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On the Order of Approximation of Unbounded Functions by the Modified Rational type Baskakov Operators

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Abstract

In the present paper, we give a generalization of rational type Baskakov operators. We obtain the order of approximation of continuous functions having polynomials growth at infinity by using the modulus of continuity and prove the theorem on weighted approximation on all positive semi-axis for these operators.

1. INTRODUCTION

Let $\varphi_n : \mathbb{C} \to \mathbb{C}$, $n = 1,2,3,..,$ be a sequence of functions having following properties:

$i)$ φ_n $(n = 1,2,3,...,)$ is analytic on domain D_n containing the disk $B_n = \{z \in \mathbb{C} : |z - b_n| \le b_n\}$ such that $\lim\limits_{n\to\infty} b_n = \infty$;

$ii)$ $\varphi_n(0) = 1$, for all $n \in \mathbb{N}$;

$iii)$ φ_n is completely monotone on $[0, b_n]$, i.e., $(-1)^k \varphi_n^{(k)}(x) \ge 0$ for $k = 0,1,2,...$;

$iv)$ there exists a positive integer $m(n)$, such that $\varphi_n^{(k)}(\alpha_n x) = -n\alpha_n \varphi_{m(n)}^{(k-1)}(\alpha_n x)(1 + \gamma_{k,n}(\alpha_n x))$, $x \in [0, b_n]$ and $\gamma_{k,n}(0)$ convergences to zero as $n \to \infty$ uniformly in k $(k = 1,2,...)$;

$v)$ $\lim\limits_{n\to\infty} \dfrac{n}{m(n)} = 1$ and $\dfrac{\varphi_n^{(k)}(\alpha_n x)}{n^k \varphi_n(\alpha_n x)} = 1 + O(\dfrac{1}{n\alpha_n})$.

Under above conditions, we will considered rational type Baskakov operators as follow:

$$L_n^{\alpha_n,\beta_n}(f;x) = \frac{1}{\varphi_n(\alpha_n x)} \sum_{k=0}^{\infty} \frac{\varphi_n^{(k)}(0)}{k!} (\alpha_n x)^k f(\frac{k}{\beta_n}),\ 0 \le x \le b_n. \tag{1}$$

Obvious that $L_n^{\alpha_n,\beta_n}$ translated continuous function with the growth condition $f(x) = O(x^2)$ at infinity. Note that, if we substitute $b_n = b$ in the conditions $(i)-(iv)$, then we obtain following the generalized rational type Baskakov operators defined by İspir and Atakut [1]

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$$L_n^{\alpha_n,\beta_n}(f;x)=\frac{1}{\varphi_n(\alpha_n x)}\sum_{k=0}^{\infty}\frac{\varphi_n^{(k)}(0)}{k!}(\alpha_n x)^k f(\frac{k}{\beta_n}),\ x\geq 0$$

and by simple calculation one have,

$$L_n^{\alpha_n,\beta_n}(1;x)=1,$$

$$L_n^{\alpha_n,\beta_n}(t;1)=\frac{\alpha_n}{\beta_n}x\frac{\varphi_n'(\alpha_n x)}{\varphi_n(\alpha_n x)},$$

$$L_n^{\alpha_n,\beta_n}(t^2;x)=\frac{\alpha_n^2}{\beta_n^2}x^2\frac{\varphi_n''(\alpha_n x)}{\varphi_n(\alpha_n x)}+\frac{\alpha_n}{\beta_n^2}x\frac{\varphi_n'(\alpha_n x)}{\varphi_n(\alpha_n x)}.$$

Baskakov type operators and their generalizations were investigated by many author (see, [1,2-7]). In [1], the authors estimated the order of approximation for the operators defined by (1) and obtained a Voronovskaja type asymptotic formula and pointwise convergence in simultaneous approximation in the case $b_n = b$. In the present paper we study convergence properties of the of the operators $L_n^{\alpha_n,\beta_n}$ in polynomial weighted spaces when the interval of convergence grows as $n\to\infty$. We will obtain an estimate for these operators on any finite interval by using the modulus of continuity of function f and will prove the theorem on weighted approximation on all positive semi-axis.

Note that, as the classical Korovkin theorem [8], the weighted Korovkin theorem(Theorem 1.) proved by A. D. Gadjiev in [9,10] plays an important role for weighted approximation in the weighted spaces and we will use special case of this theorem.

