used in special relativity, kinematics, and theoretical physics.
The set of hyperbolic number $\mathbb{H}$ can be expressed as [13]:

$$\mathbb{H} = \{z = q_1 + hq_2|\, h \notin \mathbb{R}, h^2 = 1, q_1, q_2 \in \mathbb{R}\},$$

while, dual-hyperbolic numbers $D\mathbb{H}$ can be expressed as [14]:

$$D\mathbb{H} = \{w = z_1 + \varepsilon z_2 \,|z_1, z_2 \in \mathbb{H}, j^2 = 1, \varepsilon^2 = 0, \varepsilon \neq 0, (j\varepsilon)^2 = 0\}.$$

Here $z_1 = q_1 + q_2 j$, $z_2 = q_3 + q_4 j$ then any dual hyperbolic number can be written as : $w = (q_1 + q_2 j) + \varepsilon(q_3 + q_4 j)$ where $j^2 = 1, \varepsilon^2 = 0, \varepsilon \neq 0, (j\varepsilon)^2 = 0$.
Many areas of mathematics and physics have been transformed by quaternion algebra, leading to new insights and creativity. These adaptable

Email addresses: **pankajthakur28.85@hpcu.ac.in, kapoorshilpa063@gmail.com**
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structures endowed researchers and practitioners with the power to gain deep insights into intricate systems and space transformations. Aydin [15] introduced the concept of dual hyperbolic Pell quaternions along with several identities related to these quaternions. Additionally, a number of researchers have conducted investigations on dual hyperbolic quaternions, and for more details, one can refer to [16, 17]. Recently, the concept of hybrid numbers was introduced by Özdemir [18], which are a combination of real, complex, dual, and hyperbolic numbers. The set of hybrid numbers H is defined as:

$$H = \{z = a + b\iota + c\varepsilon + dh; a,b,c,d \in \mathbb{R}\},$$

where ι, ε, h are operators such that $\iota^2 = -1, \varepsilon^2 = 0, h^2 = 1, \iota h = -h\iota = \varepsilon + \iota$. The conjugate of a hybrid numbers z is defined as: $\bar{z} = \overline{a + b\iota + c\varepsilon + dh} = a - b\iota - c\varepsilon - dh$. The character of hybrid number z is defined as the real number $C(z) = z\bar{z} = \bar{z}z = a^2 + b^2 - 2bc - d^2$ and the norm of hybrid number z is defined as $\sqrt{|C(z)|}$ and denoted by $\|z\|$ [18]. Hybrid numbers, as an extension of these sequences, offer versatile tools that have proven valuable in diverse mathematical domains, including algebra, number theory, and geometry. Beyond theoretical mathematics, hybrid numbers and their associated sequences have found meaningful applications in fields such as linear algebra, kinematics, engineering design, and theoretical physics. Their widespread utility underscores their importance in both abstract theory and real-world problem-solving. Hybrid numbers with various sequences have earned a lot of interest in recent years [19–25]. Through a comprehensive literature review, we found no prior studies addressing dual hyperbolic Narayana and Narayana-Lucas hybrid quaternions. This gap highlights the distinctive novelty of our work.

The primary objective of this study is to introduce the concepts of hyperbolic and dual hyperbolic Narayana and Narayana-Lucas quaternions, and to associate with these structures the concept of hybrid number in order to form the respective hybrid quaternion structures. Additionally, we aim to investigate the relationships between these constructs. For these newly introduced quaternions, we also derive recurrence relations, Binet formulas, generating functions, exponential generating functions, and various significant identities.

2. Preliminaries

In order to address the problem in hand related to the construction of dual hyperbolic Narayana and Narayana- Lucas hybrid quaternions, here in this preliminary section we are re-calling certain definitions which will serve as a foundational framework for the subsequent analysis to carried out and interpretation of the results.

Definition 2.1. *The Narayana sequence N_n and Narayana-Lucas sequence U_n are defined as [25]:*
For the Narayana Sequence:

$$N_{n+3} = N_{n+2} + N_n, N_0 = 0, N_1 = 1, N_2 = 1.$$

For the Narayana-Lucas Sequence:

$$U_{n+3} = U_{n+2} + U_n, U_0 = 3, U_1 = 1, U_2 = 1,$$

for $n \geq 3$ respectively.

Definition 2.2. *The Narayana hybrid sequence N_n^H and Narayana-Lucas hybrid sequence U_n^H are defined as [19]:*

$$N_n^H = N_n + \iota N_{n+1} + \varepsilon N_{n+2} + hN_{n+3},$$
$$U_n^H = U_n + \iota U_{n+1} + \varepsilon U_{n+2} + hU_{n+3},$$

where ι, ε, h are operators such that $\iota^2 = -1, \varepsilon^2 = 0, h^2 = 1, \iota h = -h\iota = \varepsilon + \iota$.

3. Dual Hyperbolic Narayana and Narayana-Lucas Quaternions

In this section, we introduce the definitions of Narayana quaternions and Narayana-Lucas quaternions, as well as dual-hyperbolic Narayana and dual-hyperbolic Narayana-Lucas quaternions, by using the basic definitions as mentioned in the previous Section 2. We substantiate our discussion with theorems and provide valuable identities related to these newly introduced dual hyperbolic Narayana and dual hyperbolic Narayana-Lucas quaternions structures.

Definition 3.1. *Let Z_n and T_n be the Narayana quaternions and Narayana-Lucas quaternions, respectively, which can be defined as:*

$$Z_n = N_n + iN_{n+1} + jN_{n+2} + kN_{n+3},$$
$$T_n = U_n + iU_{n+1} + jU_{n+2} + kU_{n+3},$$

respectively, where $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$.

Definition 3.2. *Let $D\mathbb{H}Z_n$ and $D\mathbb{H}T_n$ be the dual hyperbolic Narayana quaternions and dual hyperbolic Narayana-Lucas quaternions, respectively, which can be defined as:*

$$D\mathbb{H}Z_n = N_n + jN_{n+1} + \varepsilon(N_{n+2} + jN_{n+3}), \tag{3.1}$$
$$D\mathbb{H}T_n = U_n + jU_{n+1} + \varepsilon(U_{n+2} + jU_{n+3}), \tag{3.2}$$

where N_n denotes the n^{th} terms of Narayana sequence and U_n denotes the n^{th} terms of Narayana-Lucas sequence and $\{j, \varepsilon, j\varepsilon\}$ denote the dual hyperbolic quaterionic units which satisfy the non commutative multiplication rules:

$$j^2 = 1, \varepsilon \neq 0, \varepsilon^2 = 0, \varepsilon.j = j.\varepsilon, (j.\varepsilon)^2 = 0 \tag{3.3}$$

The addition, subtraction, and multiplication of dual hyperbolic Narayana and Narayana-Lucas quaternions can be defined as:

$$\begin{aligned}(D\mathbb{H}Z_n \mp D\mathbb{H}T_n) =&(N_n \mp U_n) + j(N_{n+1} \mp U_{n+1}) + \varepsilon(N_{n+2} \mp U_{n+2}) + j.\varepsilon(N_{n+3} \mp U_{n+3})\\ (D\mathbb{H}N_n \cdot D\mathbb{H}U_n) =&(N_n + jN_{n+1} + \varepsilon N_{n+2} + j.\varepsilon N_{n+3}) \cdot (U_n + jU_{n+1} + \varepsilon U_{n+2} + j.\varepsilon U_{n+3})\\ =&(N_nU_n + N_{n+1}U_{n+1}) + j(N_nU_{n+1} + N_{n+1}U_n) + \varepsilon(N_nU_{n+3} + N_{n+1}U_{n+3} + N_{n+2}U_n + N_{n+3}U_{n+1})\\ &+ j\cdot\varepsilon(N_nU_{n+3} + N_{n+1}U_{n+2} + N_{n+2}U_{n+1} + N_{n+3}U_n)\end{aligned}$$

