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The existence of polynomials which are unrepresentable in Kolmogorov-Arnold superposition representation

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Abstract

In this paper, it is proved that there exist polynomials of three complex variables which cannot be represented as any Kolmogorov-Arnold superposition, which has played important roles in the original version of Hilbert's 13th problem.

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1. Introduction

The purpose of this paper is to show that the roles which Kolmogorov-Arnold superposition representations play in the version of Hilbert's 13th problem for continuous functions of several real variables are definitely different from the roles which these representations play in the version of this problem for analytic functions of several complex variables. The original result obtained by Kolmogorov and Arnold [4] is that, for any continuous function f of three real variables, we can choose a family of seven continuous functions of one real variable $\{g_i; 0 \leq i \leq 6\}$ and a family of twenty one continuous functions of one real variable $\{\phi_{ij}; 0 \leq i \leq 6, 1 \leq j \leq 3\}$ satisfying

$$f(x_1, x_2, x_3) = \sum_{i=0}^{6} g_i \left(\sum_{j=1}^{3} \phi_{ij}(x_j) \right).$$

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This result, which is called Kolmogorov-Arnold representation theorem, immediately implies that any continuous function of three real variables can be represented as a ten times nested superposition of continuous functions of two real variables, because the following equality holds:

$$\begin{aligned} f(z_1, z_2, z_3) \;\; = \;\; & \Big(\Big(\Big(\Big(\Big(\Big(g_0\big((\phi_{01}(z_1) + \phi_{02}(z_2)) + \phi_{03}(z_3)\big) \\ & +g_1\big((\phi_{11}(z_1) + \phi_{12}(z_2)) + \phi_{13}(z_3)\big)\Big) \\ & +g_2\big((\phi_{21}(z_1) + \phi_{22}(z_2)) + \phi_{23}(z_3)\big)\Big) \\ & +g_3\big((\phi_{31}(z_1) + \phi_{32}(z_2)) + \phi_{33}(z_3)\big)\Big) \\ & +g_4\big((\phi_{41}(z_1) + \phi_{42}(z_2)) + \phi_{43}(z_3)\big)\Big) \\ & +g_5\big((\phi_{51}(z_1) + \phi_{52}(z_2)) + \phi_{53}(z_3)\big)\Big) \\ & +g_6\big((\phi_{61}(z_1) + \phi_{62}(z_2)) + \phi_{63}(z_3)\big)\Big). \end{aligned}$$

Precisely speaking, the problem asking if functions of three variables can be represented as Kolmogorov-Arnold superposition representations consisting of several functions of two variables, can be classified into the following two cases:

Case I. For any positive integer k, there exists a function of 3 variables which cannot be represented as any k-time nested superposition representation consisting of several functions of 2 variables.

Case II.There exists function of 3 variables which cannot be represented as any finite-time nested superposition representation consisting of several functions of 2 variables.

For example, let $f(\cdot,\cdot,\cdot)$ be the complex-valued function of three complex variable defined as

$$f(x, y, z) \;=\; xy + yz + zx, \quad x, y, z \in \mathbb{C}.$$

Then, we can prove that there do not exist any family of three functions of two complex variables $g(\cdot,\cdot)$, $u(\cdot,\cdot)$ and $v(\cdot,\cdot)$ satisfying the following equality:

$$f(x, y, z) = g(u(x, y), v(x, z)), \quad x, y, z \in \mathbb{C}.$$

This result shows us that f cannot be represented any one time nested superposition representation consisting of complex-valued functions of two variables. Actually, it is very clear to prove that the following equality:

$$f(x, y, z) = x(y + z) + (yz)$$

holds for all $x, y, z \in \mathbb{C}$. This result shows us that f can be represented as a two times nested superposition representation. Hilbert proved that, in case of analytic functions, Case I holds and Akashi [1] proved that, in case of analytic functions, Case II holds.

In this paper, it is proved that there exists a polynomial of three complex variables which cannot be represented as any Kolmogorov-Arnold superposition consisting of several functions of two complex variables.

2. Preliminaries

Throughout this paper, $\mathbb{Z}_+$, $\mathbb{R}_+$ and $\mathbb{C}$ denote the set of all nonnegative integers, the set of all nonnegative real numbers and the set of all complex numbers, respectively.