Let $\sigma_m(x)=\left(1+x^{2m}\right)^{-1}$ $(m\in \mathrm{N})$ and $B_{2m}[0,\infty)$ be the space of all functions, satisfying the inequality

$\sigma_m(x)\left|f(x)\right|\leq M_f,\quad x\geq 0$

where M_f is a constant depending on function f. We denote by $C_{2m}[0,\infty)$ the spaces of all continuous functions belonging to $B_{2m}[0,\infty)$ and denote by $C_{2m}^*[0,\infty)$ the spaces of functions belonging to $C_{2m}[0,\infty)$ such that $\lim\limits_{x\to\infty}\sigma_m(x)f(x)<\infty.$

Theorem 1.([9,10]) *Let the sequences of linear positive operators* L_n, *acting from* $C_{2m}[0,\infty)$ *to* $B_{2m}[0,\infty)$ *satisfy the conditions:*

$$\lim_{n\to\infty}\left\|L_n(t^\nu;x)-x^\nu\right\|_{C_{2m}[0,\infty)}=0,\quad \nu=0,m,2m$$

where

$$\left\|f\right\|_{C_{2m}[0,\infty)}=\sup_{x\geq 0}\sigma_m(x)\left|f(x)\right|$$

then for any function $f\in C_{2m}^*[0,\infty)$

$$\lim_{n\to\infty}\left\|L_n(f)-f\right\|_{C_{2m}[0,\infty)}=0$$

and there exists a function $f^*\in C_{2m}[0,\infty)/C_{2m}^*[0,\infty)$ *such that*

$$\lim_{n\to\infty}\left\|L_n(f^*)-f^*\right\|_{C_{2m}[0,\infty)}>1$$

holds.

As we shall see in the next section, we will obtain approximation properties for the operators (1) on whole positive semi-axis by using this theorem and the order of approximation on any finite interval will be given by the modulus of continuity denote by: for $f\in C_{2m}[0,\infty)$

$$\omega_\lambda(f;\delta)=\sup_{t,x\in[0,\lambda]}\left\{|f(t)-f(x)|\ :\ |t-x|\le\delta\right\}.$$

A well-known properties of modulus of continuity is: if f is uniformly continuous on $[0,\lambda]$, then $\lim_{\delta\to 0}\omega_\lambda(f;\delta)=0.$

2. AUXILIARY RESULTS

To obtain approximation properties of the operators defined by (1), we need some useful properties of these operators given by following Lemmas.

Lemma 1. *For any* $m\in\mathrm{N}$ *and* $x\in[0,b_n]$, *one have*

$$L_n^{\alpha_n,\beta_n}(t^m;x)=\phi(m,\alpha_n)x^m+\sum_{k=1}^{m-1}\frac{\psi_{k,m}(x)}{\beta_n^k} \tag{2}$$

where $\psi_{k,m}(x)$ $(k=1,2,...,m)$ *is bounded function on any finite closed interval and* $\lim_{n\to\infty}\phi(m,\alpha_n)=1$

.

Proof. We prove by using the induction method. It is easily seen that the result is true when $m=1$. Now, suppose that (2) hold for any positive integer m. We shall show that the result (2) is true for $m+1$. For each m, choose an constant a_i ($i=1,2,...,m$) such that the equality

$$\left(\frac{k}{\beta_n}\right)^{m+1}=\frac{k(k-1)...(k-m)}{\beta_n^{m+1}}+\sum_{i=1}^{m}\frac{a_i}{\beta_n^{m+1-i}}\left(\frac{k}{\beta_n}\right)^i \tag{3}$$