Definition 3.3. *Let $\overline{D\mathbb{H}Z_n}$ and $\overline{D\mathbb{H}T_n}$ be the conjugates of dual hyperbolic Narayana quaternions and dual hyperbolic Narayana-Lucas quaternions, respectively, which can be defined as:*

$$\begin{aligned}\overline{D\mathbb{H}Z_n} &= N_n - jN_{n+1} - \varepsilon(N_{n+2} + jN_{n+3}),\\ \overline{D\mathbb{H}T_n} &= U_n - jU_{n+1} - \varepsilon(U_{n+2} + jU_{n+3}),\end{aligned}$$

where N_n denotes the n^{th} terms of Narayana sequence and U_n denotes the n^{th} terms of Narayana-Lucas sequence and $\{j, \varepsilon, j\varepsilon\}$ denote the dual hyperbolic quaterionic units which satisfy the non commutative multiplication rules:

$$j^2 = 1, \varepsilon \neq 0, \varepsilon^2 = 0, \varepsilon.j = j.\varepsilon, (j.\varepsilon)^2 = 0$$

Theorem 3.4. *Let $D\mathbb{H}Z_n$ and $D\mathbb{H}T_n$ be the dual hyperbolic Narayana quaternions and dual hyperbolic Narayana-Lucas quaternions respectively. Then the recurrence relations for these quaternions hold:*

$$\begin{aligned}D\mathbb{H}Z_n &= D\mathbb{H}Z_{n-1} + D\mathbb{H}Z_{n-3},\\ D\mathbb{H}T_n &= D\mathbb{H}T_{n-1} + D\mathbb{H}T_{n-3}.\\ D\mathbb{H}Z_n + 3D\mathbb{H}Z_{n-2} &= D\mathbb{H}T_n,\\ 3D\mathbb{H}Z_{n+1} - 2D\mathbb{H}Z_n &= D\mathbb{H}T_n.\end{aligned}$$

Proof. From Eq. (3.1), we have

$$\begin{aligned}D\mathbb{H}Z_{n-1} + D\mathbb{H}Z_{n-3} &= N_{n-1} + jN_n + \varepsilon(N_{n+1} + jN_{n+2}) + N_{n-3} + jN_{n-2} + \varepsilon(N_{n-1} + jN_n),\\ &= (N_{n-1} + N_{n-3}) + j(N_n + N_{n-2}) + \varepsilon((N_{n+1} + N_{n-1}) + j(N_{n+2} + N_n)),\\ &= N_n + jN_{n+1} + \varepsilon(N_{n+2} + jN_{n+3}),\\ D\mathbb{H}Z_{n-1} + D\mathbb{H}Z_{n-3} &= D\mathbb{H}Z_n.\end{aligned}$$

Similarly from Eq. (3.2), we obtained that

$$D\mathbb{H}T_n = D\mathbb{H}T_{n-1} + D\mathbb{H}T_{n-3}.$$

From Eq. (3.1), we have

$$\begin{aligned}D\mathbb{H}Z_n + 3D\mathbb{H}Z_{n-2} =&N_n + jN_{n+1} + \varepsilon N_{n+2} + j\varepsilon N_{n+3} + 3(N_{n-2} + jN_{n-1} + \varepsilon N_n + j\varepsilon N_{n+1}),\\ =&(N_n + 3N_{n-2}) + j(N_{n+1} + 3N_{n-1}) + \varepsilon((N_{n+2} + 3N_n) + j(N_{n+3} + 3N_{n+1})),\end{aligned}$$

by using the identity of Narayana sequence $U_n = N_n + 3N_{n-2}$ (see [25]) in the above equation, we obtain

$$\begin{aligned}D\mathbb{H}Z_n + 3D\mathbb{H}Z_{n-2} &= U_n + jU_{n+1} + \varepsilon U_{n+2} + j\varepsilon U_{n+3},\\ D\mathbb{H}Z_n + 3D\mathbb{H}Z_{n-2} &= D\mathbb{H}T_n,\ n \geq 2.\end{aligned}$$

In a similar manner, we can see that

$$\begin{aligned}3D\mathbb{H}Z_{n+1} - 2D\mathbb{H}Z_n =&3(N_{n+1} + jN_{n+2} + \varepsilon N_{n+3} + j\varepsilon N_{n+4}) - 2(N_n + jN_{n+1} + \varepsilon N_{n+2} + j\varepsilon N_{n+3}),\\ =&(3N_{n+1} - 2N_n) + j(3N_{n+2} - 2N_{n+1}) + \varepsilon(3N_{n+3} - 2N_{n+2}) + j\varepsilon(3N_{n+4} - 2N_{n+3}),\end{aligned}$$

and by utilising the identity $U_n = 3N_{n+1} - 2N_n$ (see [25]) in the above equation, this equation reduces to the following form

$$\begin{aligned}3D\mathbb{H}Z_{n+1} - 2D\mathbb{H}Z_n &= U_n + jU_{n+1} + \varepsilon U_{n+2} + j\varepsilon U_{n+3},\\ 3D\mathbb{H}Z_{n+1} - 2D\mathbb{H}Z_n &= D\mathbb{H}T_n.\end{aligned}$$

□

Theorem 3.5. *Let $D\mathbb{H}Z_n$ and $\overline{D\mathbb{H}Z_n}$ be dual hyperbolic Narayana quaternions and conjugate of $D\mathbb{H}Z_n$ respectively, then the following relations hold:*

$$\begin{aligned}D\mathbb{H}Z_n\overline{D\mathbb{H}Z_n} =&N_n^2 + N_{n+1}^2 - N_{n+2}^2 - N_{n+3}^2,\\ D\mathbb{H}Z_n + \overline{D\mathbb{H}Z_n} =&2N_n,\\ D\mathbb{H}Z_n^2 =&2D\mathbb{H}Z_nN_n - D\mathbb{H}Z_n\overline{D\mathbb{H}Z_n}.\end{aligned}$$

Proof. From Eq. (3.1) and using conditions (3.3), we have

$$D\mathbb{H}Z_n\overline{D\mathbb{H}Z_n} =(N_n+jN_{n+1}+\varepsilon N_{n+2}+j\varepsilon N_{n+3})(N_n-jN_{n+1}-\varepsilon N_{n+2}-j\varepsilon N_{n+3}),$$
$$D\mathbb{H}Z_n\overline{D\mathbb{H}Z_n} =N_n^2+N_{n+1}^2-N_{n+2}^2-N_{n+3}^2.$$

In a similar manner, we can see that

$$D\mathbb{H}Z_n+\overline{D\mathbb{H}Z_n} = (N_n+jN_{n+1}+\varepsilon N_{n+2}+j\varepsilon N_{n+3})+(N_n-jN_{n+1}-\varepsilon N_{n+2}-j\varepsilon N_{n+3}),$$
$$D\mathbb{H}Z_n+\overline{D\mathbb{H}Z_n} = 2N_n. \tag{3.4}$$

From Eq. (3.4), it follows that

$$D\mathbb{H}Z_n^2 = D\mathbb{H}Z_nD\mathbb{H}Z_n = D\mathbb{H}Z_n(2N_n-\overline{D\mathbb{H}Z_n}) = 2D\mathbb{H}Z_nN_n - D\mathbb{H}Z_n\overline{D\mathbb{H}Z_n}.$$

□

Theorem 3.6. *Let $D\mathbb{H}Z_n$ and $D\mathbb{H}T_n$ be the dual hyperbolic Narayana quaternions and dual hyperbolic Narayana-Lucas quaternions, respectively. Then the Binet formulas can be written as:*

$$D\mathbb{H}Z_n =\frac{\mu^a\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)}+\frac{\nu^a\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)}+\frac{\lambda^a\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)},$$
$$D\mathbb{H}T_n =\mu^a\mu^n+\nu^a\nu^n+\lambda^a\lambda^n,$$

where $\mu^a=(1+j\mu+\varepsilon\mu^2+j\cdot\varepsilon\mu^3), \nu^a=(1+j\nu+\varepsilon\nu^2+j\cdot\varepsilon\nu^3), \lambda^a=(1+j\lambda+\varepsilon\lambda^2+j\cdot\varepsilon\lambda^3)$.