Let U be the open unit disc of $\mathbb{C}$ and, for any positive integer n, let $\mathcal{B}(U^n)$ be the Banach space with the supremum norm $\|\cdot\|_n$ consisting of all continuous functions of n variables defined on $\overline{U}^n$ that are also analytic on U^n. Then, for any positive number s which is greater than 1, sU denotes $\{sz; z \in U\}$ and $\mathcal{A}_n(s)$ denotes the set of all analytic functions of n variables defined on $(sU)^n$, and especially, for any positive number M, $\mathcal{A}_n(s, M)$ is defined as $\{f \in \mathcal{A}_n(s); \sup_{z_1,\cdots,z_n \in sU} |f(z_1, \cdots, z_n)| \leq M\}$. Let k be a positive integer. Then, $\mathcal{I}_k(\mathcal{A}_n(s, M))$ denotes the set of all functions of $n+1$ complex variables which can be represented as k-time nested superpositions of $\mathcal{A}_n(s, M)$. Especially, $\mathcal{I}_0(\mathcal{A}_n(s, M))$ is defined as $\mathcal{A}_n(s, M)$.

Let $\mathcal{X}$ be a metric space. Then, for any positive number ε and for any relatively compact subset $\mathcal{F}$ of $\mathcal{X}$, the ε-entropy of $\mathcal{F}$, which is denoted by $S(\mathcal{F}, \varepsilon)$, is defined as the base-2 logarithm of the minimum number of the cardinal numbers corresponding to all ε-nets of $\mathcal{F}$, and the ε-capacity of $\mathcal{F}$, which is denoted by $C(\mathcal{F}, \varepsilon)$, is defined as the base-2 logarithm of the maximum number of the cardinal numbers corresponding to all 2ε-separated sets of $\mathcal{F}$.

After K. I. Babenko [2] and V. D. Erohin [3] had proved that, for any positive number s which is greater than 1 and for any positive number M, the following equality:

$$\lim_{\varepsilon \to 0} \frac{S(\{f \in \mathcal{A}_2(s); \|f\|_2 \leq M\}, \varepsilon)}{\left(\log \frac{1}{\varepsilon}\right)^3} = \frac{2}{3!(\log s)^2}$$

holds, A. G. Vitushkin [8] gave the following generalization:

$$\lim_{\varepsilon \to 0} \frac{S(\{f \in \mathcal{A}_n(s); \|f\|_n \leq M\}, \varepsilon)}{\left(\log \frac{1}{\varepsilon}\right)^{n+1}} = \frac{2}{(n+1)!(\log s)^n},$$

where n is any positive integer that is greater than 1. Here we have the following:

Lemma 1. For any positive integer n, the following equality:

$$\lim_{\varepsilon \to 0} \frac{S\left(\overline{\mathcal{A}_n(s, M)}, \varepsilon\right)}{\left(\log \frac{1}{\varepsilon}\right)^{n+1}} = \frac{2}{(n+1)!(\log s)^n}$$

holds.

Proof. Since we have

$$\mathcal{A}_n(s, M) \subset \{f \in \mathcal{A}_n(s); \|f\|_n \leq M\},$$

we can obtain

$$\limsup_{\varepsilon \to 0} \frac{S(\mathcal{A}_n(s, M), \varepsilon)}{\left(\log \frac{1}{\varepsilon}\right)^{n+1}} \leq \frac{2}{(n+1)!(\log s)^n}.$$

This result implies that the following inequality holds:

$$\limsup_{\varepsilon \to 0} \frac{S\left(\overline{\mathcal{A}_n(s, M)}, \varepsilon\right)}{\left(\log \frac{1}{\varepsilon}\right)^{n+1}} \leq \frac{2}{(n+1)!(\log s)^n}.$$

Let $N(\varepsilon)$ be the positive integer defined as

$$N(\varepsilon) = \left[\log \frac{1}{\varepsilon} + 1\right].$$

Moreover, let $D(\varepsilon)$ be the subset of $\mathbb{Z}_+^n$ defined as

$$D(\varepsilon) = \left\{(k_1, \cdots, k_n) \in \mathbb{Z}_+^n; \sum_{i=1}^{n} k_i \leq \frac{N(\varepsilon)}{\log s}\right\}.$$