holds. Therefore, by using (ii), (iv), (v) and (3) we obtain

$$\begin{aligned}
L_n^{\alpha_n,\beta_n}(t^{m+1};x)&=\frac{1}{\varphi_n(\alpha_n x)}\sum_{k=0}^{\infty}\frac{\varphi_n^{(k)}(0)}{k!}(\alpha_n x)^k\left(\frac{k}{\beta_n}\right)^{m+1}\\
&=\frac{1}{\varphi_n(\alpha_n x)}\sum_{k=0}^{\infty}\frac{\varphi_n^{(k)}(0)}{k!}(\alpha_n x)^k\left\{\frac{k(k-1)...(k-m)}{\beta_n^{m+1}}+\sum_{i=1}^{m}\frac{a_i}{\beta_n^{m+1-i}}\left(\frac{k}{\beta_n}\right)^i\right\}\\
&=\frac{1}{\beta_n^{m+1}}\frac{1}{\varphi_n(\alpha_n x)}\sum_{k=0}^{\infty}\frac{\varphi_n^{(k)}(0)}{k!}(\alpha_n x)^k k(k-1)...(k-m)\\
&\quad+\sum_{i=1}^{m}\frac{a_i}{\beta_n^{m+1-i}}\frac{1}{\varphi_n(\alpha_n x)}\sum_{k=0}^{\infty}\frac{\varphi_n^{(k)}(0)}{k!}(\alpha_n x)^k\left(\frac{k}{\beta_n}\right)^i\\
&=\frac{1}{\beta_n^{m+1}}\frac{1}{\varphi_n(\alpha_n x)}\sum_{k=0}^{\infty}\frac{\varphi_n^{(k+m+1)}(0)}{k!}(\alpha_n x)^{k+m+1}+\sum_{i=1}^{m}\frac{a_i}{\beta_n^{m+1-i}}L_n^{\alpha_n,\beta_n}(t^i;x)\\
&=\left(\frac{\alpha_n}{\beta_n}x\right)^{m+1}\frac{\varphi_n^{(m+1)}(\alpha_n x)}{\varphi_n(\alpha_n x)}+\sum_{i=1}^{m}\frac{a_i}{\beta_n^{m+1-i}}L_n^{\alpha_n,\beta_n}(t^i;x).
\end{aligned}$$

setting $\alpha_n=\dfrac{\beta_n}{n}$ and using condition (v) we have

$$\phi(m+1,\alpha_n)=\left(\frac{\alpha_n}{\beta_n}\right)^{m+1}\frac{\varphi_n{}^{(m+1)}(\alpha_n x)}{\varphi_n(\alpha_n x)}$$
$$=\frac{\varphi_n{}^{(m+1)}(\alpha_n x)}{n^{m+1}\varphi_n(\alpha_n x)}$$
$$=1+O(\frac{1}{n\alpha_n}).$$

so we can write,

$$L_n^{\alpha_n,\beta_n}(t^{m+1};x)=\phi(m+1,\alpha_n)x^{m+1}+\sum_{i=1}^{m}\frac{a_i}{\beta_n^{m+1-i}}L_n^{\alpha_n,\beta_n}(t^i;x)$$
$$=\phi(m+1,\alpha_n)x^{m+1}+\sum_{i=1}^{m}\frac{a_i}{\beta_n^{m+1-i}}\left[\phi(i,\alpha_n)x^i+\sum_{k=1}^{i-1}\frac{\psi_{k,i}(x)}{\beta_n^k}\right]$$

where each $\psi_{k,i}(x)$ ($k=1,...,i-1$; $i=1,...,m$) is bounded function. Since $\phi(i,\alpha_n)x^i$ is also a bounded function on any finite interval, we can also denote $\psi_{0,i}(x)=\phi(i,\alpha_n)x^i$. Then we have

$$L_n^{\alpha_n,\beta_n}(t^{m+1};x)=\phi(m+1,\alpha_n)x^{m+1}+\sum_{i=1}^{m}\frac{a_i}{\beta_n^{m+1-i}}\sum_{k=0}^{i-1}\frac{\psi_{k,i}(x)}{\beta_n^k}.$$

On the other hand, by expanding the sum in the last equality, we get the following form:

$$L_n^{\alpha_n,\beta_n}(t^{m+1};x)=\phi(m+1,\alpha_n)x^{m+1}+\frac{a_1}{\beta_n^m}\psi_{0,1}(x)+\frac{a_2}{\beta_n^{m-1}}\left(\psi_{0,2}(x)+\frac{\psi_{1,2}(x)}{\beta_n}\right)+...+$$
$$+\frac{a_{m-1}}{\beta_n^2}\left(\psi_{0,m-1}(x)+\frac{\psi_{1,m-1}(x)}{\beta_n}+...+\frac{\psi_{m-2,m-1}(x)}{\beta_n^{m-2}}\right)$$
$$+\frac{a_m}{\beta_n}\left(\psi_{0,m}(x)+\frac{\psi_{1,m}(x)}{\beta_n}+...+\frac{\psi_{m-1,m}(x)}{\beta_n^{m-1}}\right)$$
$$=\phi(m+1,\alpha_n)x^{m+1}+\frac{a_m\psi_{0,m}(x)}{\beta_n}+\frac{a_{m-1}\psi_{0,m-1}(x)+a_m\psi_{1,m}(x)}{\beta_n^2}+...+$$
$$+\frac{a_1\psi_{0,1}(x)+a_2\psi_{1,2}(x)+...+a_m\psi_{m-1,m}(x)}{\beta_n^m}.$$