Proof. Let $D\mathbb{H}Z_n$ be the dual hyperbolic Narayana quaternions. From Eq. (3.1), we have

$$D\mathbb{H}Z_n = N_n+jN_{n+1}+\varepsilon N_{n+2}+j.\varepsilon N_{n+3}.$$

Therefore, by using the Binet formulas for the Narayana sequence (see [25]) in the above relation we have,

$$\begin{aligned}D\mathbb{H}Z_n =&\left(\frac{\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)}+\frac{\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)}+\frac{\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}\right)+j\left(\frac{\mu^{n+2}}{(\mu-\nu)(\mu-\lambda)}+\frac{\nu^{n+2}}{(\nu-\mu)(\nu-\lambda)}+\frac{\lambda^{n+2}}{(\lambda-\mu)(\lambda-\nu)}\right)\\&+\varepsilon\left(\frac{\mu^{n+3}}{(\mu-\nu)(\mu-\lambda)}+\frac{\nu^{n+3}}{(\nu-\mu)(\nu-\lambda)}+\frac{\lambda^{n+3}}{(\lambda-\mu)(\lambda-\nu)}\right)+j.\varepsilon\left(\frac{\mu^{n+4}}{(\mu-\nu)(\mu-\lambda)}+\frac{\nu^{n+4}}{(\nu-\mu)(\nu-\lambda)}+\frac{\lambda^{n+4}}{(\lambda-\mu)(\lambda-\nu)}\right),\\=&\frac{\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)}(1+j\mu+\varepsilon\mu^2+j.\varepsilon\mu^3)+\frac{\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)}(1+j\nu+\varepsilon\nu^2+j.\varepsilon\nu^3)+\frac{\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}(1+j\lambda+\varepsilon\lambda^2+j.\varepsilon\lambda^3),\end{aligned}$$

$$D\mathbb{H}Z_n = \frac{\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)}\mu^a+\frac{\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)}\nu^a+\frac{\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}\lambda^a. \tag{3.5}$$

Moreover, from Eq. (3.2) we have

$$D\mathbb{H}T_n = U_n+jU_{n+1}+\varepsilon U_{n+2}+j.\varepsilon U_{n+3}.$$

Again, by using the Binet formulas for the Narayana-Lucas sequence (see [25]) in the above relation we have,

$$\begin{aligned}D\mathbb{H}T_n =&\mu^n+\nu^n+\lambda^n+j(\mu^{n+1}+\nu^{n+1}+\lambda^{n+1})+\varepsilon(\mu^{n+2}+\nu^{n+2}+\lambda^{n+2})\\&+j.\varepsilon(\mu^{n+3}+\nu^{n+3}+\lambda^{n+3}),\\=&\mu^n(1+j\mu+\varepsilon\mu^2+j.\varepsilon\mu^3)+\nu^n(1+j\nu+\varepsilon\nu^2+j.\varepsilon\nu^3)+\lambda^n(1+j\lambda+\varepsilon\lambda^2+j.\varepsilon\lambda^3),\end{aligned}$$
$$D\mathbb{H}T_n =\mu^a\mu^n+\nu^a\nu^n+\lambda^a\lambda^n. \tag{3.6}$$

□

Theorem 3.7. *Let $D\mathbb{H}Z_n$ and $D\mathbb{H}T_n$ be the dual hyperbolic Narayana quaternions and dual hyperbolic Narayana-Lucas quaternions, respectively. Then the generating functions can be written as:*

$$G(t)=\sum_{n=0}^{\infty}D\mathbb{H}Z_n(r)t^n=\frac{D\mathbb{H}Z_0(r)+t(D\mathbb{H}Z_1(r)-D\mathbb{H}Z_0(r))+t^2(D\mathbb{H}Z_2(r)-D\mathbb{H}Z_1(r))}{1-rt-t^3},$$
$$G(t)=\sum_{n=0}^{\infty}D\mathbb{H}T_n(r)t^n=\frac{D\mathbb{H}T_0(r)+t(D\mathbb{H}T_1(r)-D\mathbb{H}T_0(r))+t^2(D\mathbb{H}T_2(r)-D\mathbb{H}T_1(r))}{1-rt-t^3}.$$

Proof. Let us consider the following formal power series to be the generating function for the dual hyperbolic Narayana quaternions as:

$$G(t)=\sum_{n=0}^{\infty}D\mathbb{H}Z_n(r)t^n=D\mathbb{H}Z_0(r)+D\mathbb{H}Z_1(r)t+D\mathbb{H}Z_2(r)t^2+\dots.$$

Then we have

$$\begin{aligned}rtG(t)&=rD\mathbb{H}Z_0(r)t+rD\mathbb{H}Z_1(r)t^2+rD\mathbb{H}Z_2(r)t^3+\dots,\\ t^3G(t)&=D\mathbb{H}Z_0(r)t^3+D\mathbb{H}Z_1(r)t^4+D\mathbb{H}Z_2(r)t^5+\dots.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}G(t)-rtG(t)-t^3G(t)=&(D\mathbb{H}Z_0(r)+D\mathbb{H}Z_1(r)t+D\mathbb{H}Z_2(r)t^2+\dots)-(rD\mathbb{H}Z_0(r)t+rD\mathbb{H}Z_1(r)t^2+rZ_2(r)t^3+\dots)\\ &-(D\mathbb{H}Z_0(r)t^3+D\mathbb{H}Z_1(r)t^4+D\mathbb{H}Z_2(r)t^5+\dots).\end{aligned}$$

From the above relation, we further deduce that

$$G(t)(1-rt-t^3)=D\mathbb{H}Z_0(r)+(D\mathbb{H}Z_1(r)-rD\mathbb{H}Z_0(r))t+(D\mathbb{H}Z_2(r)-rD\mathbb{H}Z_1(r))t^2,$$

$$\begin{aligned}G(t)&=\frac{D\mathbb{H}Z_0(r)+t(D\mathbb{H}Z_1(r)-rD\mathbb{H}Z_0(r))+t^2(D\mathbb{H}Z_2(r)-rD\mathbb{H}Z_1(r))}{(1-rt-t^3)},\\ G(t)&=\sum_{n=0}^{\infty}D\mathbb{H}Z_n(r)t^n=\frac{D\mathbb{H}Z_0(r)+t(D\mathbb{H}Z_1(r)-rD\mathbb{H}Z_0(r))+t^2(D\mathbb{H}Z_2(r)-rD\mathbb{H}Z_1(r))}{(1-rt-t^3)}.\end{aligned}$$

Thus, the proof is completed for the dual hyperbolic Narayana quaternions.
In a similar manner, we can also prove the following generating function for dual hyperbolic Narayana-Lucas quaternions as:

$$G(t)=\sum_{n=0}^{\infty}D\mathbb{H}T_n(r)t^n=\frac{D\mathbb{H}T_0(r)+t(D\mathbb{H}T_1(r)-D\mathbb{H}T_0(r))+t^2(D\mathbb{H}T_2(r)-D\mathbb{H}T_1(r))}{1-rt-t^3}.$$

□

Theorem 3.8. *Let $D\mathbb{H}Z_n$ and $D\mathbb{H}T_n$ be the dual hyperbolic Narayana quaternions and dual hyperbolic Narayana-Lucas quaternions, respectively. Then the exponential generating functions can be written as:*

$$\begin{aligned}\sum_{n=0}^{\infty}D\mathbb{H}Z_n(r)\frac{t^n}{n!}&=\frac{\mu^a\mu}{(\mu-\nu)(\mu-\lambda)}e^{\mu t}+\frac{\nu^a\nu}{(\nu-\mu)(\nu-\lambda)}e^{\nu t}+\frac{\lambda^a\lambda}{(\lambda-\mu)(\lambda-\nu)}e^{\lambda t},\\ \sum_{n=0}^{\infty}D\mathbb{H}T_n(r)\frac{t^n}{n!}&=\mu^a\mu e^{\mu t}+\nu^a\nu e^{\nu t}+\lambda^a\lambda e^{\lambda t},\end{aligned}$$

where $\mu^a=(1+j\mu+\varepsilon\mu^2+j\cdot\varepsilon\mu^3)$, $\nu^a=(1+j\nu+\varepsilon\nu^2+j\cdot\varepsilon\nu^3)$, $\lambda^a=(1+j\lambda+\varepsilon\lambda^2+j\cdot\varepsilon\lambda^3)$.