Let ϕ be a mapping defined on $D(\varepsilon)$ with values in $\mathbb{C}$ satisfying

$$\left|\frac{Re(\phi(k_1,\cdots,k_n))}{2\varepsilon}\right| \in \mathbb{Z}_+, \quad \left|\frac{Re(\phi(k_1,\cdots,k_n))}{2\varepsilon}\right| \le \left[\frac{M/\sqrt{2}}{2^{n+1}\varepsilon\prod_{i=1}^n(k_i+1)^2}\prod_{i=1}^n\left(\frac{1}{s}\right)^{k_i}\right],$$
$$\left|\frac{Im(\phi(k_1,\cdots,k_n))}{2\varepsilon}\right| \in \mathbb{Z}_+, \quad \left|\frac{Im(\phi(k_1,\cdots,k_n))}{2\varepsilon}\right| \le \left[\frac{M/\sqrt{2}}{2^{n+1}\varepsilon\prod_{i=1}^n(k_i+1)^2}\prod_{i=1}^n\left(\frac{1}{s}\right)^{k_i}\right],$$

where $(k_1,\cdots,k_n)$ is an element belonging to $D(\varepsilon)$, and let $\Phi(\varepsilon)$ be the set of all mappings satisfying the above conditions. For any $\phi \in \Phi(\varepsilon)$, $g_\phi(\cdot)$ denotes the polynomial of n complex variables which is defined as

$$g_\phi(z_1,\cdots,z_n) = \sum_{(k_1,\cdots,k_n)\in D(\varepsilon)} \phi(k_1,\cdots,k_n)\prod_{i=1}^n z_i^{k_i}, \quad z_1,\cdots,z_n \in \mathbb{C}.$$

If $(z_1,\cdots,z_n) \in sU$ holds, then we have

$$\begin{aligned}|g_\phi(z_1,\cdots,z_n)| &\le \sum_{(k_1,\cdots,k_n)\in D(\varepsilon)} |\phi(k_1,\cdots,k_n)|\prod_{i=1}^n s^{k_i} \\ &\le \sum_{(k_1,\cdots,k_n)\in D(\varepsilon)} \frac{M}{2^n\prod_{i=1}^n(k_i+1)^2} \\ &\le \frac{M}{2^n}\left(1+\sum_{k=1}^\infty\frac{1}{k(k+1)}\right)^n \\ &\le M.\end{aligned}$$

Therefore, g_ϕ is an element belonging to $\mathcal{A}_n(s,M)$. Let ϕ_1 and ϕ_2 be two elements belonging to $\Phi(\varepsilon)$. Then, there exists an element $(k_1,\cdots,k_n) \in D(\varepsilon)$ satisfying

$$|\phi_1(k_1,\cdots,k_n) - \phi_2(k_1,\cdots,k_n)| \ge 2\varepsilon.$$

This inequality implies that

$$\|g_{\phi_1} - g_{\phi_2}\|_n \ge 2\varepsilon$$

holds, and $\{g_\phi; \phi \in \Phi(\varepsilon)\}$ is a 2ε-separated set of $\mathcal{A}_n(s,M)$. Let $\Delta(\varepsilon)$ be the subset of $\mathbb{R}_+^n$ defined as

$$\Delta(\varepsilon) = \left\{(x_1,\cdots,x_n) \in \mathbb{R}_+^n; \sum_{i=1}^n x_i \le \frac{N(\varepsilon)}{\log s}\right\}.$$

If $(k_1,\cdots,k_n)$ is an element belonging to $D(\varepsilon)$, then, for any sufficiently small ε, we obtain the following inequalities:

$$\prod_{i=1}^n(k_i+1)^2 \le \left(\frac{2}{\log s}\right)^{2n} N(\varepsilon)^{2n},$$

$$\begin{aligned}card(D(\varepsilon)) &\ge \int\cdots\int_{\Delta(\varepsilon)} dx_1\cdots dx_n \\ &= \frac{N(\varepsilon)^n}{n!(\log s)^n}\end{aligned}$$

and

$$\begin{aligned}\sum_{(k_1,\cdots,k_n)\in D(\varepsilon)} \log\left(\prod_{i=1}^{n}\frac{1}{s^{k_i}}\right) &= -\sum_{(k_1,\cdots,k_n)\in D(\varepsilon)} \log s \sum_{i=1}^{n} k_i \\ &\geq -\int\cdots\int_{\Delta(\varepsilon)} \log s \sum_{i=1}^{n}(x_i+1)\, dx_1\cdots dx_n \\ &\geq \frac{-nN(\varepsilon)^{n+1}}{(n+1)!(\log s)^n} + \mathcal{O}\left(N(\varepsilon)^n\right).\end{aligned}$$