if we denote,

$$\psi_{1,m+1}(x)=a_m\psi_{0,m}(x),$$
$$\psi_{2,m+1}(x)=a_{m-1}\psi_{0,m-1}(x)+a_m\psi_{1,m}(x),$$
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$$\psi_{m,m+1}(x)=a_1\psi_{0,1}(x)+a_2\psi_{1,2}(x)+...+a_m\psi_{m-1,m}(x),$$

then we have

$$L_n^{\alpha_n,\beta_n}(t^{m+1};x)=\phi(m+1,\alpha_n)x^{m+1}+\sum_{k=1}^{m}\frac{\psi_{k,m}(x)}{\beta_n^k}$$

this establishes desired result and completes the proof of the lemma.

Lemma 2. *The operator* $L_n^{\alpha_n,\beta_n}$ *maps from* $C_{2m}[0,b_n]$ *to* $B_{2m}[0,b_n]$.

Proof. Let $f \in C_{2m}[0,b_n]$. Then

$$\begin{aligned}\left\|L_n^{\alpha_n,\beta_n}(f)\right\|_{B[0,b_n]} &= \sup_{0\le x\le b_n} \sigma_m(x)\left|L_n^{\alpha_n,\beta_n}(f;x)\right| \\ &\le \sup_{0\le x\le b_n} \sigma_m(x) L_n^{\alpha_n,\beta_n}(|f|;x) \\ &\le M_f \sup_{0\le x\le b_n} \sigma_m(x) L_n^{\alpha_n,\beta_n}((1+t^{2m});x) \\ &\le M_f\left[1+\sup_{0\le x\le b_n} \sigma_m(x) L_n^{\alpha_n,\beta_n}(t^{2m};x)\right].\end{aligned}$$

Since $\psi_{k,m}(x)$ is bounded function, it can be shown easily that for all $k=1,...,m-1$ the following inequality

$$\left|\psi_{k,m}(x)\right| \le C_\psi (1+x^k)$$

where C_ψ is a constant depending on function $\psi_{k,m}$, holds. Then there exists an constant C such that

$$\sup_{0\le x\le b_n} \sigma_m(x) L_n^{\alpha_n,\beta_n}(t^{2m};x) \le C .$$

Therefore, we have

$$\left\|L_n^{\alpha_n,\beta_n}(f)\right\|_{B[0,b_n]} \le M_f(1+C).$$

This completes the proof of the lemma.

3. MAIN RESULTS

In this section, firstly we investigate the order of approximation of continuous function by the operator $L_n^{\alpha_n,\beta_n}$ defined by (1) on any finite interval of positive semi-axis.

Theorem 2. *Let* $f \in C_{2m}[0,b_n]$. *Then for all sufficiently large* n, *the inequality*

$$\left\|L_n^{\alpha_n,\beta_n}(f)-f\right\|_{C[0,\lambda]} \le c_f(\lambda)\omega_{\lambda+1}(f;\gamma(n,m))$$

holds, where $\gamma(n,m)=\max\left\{\left|\alpha(2m,n)-2\alpha(m,n)+1\right|^{1/2m}, \frac{1}{2m\sqrt{n}}\right\}$,

$$c_f(\lambda)=6M_f(1+\lambda^m)^2\frac{C_f^{2m}(\lambda)}{\omega_{\lambda+1}(f;1)}+2$$

and

$$C_f^{2m}(\lambda)=\max\left\{\sup_{0\le x\le\lambda}\left(\sum_{k=1}^{m-1}\left|\psi_{k,2m}(x)-2x^m\psi_{k,m}(x)\right|+\sum_{k=m}^{2m-1}\left|\psi_{k,2m}(x)\right|\right)^{1/2m},\lambda\right\}.$$