Proof. By using the definition of exponential generating functions and Binet formulas for the dual hyperbolic Narayana quaternions (see (3.5)), we obtained as:

$$\begin{aligned}\sum_{n=0}^{\infty}D\mathbb{H}Z_n(r)\frac{t^n}{n!}&=\sum_{n=0}^{\infty}\left[\frac{\mu^a\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)}+\frac{\nu^a\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)}+\frac{\lambda^a\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}\right]\frac{t^n}{n!},\\ &=\frac{\mu^a\mu}{(\mu-\nu)(\mu-\lambda)}\sum_{n=0}^{\infty}\frac{(\mu t)^n}{n!}+\frac{\nu^a\nu}{(\nu-\mu)(\nu-\lambda)}\sum_{n=0}^{\infty}\frac{(\nu t)^n}{n!}+\frac{\lambda^a\lambda}{(\lambda-\mu)(\lambda-\nu)}\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!},\end{aligned}$$

$$\sum_{n=0}^{\infty}D\mathbb{H}Z_n(r)\frac{t^n}{n!}=\frac{\mu^a\mu}{(\mu-\nu)(\mu-\lambda)}e^{\mu t}+\frac{\nu^a\nu^n}{(\nu-\mu)(\nu-\lambda)}e^{\nu t}+\frac{\lambda^a\lambda^n}{(\lambda-\mu)(\lambda-\nu)}e^{\lambda t}.$$

Thus, the proof is completed for the exponential generating function for the dual hyperbolic Narayana quaternions.
In a similar manner, we can also prove the following exponential generating function for dual hyperbolic Narayana-Lucas quaternions (see (3.6)), we obtained as:

$$\sum_{n=0}^{\infty}D\mathbb{H}T_n(r)\frac{t^n}{n!}=\mu^a\mu e^{\mu t}+\nu^a\nu e^{\nu t}+\lambda^a\lambda e^{\lambda t}.$$

□

4. Dual Hyperbolic Narayana and Narayana-Lucas Hybrid Quaternions

In this section, we extend the dual hyperbolic Narayana and dual hyperbolic Narayana-Lucas quaternion structures (as discussed in section 3) in the context of hybrid quaternions. Additionally, we have presented certain theorems along with their proofs, and identities associated with these newly introduced hybrid quaternions.

Definition 4.1. *Let the n^{th} dual hyperbolic Narayana hybrid quaternions be $D\mathbb{H}Z_n^H$ and can be defined as:*

$$\begin{aligned}D\mathbb{H}Z_n^H =&N_n^H + jN_{n+1}^H + \varepsilon N_{n+2}^H + j.\varepsilon N_{n+3}^H,\\ =&(N_n + \iota N_{n+1} + \varepsilon N_{n+2} + hN_{n+3}) + j(N_{n+1} + \iota N_{n+2} + \varepsilon N_{n+3} + hN_{n+4}) + \varepsilon(N_{n+2} + \iota N_{n+3} + \varepsilon N_{n+4} + hN_{n+5})\\ &+ j.\varepsilon(N_{n+3} + \iota N_{n+4} + \varepsilon N_{n+5} + hN_{n+6}),\end{aligned}$$

where $\{\iota, \varepsilon, h\}$ are operators and $\{j, \varepsilon, j.\varepsilon\}$ are dual hyperbolic quaternionic units.

$$D\mathbb{H}Z_n^H = D\mathbb{H}Z_n + \iota D\mathbb{H}Z_{n+1} + \varepsilon D\mathbb{H}Z_{n+2} + hD\mathbb{H}Z_{n+3}. \tag{4.1}$$

Definition 4.2. *Let the n^{th} dual hyperbolic Narayana-Lucas hybrid quaternions be $D\mathbb{H}T_n^H$ and can be defined as:*

$$\begin{aligned}D\mathbb{H}T_n^H =&U_n^H + jU_{n+1}^H + \varepsilon U_{n+2}^H + j.\varepsilon U_{n+3}^H,\\ =&(U_n + \iota U_{n+1} + \varepsilon U_{n+2} + hU_{n+3}) + j(U_{n+1} + \iota U_{n+2} + \varepsilon U_{n+3} + hU_{n+4}) + \varepsilon(U_{n+2} + \iota U_{n+3} + \varepsilon U_{n+4} + hU_{n+5})\\ &+ j.\varepsilon(U_{n+3} + \iota U_{n+4} + \varepsilon U_{n+5} + hU_{n+6}),\end{aligned}$$

where $\{\iota, \varepsilon, h\}$ are operators and $\{j, \varepsilon, j.\varepsilon\}$ are dual hyperbolic quaternionic units.

$$D\mathbb{H}T_n^H = D\mathbb{H}T_n + \iota D\mathbb{H}T_{n+1} + \varepsilon D\mathbb{H}T_{n+2} + hD\mathbb{H}T_{n+3}. \tag{4.2}$$

Lemma 4.3. *Let $D\mathbb{H}Z_n^H$ and $D\mathbb{H}T_n^H$ be the dual hyperbolic Narayana hybrid quaternions and dual hyperbolic Narayana-Lucas hybrid quaternions, respectively. Then the recurrence relation for these quaternions holds:*

$$\begin{aligned}D\mathbb{H}Z_n^H &= D\mathbb{H}Z_{n-1}^H + D\mathbb{H}Z_{n-3}^H,\\ D\mathbb{H}T_n^H &= D\mathbb{H}T_{n-1}^H + D\mathbb{H}T_{n-3}^H.\end{aligned}$$

Proof. From Eq. (4.1), we have

$$\begin{aligned}D\mathbb{H}Z_{n-1}^H + D\mathbb{H}Z_{n-3}^H &= D\mathbb{H}Z_{n-1} + \iota D\mathbb{H}Z_n + \varepsilon D\mathbb{H}Z_{n+1} + hD\mathbb{H}Z_{n+2} + (D\mathbb{H}Z_{n-3} + \iota D\mathbb{H}Z_{n-2} + \varepsilon D\mathbb{H}Z_{n-1} + hD\mathbb{H}Z_n),\\ &= (D\mathbb{H}Z_{n-1} + D\mathbb{H}Z_{n-3}) + \iota(D\mathbb{H}Z_n + D\mathbb{H}Z_{n-2}) + \varepsilon(D\mathbb{H}Z_{n+1} + D\mathbb{H}Z_{n-1}) + h(D\mathbb{H}Z_{n+2} + D\mathbb{H}Z_n),\\ &= D\mathbb{H}Z_n + \iota D\mathbb{H}Z_{n+1} + \varepsilon D\mathbb{H}Z_{n+2} + hD\mathbb{H}Z_{n+3},\\ D\mathbb{H}Z_{n-1}^H + D\mathbb{H}Z_{n-3}^H &= D\mathbb{H}Z_n^H.\end{aligned}$$

Similarly, from Eq. (4.2), we obtain

$$D\mathbb{H}T_n^H = D\mathbb{H}T_{n-1}^H + D\mathbb{H}T_{n-3}^H.$$

□

Theorem 4.4. *Let $D\mathbb{H}Z_n^H$ and $D\mathbb{H}T_n^H$ be the dual hyperbolic Narayana hybrid quaternions and dual hyperbolic Narayana-Lucas hybrid quaternions, respectively. Then the Binet formulas for these quaternions can be written as:*

$$\begin{aligned}D\mathbb{H}Z_n^H =&\frac{\mu^a\mu^b\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)} + \frac{\nu^a\nu^b\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)} + \frac{\lambda^a\lambda^b\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)},\\ D\mathbb{H}T_n^H =&\mu^a\mu^b\mu^n + \nu^a\nu^b\nu^n + \lambda^a\lambda^b\lambda^n,\end{aligned}$$

where $\mu^a = (1 + j\mu + \varepsilon\mu^2 + j\cdot\varepsilon\mu^3), \nu^a = (1 + j\nu + \varepsilon\nu^2 + j\cdot\varepsilon\nu^3), \lambda^a = (1 + j\lambda + \varepsilon\lambda^2 + j\cdot\varepsilon\lambda^3), \mu^b = (1 + \iota\mu + \varepsilon\mu^2 + h\mu^3), \nu^b = (1 + \iota\nu + \varepsilon\nu^2 + h\nu^3), \lambda^b = (1 + \iota\lambda + \varepsilon\lambda^2 + h\lambda^3)$.