Therefore, a lower bound of $C(\mathcal{A}_n(s,M).,\varepsilon)$ can be estimated as follows:

$$\begin{aligned}C(\mathcal{A}_n(s,M),\varepsilon) &\geq \log card(\Phi(\varepsilon)) \\ &\geq \log \prod_{(k_1,\cdots,k_n)\in D(\varepsilon)} \left(\frac{2M}{2^{n+1}\varepsilon\prod_{i=1}^{n}(k_i+1)^2}\prod_{i=1}^{n}\frac{1}{s^{k_i}}+1\right)^2 \\ &\geq \frac{2N(\varepsilon)^n}{n!(\log s)^n}\log\frac{1}{\varepsilon} - \frac{2nN(\varepsilon)^{n+1}}{(n+1)!(\log s)^n} + \mathcal{O}\left(\left(\log\frac{1}{\varepsilon}\right)^n \log\log\frac{1}{\varepsilon}\right).\end{aligned}$$

These results imply that the following inequality:

$$\liminf_{\varepsilon\to 0}\frac{C(\mathcal{A}_n(s,M),\varepsilon)}{\left(\log\frac{1}{\varepsilon}\right)^{n+1}} \geq \frac{2}{(n+1)!(\log s)^n}$$

holds, and therefore, we can conclude the proof. □

3. Finite-time nested superpositions of analytic functions of two complex variables

For any two positive numbers s and M which are greater than one, let $\mathcal{N}_\varepsilon(\mathcal{A}_2(s,M))$ be an ε-net of $\mathcal{A}_2(s,M)$. Then, we obtain

Lemma 2. For any fixed positive integer n, there exists a positive constant c_n, which is independent of ε, satisfying the condition that $\mathcal{I}_n(\mathcal{N}_\varepsilon(\mathcal{A}_2(s,M)))$ is a $c_n\varepsilon$-net of $\overline{\mathcal{I}_n(\mathcal{A}_2(s,M))}$.

Proof. We prove this lemma by induction. Assume that $\mathcal{I}_n(\mathcal{N}_\varepsilon(\mathcal{A}_2(s,M)))$ is a $c_n\varepsilon$-net. It is clearly sufficient to prove the case of $n=0$. Let g be an element belonging to $\overline{\mathcal{I}_{n+1}(\mathcal{A}_2(s,M))}$. Then, there exist three functions $f\in\mathcal{A}_2(s,M)$, $u\in\mathcal{I}_n(\mathcal{A}_2(s,M))$ and $v\in\mathcal{I}_n(\mathcal{A}_2(s,M))$ satisfying

$$\|g-f(u,v)\|_3<\varepsilon.$$

Moreover, there exist three functions $f'\in\mathcal{N}_\varepsilon(\mathcal{A}_2(s,M))$, $u'\in\mathcal{I}_n(\mathcal{N}_\varepsilon(\mathcal{A}_2(s,M)))$ and $v'\in\mathcal{I}_n(\mathcal{N}_\varepsilon(\mathcal{A}_2(s,M)))$ satisfying

$$\begin{aligned}\|f-f'\|_2 &< \varepsilon, \\ \|u-u'\|_3 &< c_n\varepsilon, \\ \|v-v'\|_3 &< c_n\varepsilon.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\|g-f'(u',v')\|_3 &\leq \|g-f(u,v)\|_3+\|f(u,v)-f'(u',v')\|_3 \\ &\leq \|g-f(u,v)\|_3 \\ &\quad +\|f(u,v)-f(u',v)\|_3+\|f(u',v)-f(u',v')\|_3+\|f(u',v')-f'(u',v')\|_3 \\ &< \left(\frac{2c_nM}{s-1}+2\right)\varepsilon,\end{aligned}$$

because the following inequalities:

$$\sup\left\{\left|\frac{\partial f}{\partial z_1}(z_1,z_2)\right|; (z_1,z_2)\in U^2\right\} \le \frac{M}{s-1}$$

and

$$\sup\left\{\left|\frac{\partial f}{\partial z_2}(z_1,z_2)\right|; (z_1,z_2)\in U^2\right\} \le \frac{M}{s-1}$$

hold. It follows that $\mathcal{I}_{n+1}(\mathcal{N}_\varepsilon(\mathcal{A}_2(s,M)))$ is a $((2c_nM)/(s-1)+2)\varepsilon$-net of $\overline{\mathcal{I}_n(\mathcal{A}_2(s,M))}$. □

Now, we obtain

Theorem 3. For any two positive number s and M, which are greater than one, there exists an polynomial belonging to $\mathcal{B}(U^3)$ which does not belong to $\overline{\{f|_{U^3}; f\in\mathcal{I}_n(\mathcal{A}_2(s,M))\}}$.