Proof. Clearly, for $x\in[0,\lambda]$ and $t\in(0,\infty)$ we can divide the $[0,\lambda]\times(0,\infty)$ into the two subsets as follows:

$$B_1=\{(x,t):\ 0\le x\le\lambda,\ \lambda+1<t<\infty\},$$

$$B_2=\{(x,t):\ 0\le x\le\lambda,\ 0<t\le\lambda+1\}.$$

Firstly, we consider $(x,t)\in B_1$. It is easily seen that

$$\begin{aligned}|f(t)-f(x)| &\le |f(t)|+|f(x)| \\ &\le M_f(2+t^{2m}+x^{2m}) \\ &\le M_f(2+(t^m-x^m)^2+2\lambda^m(t^m-x^m)+2\lambda^{2m})\end{aligned}$$

since $x \le \lambda$ and $t > \lambda+1,$ we have $x^m \le \lambda^m$ and $t^m > (\lambda+1)^m$, thus we get $t^m - x^m > 1.$ Using the this conclusion, we obtain

$$\begin{aligned}|f(t)-f(x)| &\le M_f((1+\lambda^m)^2(t^m-x^m)^2+2(1+\lambda^m)^2(t^m-x^m)^2) \\ &\le 3M_f(1+\lambda^m)^2(t^m-x^m)^2.\end{aligned} \tag{4}$$

If $(x,t)\in B_2,$ we get

$$\begin{aligned}|f(t)-f(x)| &\le \omega_{\lambda+1}(f;|t-x|) \\ &\le \left(1+\frac{|t-x|}{\delta_n}\right)\omega_{\lambda+1}(f;\delta_n).\end{aligned} \tag{5}$$

by using (4) and (5), we have

$$|f(t)-f(x)| \le 3M_f(1+\lambda^m)^2(t^m-x^m)^2+\left(1+\frac{|t-x|}{\delta_n}\right)\omega_{\lambda+1}(f;\delta_n) \tag{6}$$

for any $t \ge 0$ and $0 \le x \le \lambda.$ Now, by applying the operator $L_n^{\alpha_n,\beta_n}$ to both sides of (6) and, use Hölder's inequality to get

$$\begin{aligned}L_n^{\alpha_n,\beta_n}(|f(t)-f(x)|;x) &\le 3M_f(1+\lambda^m)^2L_n^{\alpha_n,\beta_n}((t^m-x^m)^2;x) \\ &+\omega_{\lambda+1}(f;\delta_n)\left(1+\frac{1}{\delta_n}\left[L_n^{\alpha_n,\beta_n}((t-x)^{2m})\right]^{1/2m}\right)\end{aligned}$$

and using the inequality $(t-x)^{2m} \le (t^m-x^m)^2,$ we have

$$\begin{aligned}L_n^{\alpha_n,\beta_n}(|f(t)-f(x)|;x) &\le 3M_f(1+\lambda^m)^2L_n^{\alpha_n,\beta_n}((t^m-x^m)^2;x) \\ &+\omega_{\lambda+1}(f;\delta_n)\left(1+\frac{1}{\delta_n}\left[L_n^{\alpha_n,\beta_n}((t^m-x^m)^2)\right]^{1/2m}\right).\end{aligned} \tag{7}$$

Now, we consider the term $L_n^{\alpha_n,\beta_n}((t^m-x^m)^2).$ Using (2), one get

$$\begin{aligned}L_n^{\alpha_n,\beta_n}((t^m-x^m)^2) &= \phi(2m,\alpha_n)x^{2m}+\sum_{k=1}^{2m-1}\frac{\psi_{k,2m}(x)}{\beta_n^k} \\ &-2x^m\left(\phi(m,\alpha_n)x^m+\sum_{k=1}^{m-1}\frac{\psi_{k,m}(x)}{\beta_n^k}\right)+x^{2m} \\ &\le |\phi(2m,\alpha_n)-2\phi(m,\alpha_n)+1|\,x^{2m} \\ &+\frac{1}{\beta_n}\left[\left|\sum_{k=1}^{m-1}\psi_{k,2m}(x)-2x^m\psi_{k,m}(x)\right|+\sum_{k=m}^{m-1}|\psi_{k,m}(x)|\right]\end{aligned}$$