Proof. Let $D\mathbb{H}Z_n^H$ be the dual hyperbolic Narayana hybrid quaternions. From Eq. (4.1), we have

$$D\mathbb{H}Z_n^H = D\mathbb{H}Z_n + \iota D\mathbb{H}Z_{n+1} + \varepsilon D\mathbb{H}Z_{n+2} + hD\mathbb{H}Z_{n+3}.$$

Therefore, utilising Eq. (3.5) in above relation for each term, we have

$$\begin{aligned}D\mathbb{H}Z_n^H =&\left(\frac{\mu^a\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)} + \frac{\nu^a\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)} + \frac{\lambda^a\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}\right) + \iota\left(\frac{\mu^a\mu^{n+2}}{(\mu-\nu)(\mu-\lambda)} + \frac{\nu^a\nu^{n+2}}{(\nu-\mu)(\nu-\lambda)} + \frac{\lambda^a\lambda^{n+2}}{(\lambda-\mu)(\lambda-\nu)}\right)\\ &+\varepsilon\left(\frac{\mu^a\mu^{n+3}}{(\mu-\nu)(\mu-\lambda)} + \frac{\nu^a\nu^{n+3}}{(\nu-\mu)(\nu-\lambda)} + \frac{\lambda^a\lambda^{n+3}}{(\lambda-\mu)(\lambda-\nu)}\right) + h\left(\frac{\mu^a\mu^{n+4}}{(\mu-\nu)(\mu-\lambda)} + \frac{\nu^a\nu^{n+4}}{(\nu-\mu)(\nu-\lambda)} + \frac{\lambda^a\lambda^{n+4}}{(\lambda-\mu)(\lambda-\nu)}\right),\\ =&\frac{\mu^a\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)}(1 + \iota\mu + \varepsilon\mu^2 + h\mu^3) + \frac{\nu^a\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)}(1 + \iota\nu + \varepsilon\nu^2 + h\nu^3)\\ &+ \frac{\lambda^a\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}(1 + \iota\lambda + \varepsilon\lambda^2 + h\lambda^3),\end{aligned}$$

$$D\mathbb{H}Z_n^H = \frac{\mu^a\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)}\mu^b + \frac{\nu^a\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)}\nu^b + \frac{\lambda^a\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}\lambda^b,$$

where $\mu^b = (1+\iota\mu+\varepsilon\mu^2+h\mu^3), \nu^b = (1+\iota\nu+\varepsilon\nu^2+h\nu^3), \lambda^b = (1+\iota\lambda+\varepsilon\lambda^2+h\lambda^3)$.
Moreover, from Eq. (4.2), we have

$$D\mathbb{H}T_n^H = D\mathbb{H}T_n + \iota D\mathbb{H}T_{n+1} + \varepsilon D\mathbb{H}T_{n+2} + hD\mathbb{H}T_{n+3}.$$

Therefore, utilising Eq. (3.6) in the above relation for each term, we have

$$\begin{aligned} D\mathbb{H}T_n^H =&(\mu^a\mu^n+\nu^a\nu^n+\lambda^a\lambda^n)+\iota(\mu^a\mu^{n+1}+\nu^a\nu^{n+1}+\lambda^a\lambda^{n+1})+\varepsilon(\mu^a\mu^{n+2}+\nu^{n+2}+\lambda^a\lambda^{n+2}) \\ &+h(\mu^a\mu^{n+3}+\nu^a\nu^{n+3}+\lambda^{n+3}), \\ =&\mu^a\mu^n(1+\iota\mu+\varepsilon\mu^2+h\mu^3)+\nu^a\nu^n(1+\iota\nu+\varepsilon\nu^2+h\nu^3)+\lambda^a\lambda^n(1+\iota\lambda+\varepsilon\lambda^2+h\lambda^3), \\ D\mathbb{H}T_n^H =&\mu^a\mu^n\mu^b+\nu^a\nu^n\nu^b+\lambda^a\lambda^n\lambda^b. \end{aligned}$$

□

Theorem 4.5. *Let $D\mathbb{H}Z_n^H$ and $D\mathbb{H}T_n^H$ be the dual hyperbolic Narayana hybrid quaternions and dual hyperbolic Narayana-Lucas hybrid quaternions, respectively. Then the generating functions for these quaternions can be written as:*

$$G(t) = \sum_{n=0}^{\infty} D\mathbb{H}Z_n^H(r)t^n = \frac{D\mathbb{H}Z_0^H(r)+t(D\mathbb{H}Z_1^H(r)-D\mathbb{H}Z_0^H(r))+t^2(D\mathbb{H}Z_2^H(r)-D\mathbb{H}Z_1^H(r))}{(1-rt-t^3)},$$

$$G(t) = \sum_{n=0}^{\infty} D\mathbb{H}T_n^H(r)t^n = \frac{D\mathbb{H}T_0^H(r)+t(D\mathbb{H}T_1^H(r)-D\mathbb{H}T_0^H(r))+t^2(D\mathbb{H}T_2^H(r)-D\mathbb{H}T_1^H(r))}{(1-rt-t^3)}.$$

Proof. Let us consider the following formal power series to be the generating function for the hyperbolic Narayana hybrid quaternions as:

$$G(t) = \sum_{n=0}^{\infty} D\mathbb{H}Z_n^H(r)t^n = D\mathbb{H}Z_0^H(r)+D\mathbb{H}Z_1^H(r)t+D\mathbb{H}Z_2^H(r)t^2+\dots.$$

Then, we have

$$rtG(t) = rD\mathbb{H}Z_0^H(r)t+rD\mathbb{H}Z_1^H(r)t^2+rD\mathbb{H}Z_2^H(r)t^3+\dots,$$
$$t^3G(t) = D\mathbb{H}Z_0^H(r)t^3+D\mathbb{H}Z_1^H(r)t^4+D\mathbb{H}Z_2^H(r)t^5+\dots.$$

Thus, we obtain

$$\begin{aligned} G(t)-rtG(t)-t^3G(t) =&(D\mathbb{H}Z_0^H(r)+D\mathbb{H}Z_1^H(r)t+D\mathbb{H}Z_2^H(r)t^2+\dots)-(rD\mathbb{H}Z_0^H(r)t+rD\mathbb{H}Z_1^H(r)t^2+rD\mathbb{H}Z_2^H(r)t^3+\dots) \\ &-(D\mathbb{H}Z_0^H(r)t^3+D\mathbb{H}Z_1^H(r)t^4+D\mathbb{H}Z_2^H(r)t^5+\dots). \end{aligned}$$

From the above relation we further deduce that

$$G(t)(1-rt-t^3) = D\mathbb{H}Z_0^H(r)+t(D\mathbb{H}Z_1^H(r)-rD\mathbb{H}Z_0^H(r))+t^2(D\mathbb{H}Z_2^H(r)-rD\mathbb{H}Z_1^H(r)).$$

Therefore, we have

$$G(t) = \frac{D\mathbb{H}Z_0^H(r)+t(D\mathbb{H}Z_1^H(r)-rD\mathbb{H}Z_0^H(r))+t^2(D\mathbb{H}Z_2^H(r)-rD\mathbb{H}Z_1^H(r))}{(1-rt-t^3)},$$

$$\begin{aligned} G(t) =& \sum_{n=0}^{\infty} D\mathbb{H}Z_n^H(r)t^n \\ =&\frac{D\mathbb{H}Z_0^H(r)+t(D\mathbb{H}Z_1^H(r)-rD\mathbb{H}Z_0^H(r))+t^2(D\mathbb{H}Z_2^H(r)-rD\mathbb{H}Z_1^H(r))}{(1-rt-t^3)}. \end{aligned}$$