Proof. It is sufficient to prove that $\overline{\{f|_{U^3}; f\in\mathcal{I}_n(\mathcal{A}_2(s,M))\}}$ is a subset of the first category. If we assume that $\{f|_{U^3};\ f\in\mathcal{I}_n(\mathcal{A}_2(s,M))\}$ is not nowhere dense for a certain $n\in\mathbb{Z}_+$, then there exist a positive number δ and an element g_δ belonging to $\mathcal{B}(U^3)$ satisfying

$$\overline{\{f|_{U^3}; f\in\mathcal{I}_n(\mathcal{A}_2(s,M))\}} \supset \{g\in\mathcal{B}(U^3); \|g\|_3<\delta\}+g_\delta.$$

Since the following inclusions:

$$\begin{aligned}\{g\in\mathcal{B}(U^3); \|g\|_3<\delta\} &\supset \left\{h|_{U^3}; h\in\mathcal{A}_3(s), \|h|_{U_3}\|_3\le\frac{\delta}{2}\right\}\\ &\supset \mathcal{A}_3\left(s,\frac{\delta}{2}\right)\end{aligned}$$

hold, we have

$$S\left(\overline{\{f|_{U^3}; f\in\mathcal{I}_n(\mathcal{A}_2(s,M))\}},\varepsilon\right) \ge S\left(\mathcal{A}_3\left(s,\frac{\delta}{2}\right),\varepsilon\right),\quad \varepsilon>0.$$

Actually, the above inequality contradicts the results which were obtained by Babenko [2], Erohin [3] and Vitushkin [8], because their results and Lemma 2 assure that, for any sufficiently small number α, there exists a certain positive number ε_α satisfying the following two inequalities:

$$S\left(\overline{\{f|_{U^3}; f\in\mathcal{I}_n(\mathcal{A}_2(s,M))\}},\varepsilon\right) \le \left(\frac{2}{3!(\log s)^2}+\alpha\right)\left(\log\frac{1}{c_n\varepsilon}\right)^3,\quad 0<\varepsilon<\varepsilon_\alpha,$$

$$S\left(\mathcal{A}_3\left(s,\frac{\delta}{2}\right),\varepsilon\right) \ge \left(\frac{2}{4!(\log s)^3}-\alpha\right)\left(\log\frac{1}{\varepsilon}\right)^4,\quad 0<\varepsilon<\varepsilon_\alpha,$$

which contradict each other. □

Theorem 4. For any positive integer n and for any two positive numbers s and M which are greater than one, there exists a polynomial, which cannot be represented as n-time nested superposition consisting of elements belonging to $\mathcal{I}_n(\mathcal{A}_2(s,M))\}$.

Proof. Let $g_{n,s,M}$ be an element belonging to $\mathcal{B}(U^3)$ which does not belong to $\overline{\{f|_{U^3}; f\in\mathcal{I}_n(\mathcal{A}_2(s,M))\}}$. Then it is clear that the following inequality:

$$\inf\left\{\|g_{n,s,M}-h\|_3; h\in\overline{\{f|_{U^3}; f\in\mathcal{I}_n(\mathcal{A}_2(s,M))\}}\right\} > 0$$

holds. Therefore, it follows from Mergelyan's theorem [6, 7] that there exists a polynomial p satisfying $\|p - g_{n,s,M}\| \leq \frac{\delta}{2}$. This concludes the proof. □

Remarks. Theorem 4 assures that, for any positive integer k, and for any two positive numbers s and M which are greater than one, there exists a polynomial, which cannot be represented as Kolmogorov-Arnold superposition representation, because the fact that any analytic functions can be represented as a Kolmogorov-Arnold superposition necessarily implies the fact that they can be represented as ten-time nested superpositions consisting of several analytic functions of two complex variables.

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References

[1] S. Akashi, A version of Hilbert's 13th problem for analytic functions, The Bulletin of the London Mathematical Society, 35(2003), 8-14.
[2] K. I. Babenko, On the best approximation of a class of analytic functions, Izv. 22(1958), 631-640.
[3] V. D. Erohin, On the asymptotic behavior of the ε-entropy of analytic functions, Dokl., 120(1958), 949-952.
[4] G. G. Lorentz, Approximation of functions, Holt, Rinehart and Winston, New York, 1966.
[5] A. N. Kolmogorov, On the representation of continuous functions of several variables by superpositions of continuous functions of one variable and addition, Dokl., 114(1957), 679-681.
[6] S.N. Mergelyan, On the representation of functions by series of polynomials on closed sets, Transl. Amer. Math. Soc., 3 (1962), 287–293.
[7] S.N. Mergelyan, Uniform approximation to functions of a complex variable, Transl. Amer. Math. Soc., 3 (1962), 294391.
[8] A. G. Vitushkin, Some properties of linear superpositions of smooth functions, Dokl., 156(1964), 1003-1006.