hence,

$$\begin{aligned}\sup_{0\le x\le\lambda} L_n^{\alpha_n,\beta_n}((t^m-x^m)^2) &\le |\phi(2m,\alpha_n)-2\phi(m,\alpha_n)+1|\,\lambda^{2m} \\ &+\frac{1}{\beta_n}\sup_{0\le x\le\lambda}\left[\left|\sum_{k=1}^{m-1}\psi_{k,2m}(x)-2x^m\psi_{k,m}(x)\right|+\sum_{k=m}^{m-1}|\psi_{k,m}(x)|\right]\end{aligned}$$

therefore,

$$\left(\sup_{0\le x\le\lambda} L_n^{\alpha_n,\beta_n}\left((t^m-x^m)^2\right)\right)^{1/2m} \le C_f(\lambda)\left(\left|\phi(2m,\alpha_n)-2\phi(m,\alpha_n)+1\right|^{1/2m}+\left(\frac{1}{\beta_n}\right)^{1/2m}\right)$$

where $C_f(\lambda)$ given as above. Setting $\delta_n=\max\left\{\left|\phi(2m,\alpha_n)-2\phi(m,\alpha_n)+1\right|^{1/2m},\left(\frac{1}{\beta_n}\right)^{1/2m}\right\}$, we see that $L_n^{\alpha_n,\beta_n}\left((t^m-x^m)^2\right)\le(\delta_n C_f(\lambda))^{2m}$ and $\lim\limits_{n\to\infty}\delta_n=0.$

Using this result in (7),

$L_n^{\alpha_n,\beta_n}\left(\left|f(t)-f(x)\right|;x\right)\le 3M_f(1+\lambda^m)^2(\delta_n C_f(\lambda))^{2m}+2\omega_{\lambda+1}(f;\delta_n)$

and we have

$$\left\|L_n^{\alpha_n,\beta_n}(f)-f\right\|_{C[0,\lambda]}\le 3M_f(1+\lambda^m)^2(\delta_n C_f(\lambda))^{2m}+2\omega_{\lambda+1}(f;\delta_n). \tag{8}$$

Since $\delta_n\to 0$ as $n\to\infty$, for sufficiently large n, we have $\delta_n^{2m}\le\delta_n$ and by using properties of the modulus of continuity $\delta_n\le\frac{2}{\omega_{\lambda+1}(f;1)}\omega_{\lambda+1}(f;\delta_n)$. Thus, from (8) we obtain the desired result which gives the proof.

In order to result on weighted approximation by operators (1) in whole semi-axis $[0,\infty)$, we have the following theorem:

Theorem 3. *For any function* $f\in C_{2m}^*[0,\infty)$, *we have*

$$\lim_{n\to\infty}\sup_{0\le x\le b_n}\sigma_m(x)\left|L_n^{\alpha_n,\beta_n}(f;x)-f(x)\right|=0.$$

Proof. Recall that according to the theorem 1., we have to show that the conditions

$$\lim_{n\to\infty}\left\|L_n^{\alpha_n,\beta_n}(t^\nu;x)-x^\nu\right\|_{C_{2m}[0,\infty)}=0,\quad \nu=0,m,2m$$

hold. Obviously, the result is true for $\nu=0$. We will prove only for $\nu=m$, since the other case, for $\nu=2m$, can be proven easily in a similar manner. Using (2) we can write

$$\begin{aligned}\left\|L_n^{\alpha_n,\beta_n}(t^\nu;x)-x^\nu\right\|_{C_{2m}[0,b_n]} &\le \left|\phi(m,\alpha_n)-1\right|\sup_{0\le x\le b_n}\sigma_m(x)x^m+\sup_{0\le x\le b_n}\sum_{k=1}^{m-1}\sigma_m(x)\frac{\psi_{k,m}(x)}{\beta_n^k}\\ &\le \left|\phi(m,\alpha_n)-1\right|+\frac{1}{\beta_n}\sup_{0\le x\le b_n}\sum_{k=1}^{m-1}\sigma_m(x)\psi_{k,m}(x).\end{aligned}$$

Since for all $k=1,2,...,m-1$, $\left|\psi_{k,m}(x)\right|\le C_\psi(1+x^k)$, we have $\sum\limits_{k=1}^{m-1}\sigma_m(x)\psi_{k,m}(x)\le C$. Therefore, we get

$$\left\|L_n^{\alpha_n,\beta_n}(t^\nu;x)-x^\nu\right\|_{C_{2m}[0,b_n]}\le\left|\phi(m,\alpha_n)-1\right|+\frac{C}{\beta_n}$$

and this gives desired result.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

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