Thus, the proof is completed for the generating functions for dual hyperbolic Narayana hybrid quaternions.
In a similar manner, we can also prove the following generating function for dual hyperbolic Narayana-Lucas hybrid quaternions as:

$$G(t) = \sum_{n=0}^{\infty} D\mathbb{H}T_n^H(r)t^n = \frac{D\mathbb{H}T_0^H(r)+t(D\mathbb{H}T_1^H(r)-D\mathbb{H}T_0^H(r))+t^2(D\mathbb{H}T_2^H(r)-D\mathbb{H}T_1^H(r))}{(1-rt-t^3)}.$$

□

Theorem 4.6. *Let $D\mathbb{H}Z_n^H$ and $D\mathbb{H}T_n^H$ be the dual hyperbolic Narayana hybrid quaternions and dual hyperbolic Narayana-Lucas hybrid quaternions, respectively. Then the exponential generating functions for these quaternions can be written as:*

$$\sum_{n=0}^{\infty} D\mathbb{H}Z_n^H(r)\frac{t^n}{n!} = \frac{\mu^a\mu}{(\mu-\nu)(\mu-\lambda)}\mu^b e^{\mu t} + \frac{\nu^a\nu}{(\nu-\mu)(\nu-\lambda)}\nu^b e^{\nu t} + \frac{\lambda^a\lambda}{(\lambda-\mu)(\lambda-\nu)}\lambda^b e^{\lambda t},$$

$$\sum_{n=0}^{\infty} D\mathbb{H}T_n^H(r)\frac{t^n}{n!} = \mu^a\mu^b e^{\mu t}+\nu^a\nu^b e^{\nu t}+\lambda^a\lambda^b e^{\lambda t},$$

where $\mu^a = (1+j\mu+\varepsilon\mu^2+j\cdot\varepsilon\mu^3), \nu^a = (1+j\nu+\varepsilon\nu^2+j\cdot\varepsilon\nu^3), \lambda^a = (1+j\lambda+\varepsilon\lambda^2+j\cdot\varepsilon\lambda^3), \mu^b = (1+\iota\mu+\varepsilon\mu^2+h\mu^3), \nu^b = (1+\iota\nu+\varepsilon\nu^2+h\nu^3), \lambda^b = (1+\iota\lambda+\varepsilon\lambda^2+h\lambda^3)$.

Proof. By using the Binet formulas for the dual hyperbolic Narayana and Narayana-Lucas hybrid quaternions from Theorem 4.4, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} D\mathbb{H}Z_n^H(r)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}[\frac{\mu^a\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)}(1+\iota\mu+\varepsilon\mu^2+h\mu^3)+\frac{\nu^a\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)}(1+\iota\nu+\varepsilon\nu^2+h\nu^3)\\ &+\frac{\lambda^a\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}(1+\iota\lambda+\varepsilon\lambda^2+h\lambda^3)]\frac{t^n}{n!},\\ &=\frac{\mu^a\mu}{(\mu-\nu)(\mu-\lambda)}(1+\iota\mu+\varepsilon\mu^2+h\mu^3)\sum_{n=0}^{\infty}\frac{(\mu t)^n}{n!}+\frac{\nu^a\nu}{(\nu-\mu)(\nu-\lambda)}(1+\iota\nu+\varepsilon\nu^2+h\nu^3)\sum_{n=0}^{\infty}\frac{(\nu t)^n}{n!}\\ &+\frac{\lambda^a\lambda}{(\lambda-\mu)(\lambda-\nu)}(1+\iota\lambda+\varepsilon\lambda^2+h\lambda^3)\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!},\\ \sum_{n=0}^{\infty} D\mathbb{H}Z_n^H(r)\frac{t^n}{n!} &=\frac{\mu^a\mu}{(\mu-\nu)(\mu-\lambda)}\mu^b e^{\mu t}+\frac{\nu^a\nu}{(\nu-\mu)(\nu-\lambda)}\nu^b e^{\nu t}+\frac{\lambda^a\lambda}{(\lambda-\mu)(\lambda-\nu)}\lambda^b e^{\lambda t}.\end{aligned}$$

Thus, the proof is completed for the exponential generating function for the dual hyperbolic Narayana hybrid quaternions.
In a similar manner, we can also prove the exponential generating function for dual hyperbolic Narayana-Lucas hybrid quaternions, we have

$$\sum_{n=0}^{\infty} D\mathbb{H}T_n^H(r)\frac{t^n}{n!}=\mu^a\mu^b e^{\mu t}+\nu^a\nu^b e^{\nu t}+\lambda^a\lambda^b e^{\lambda t}.$$

□

Theorem 4.7. *Let m and n be any positive integers and $m\geq n$, then the following relations hold as:*

$$\begin{aligned}&D\mathbb{H}Z_m^H D\mathbb{H}T_n^H+D\mathbb{H}T_m^H D\mathbb{H}Z_n^H\\ &=\frac{\left(\lambda^a\lambda^b\lambda^n+\mu^a\mu^b\mu^n+\lambda^a\lambda^b\lambda^n\right)\left(\lambda^a\lambda^b\lambda^{1+m}(\mu-\nu)+(\lambda-\mu)\nu^{1+m}\nu^a\nu^b+(-\lambda+\nu)(\mu^{1+m}\mu^a\mu^b)\right)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}\\ &+\frac{\left(\lambda^a\lambda^b\lambda^m+\mu^a\mu^b\mu^m+\nu^a\nu^b\nu^m\right)\left(\lambda^a\lambda^b\lambda^{1+n}(\mu-\nu)+(\lambda-\mu)\nu^{1+n}\nu^a\nu^b+\mu^{1+n}\mu^a\mu^b(-\lambda+\nu)\right)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)},\\ &D\mathbb{H}Z_m^H D\mathbb{H}T_n^H-D\mathbb{H}T_m^H D\mathbb{H}Z_n^H\\ &=\frac{\left(\lambda^a\lambda^b\lambda^n+\mu^a\mu^b\mu^n+\nu^a\nu^b\nu^n\right)\left(\lambda^a\lambda^b\lambda^{1+m}(\mu-\nu)+(\lambda-\mu)\nu^{1+m}\nu^a\nu^b+(-\lambda+\nu)(\mu^{1+m}\mu^a\mu^b)\right)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}\\ &-\frac{\left(\lambda^a\lambda^b\lambda^m+\mu^a\mu^b\mu^m+\nu^a\nu^b\nu^m\right)\left(\lambda^a\lambda^b\lambda^{1+n}(\mu-\nu)+(\lambda-\mu)\nu^{1+n}\nu^a\nu^b+\mu^{1+n}\mu^a\mu^b(-\lambda+\nu)\right)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}.\end{aligned}$$

Proof. By using the Binet formulas from Theorem 4.4, we obtain

$$\begin{aligned}&D\mathbb{H}Z_m^H D\mathbb{H}T_n^H+D\mathbb{H}T_m^H D\mathbb{H}Z_n^H\\ &=\left(\frac{\mu^a\mu^b\mu^{m+1}}{(\mu-\nu)(\mu-\lambda)}+\frac{\nu^a\nu^b\nu^{m+1}}{(\nu-\mu)(\nu-\lambda)}+\frac{\lambda^a\lambda^b\lambda^{m+1}}{(\lambda-\mu)(\lambda-\nu)}\right)\left(\mu^a\mu^b\mu^n+\nu^a\nu^b\nu^n+\lambda^a\lambda^b\lambda^n\right)\\ &+\left(\mu^a\mu^b\mu^m+\nu^a\nu^b\nu^m+\lambda^a\lambda^b\lambda^m\right)\left(\frac{\mu^a\mu^b\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)}+\frac{\nu^a\nu^b\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)}+\frac{\lambda^a\lambda^b\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}\right),\\ &D\mathbb{H}Z_m^H D\mathbb{H}T_n^H+D\mathbb{H}T_m^H D\mathbb{H}Z_n^H\\ &=\frac{\left(\lambda^a\lambda^b\lambda^n+\mu^a\mu^b\mu^n+\lambda^a\lambda^b\lambda^n\right)\left(\lambda^a\lambda^b\lambda^{1+m}(\mu-\nu)+(\lambda-\mu)\nu^{1+m}\nu^a\nu^b+(-\lambda+\nu)(\mu^{1+m}\mu^a\mu^b)\right)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}\\ &+\frac{\left(\lambda^a\lambda^b\lambda^m+\mu^a\mu^b\mu^m+\nu^a\nu^b\nu^m\right)\left(\lambda^a\lambda^b\lambda^{1+n}(\mu-\nu)+(\lambda-\mu)\nu^{1+n}\nu^a\nu^b+\mu^{1+n}\mu^a\mu^b(-\lambda+\nu)\right)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&D\mathbb{H}Z_m^H D\mathbb{H}T_n^H-D\mathbb{H}T_m^H D\mathbb{H}Z_n^H\\ &=\left(\frac{\mu^a\mu^b\mu^{m+1}}{(\mu-\nu)(\mu-\lambda)}+\frac{\nu^a\nu^b\nu^{m+1}}{(\nu-\mu)(\nu-\lambda)}+\frac{\lambda^a\lambda^b\lambda^{+1}}{(\lambda-\mu)(\lambda-\nu)}\right)\left(\mu^a\mu^b\mu^n+\nu^a\nu^b\nu^n+\lambda^a\lambda^b\lambda^n\right)\\ &-\left(\mu^a\mu^b\mu^m+\nu^a\nu^b\nu^m+\lambda^a\lambda^b\lambda^m\right)\left(\frac{\mu^a\mu^b\mu^{n+1}}{(\mu-\nu)(\mu-\lambda)}+\frac{\nu^a\nu^b\nu^{n+1}}{(\nu-\mu)(\nu-\lambda)}+\frac{\lambda^a\lambda^b\lambda^{n+1}}{(\lambda-\mu)(\lambda-\nu)}\right),\end{aligned}$$

$$\begin{aligned}&D\mathbb{H}Z_m^H D\mathbb{H}T_n^H-D\mathbb{H}T_m^H D\mathbb{H}Z_n^H\\ &=\frac{\left(\lambda^a\lambda^b\lambda^n+\mu^a\mu^b\mu^n+\nu^a\nu^b\nu^n\right)\left(\lambda^a\lambda^b\lambda^{1+m}(\mu-\nu)+(\lambda-\mu)\nu^{1+m}\nu^a\nu^b+(-\lambda+\nu)(\mu^{1+m}\mu^a\mu^b)\right)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}\\ &-\frac{\left(\lambda^a\lambda^b\lambda^m+\mu^a\mu^b\mu^m+\nu^a\nu^b\nu^m\right)\left(\lambda^a\lambda^b\lambda^{1+n}(\mu-\nu)+(\lambda-\mu)\nu^{1+n}\nu^a\nu^b+\mu^{1+n}\mu^a\mu^b(-\lambda+\nu)\right)}{(\lambda-\mu)(\lambda-\nu)(\mu-\nu)}.\end{aligned}$$

□

Theorem 4.8. *(Catalan's identities) Let $D\mathbb{H}Z_n^H$ and $D\mathbb{H}T_n^H$ be the dual hyperbolic Narayana hybrid quaternions and dual hyperbolic Narayana-Lucas hybrid quaternions, respectively. Therefore, for $n \geq 1$, we have*

$$
\begin{aligned}
&D\mathbb{H}Z_{n+r}^H D\mathbb{H}Z_{n-r}^H - (D\mathbb{H}Z_n^H)^2 \\
=&\lambda^{-r}\mu^{-r}\nu^{-r}\frac{\left(-2\lambda^{1+n+r}\lambda^a\lambda^b\mu^{1+r}(\mu-\nu)\nu^{1+r}\left(\mu^{a+b}\mu^n-\nu^{a+b}\nu^n\right)+2\lambda^{2+r+n}\lambda^a\lambda^b\mu^r(\mu-\nu)\nu^r(\mu^{1+n}\mu^a\mu^b-\nu^{1+n}\nu^a\nu^b)\right)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2} \\
&+\frac{\left(-\lambda^{2+r}\mu^{1+n}\mu^a\mu^b\nu^{1+n}\nu^a\nu^b(\mu^r-\nu^r)^2-\lambda^r\mu^{2+n}\mu^a\mu^b\nu^{2+n}\nu^a\nu^b(\mu^r-\nu^r)^2+\lambda^{1+r}\mu^{1+n}\mu^a\mu^b\nu^{1+n}\nu^a\nu^b(\mu+\nu)(\mu^r-\nu^r)^2\right)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2} \\
&+\frac{\left(-\lambda^{1+n+2r}\lambda^a\lambda^b\mu(\mu-\nu)\nu(\mu^r\nu^{a+b+n}-\mu^{a+b+n}\nu^r)-\lambda^{2+n+2r}\lambda^a\lambda^b(\mu-\nu)(-\mu^r\nu^{1+n}\nu^a\nu^b+\mu^{1+n}\mu^a\mu^b\nu^r)\right)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2} \\
&+\frac{\lambda^{1+n}\lambda^a\lambda^b\mu^{1+r}(\mu-\nu)\nu^{1+r}(\mu^a\mu^b\mu^{n+r}-\nu^a\nu^b\nu^{n+r})-\lambda^{2+n}\lambda^a\lambda^b\mu^r(\mu-\nu)\nu^r(\mu^{1+n}\mu^a\mu^b\mu^r-\nu^{1+n+r}\nu^a\nu^b)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2},
\end{aligned}
$$

$$
\begin{aligned}
D\mathbb{H}T_{n+r}^H D\mathbb{H}T_{n-r}^H - (D\mathbb{H}T_n^H)^2 =& -2\lambda^a\lambda^b\lambda^n(\mu^a\mu^b\mu^n+\nu^a\nu^b\nu^n)+\lambda^a\lambda^b\lambda^{n+r}(\mu^a\mu^b\mu^{n-r}+\nu^a\nu^b\nu^{n-r})+\mu^a\mu^b\mu^{n-r} \\
&+\nu^a\nu^b\nu^{n-r}(\mu^r-\nu^r)^2+\lambda^a\lambda^b\lambda^{n-r}(\mu^a\mu^b\mu^{n+r}+\nu^a\nu^b\nu^{n+r}).
\end{aligned}
$$

Proof. By using the Binet formulas from Theorem 4.4, we obtain

$$
\begin{aligned}
&D\mathbb{H}Z_{n+r}^H D\mathbb{H}Z_{n-r}^H - (D\mathbb{H}Z_n^H)^2 \\
=&\left(\frac{\mu^a\mu^b\mu^{1+n+r}}{(\mu-\nu)(\mu-\lambda)}+\frac{\nu^a\nu^b\nu^{1+n+r}}{(\nu-\mu)(\nu-\lambda)}+\frac{\lambda^a\lambda^b\lambda^{1+n+r}}{(\lambda-\mu)(\lambda-\nu)}\right)\left(\frac{\mu^a\mu^b\mu^{1+n-r}}{(\mu-\nu)(\mu-\lambda)}+\frac{\nu^a\nu^b\nu^{1+n-r}}{(\nu-\mu)(\nu-\lambda)}+\frac{\lambda^a\lambda^b\lambda^{1+n-r}}{(\lambda-\mu)(\lambda-\nu)}\right) \\
&-\left(\frac{\mu^a\mu^b\mu^{1+n}}{(\mu-\nu)(\mu-\lambda)}+\frac{\nu^a\nu^b\nu^{1+n}}{(\nu-\mu)(\nu-\lambda)}+\frac{\lambda^a\lambda^b\lambda^{1+n}}{(\lambda-\mu)(\lambda-\nu)}\right)^2,
\end{aligned}
$$

$$
\begin{aligned}
=&\lambda^{-r}\mu^{-r}\nu^{-r}\frac{\left(-2\lambda^{1+n+r}\lambda^a\lambda^b\mu^{1+r}(\mu-\nu)\nu^{1+r}\left(\mu^{a+b}\mu^n-\nu^{a+b}\nu^n\right)+2\lambda^{2+r+n}\lambda^a\lambda^b\mu^r(\mu-\nu)\nu^r(\mu^{1+n}\mu^a\mu^b-\nu^{1+n}\nu^a\nu^b)\right)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2} \\
&+\frac{\left(-\lambda^{2+r}\mu^{1+n}\mu^a\mu^b\nu^{1+n}\nu^a\nu^b(\mu^r-\nu^r)^2-\lambda^r\mu^{2+n}\mu^a\mu^b\nu^{2+n}\nu^a\nu^b(\mu^r-\nu^r)^2+\lambda^{1+r}\mu^{1+n}\mu^a\mu^b\nu^{1+n}\nu^a\nu^b(\mu+\nu)(\mu^r-\nu^r)^2\right)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2} \\
&+\frac{\left(-\lambda^{1+n+2r}\lambda^a\lambda^b\mu(\mu-\nu)\nu(\mu^r\nu^{a+b+n}-\mu^{a+b+n}\nu^r)-\lambda^{2+n+2r}\lambda^a\lambda^b(\mu-\nu)(-\mu^r\nu^{1+n}\nu^a\nu^b+\mu^{1+n}\mu^a\mu^b\nu^r)\right)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2} \\
&+\frac{\lambda^{1+n}\lambda^a\lambda^b\mu^{1+r}(\mu-\nu)\nu^{1+r}(\mu^a\mu^b\mu^{n+r}-\nu^a\nu^b\nu^{n+r})-\lambda^{2+n}\lambda^a\lambda^b\mu^r(\mu-\nu)\nu^r(\mu^{1+n}\mu^a\mu^b\mu^r-\nu^{1+n+r}\nu^a\nu^b)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&D\mathbb{H}T_{n+r}^H D\mathbb{H}T_{n-r}^H - (D\mathbb{H}T_n^H)^2 \\
=&\left(\mu^a\mu^b\mu^{n+r}+\nu^a\nu^b\nu^{n+r}+\lambda^a\lambda^b\lambda^{n+r}\right)\left(\mu^a\mu^b\mu^{n-r}+\nu^a\nu^b\nu^{n-r}+\lambda^a\lambda^b\lambda^{n-r}\right)-\left(\mu^a\mu^b\mu^n+\nu^a\nu^b\nu^n+\lambda^a\lambda^b\lambda^n\right)^2, \\
=&-2\lambda^a\lambda^b\lambda^n\mu^a\mu^b\mu^n+\lambda^a\lambda^b\lambda^{n+r}\mu^a\mu^b\mu^{n-r}+\lambda^a\lambda^b\lambda^{n+r}\mu^a\mu^b\mu^{n+r}-2\lambda^a\lambda^b\lambda^n\nu^a\nu^b\nu^n-2\mu^a\mu^b\mu^n \\
&+\lambda^a\lambda^b\lambda^{n+r}\nu^a\nu^b\nu^{n-r}+\mu^a\mu^b\mu^{n+r}\nu^a\nu^b\nu^{n-r}+\lambda^a\lambda^b\lambda^{n-r}\nu^a\nu^b\nu^{n+r}+\mu^a\mu^b\mu^{n-r}\nu^a\nu^b\nu^{n+r}, \\
=&-2\lambda^a\lambda^b\lambda^n(\mu^a\mu^b\mu^n+\nu^a\nu^b\nu^n)+\lambda^a\lambda^b\lambda^{n+r}(\mu^a\mu^b\mu^{n-r}+\nu^a\nu^b\nu^{n-r})+\mu^a\mu^b\mu^{n-r}+\nu^a\nu^b\nu^{n-r}(\mu^r-\nu^r)^2 \\
&+\lambda^a\lambda^b\lambda^{n-r}(\mu^a\mu^b\mu^{n+r}+\nu^a\nu^b\nu^{n+r}).
\end{aligned}
$$

□

Theorem 4.9. *(Cassini's identities) Let $D\mathbb{H}Z^H$ and $D\mathbb{H}T^H$ be the dual hyperbolic Narayana and Narayana-Lucas hybrid quaternions respectively. Therefore, for $n \geq 1$, we have*

$$
\begin{aligned}
&D\mathbb{H}Z_{n+1}^H D\mathbb{H}Z_{n-1}^H - (D\mathbb{H}Z_n^H)^2 \\
=&\lambda^{-1}\mu^{-1}\nu^{-1}\frac{\left(-2\lambda^{2+n}\lambda^a\lambda^b\mu^2(\mu-\nu)\nu^2\left(\mu^{a+b}\mu^n-\nu^{a+b}\nu^n\right)+2\lambda^{3+n}\lambda^a\lambda^b\mu(\mu-\nu)\nu(\mu^{1+n}\mu^a\mu^b-\nu^{1+n}\nu^a\nu^b)\right)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2} \\
&+\frac{\left(-\lambda^3\nu^{1+n}\mu^a\mu^b\nu^{1+n}\nu^a\nu^b(\mu-\nu)^2-\lambda\mu^{2+n}\mu^a\mu^b\nu^{2+n}\nu^a\nu^b(\mu-\nu)^2+\lambda^2\mu^{1+n}\mu^a\mu^b\nu^{1+n}\nu^a\nu^b(\mu+\nu)(\mu-\nu)^2\right)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2} \\
&+\frac{\left(-\lambda^{3+n}\lambda^a\lambda^b\mu(\mu-\nu)\nu(\mu\nu^{a+b+n}-\mu^{a+b+n}\nu)-\lambda^{4+n}\lambda^a\lambda^b(\mu-\nu)(-\mu\nu^{1+n}\nu^a\nu^b+\mu^{1+n}\mu^a\mu^b\nu)\right)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2} \\
&+\frac{\lambda^{1+n}\lambda^a\lambda^b\mu^2(\mu-\nu)\nu^2(\mu^a\mu^b\mu^{n+1}-\nu^a\nu^b\nu^{n+1})-\lambda^{2+n}\lambda^a\lambda^b\mu(\mu-\nu)\nu(\mu^{1+n}\mu^a\mu^b\mu-\nu^{2+n}\nu^a\nu^b)}{(\lambda-\mu)^2(\lambda-\nu)^2(\mu-\nu)^2},
\end{aligned}
$$

$$D\mathbb{H}T^H_{n+1}D\mathbb{H}T^H_{n-1}-(D\mathbb{H}T^H_n)^2=-2\lambda^a\lambda^b\lambda^n(\mu^a\mu^b\mu^n+\nu^a\nu^b\nu^n)+\lambda^a\lambda^b\lambda^{n+1}(\mu^a\mu^b\mu^{n-1}+\nu^a\nu^b\nu^{n-1})$$
$$+\mu^a\mu^b\mu^{n-1}+\nu^a\nu^b\nu^{n-1}(\mu-\nu)^2+\lambda^a\lambda^b\lambda^{n-1}(\mu^a\mu^b\mu^{n+1}+\nu^a\nu^b\nu^{n+1}).$$

Proof. By substituting $r=1$ in Theorem 4.8, this theorem can be easily proved. □

5. Conclusion

This study explores the association of hybrid numbers within the framework of dual hyperbolic Narayana quaternions and dual hyperbolic Narayana-Lucas quaternions. We further examined their interconnections and expressed Binet formulas, generating functions, and exponential generating functions, along with an array of established identities related to these newly introduced quaternions. The novelty of this work lies in its ability to formulate hybrid wavelets that can be applied to the solution of both linear hybrid differential equations and nonlinear hybrid differential equations. Furthermore, these findings may help to analyze complex problems in graph theory. Moreover, the present work may contribute to the development of new cryptographic protocols that can provide robust security mechanisms in real-world applications.

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