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A new class activation functions with application in the theory of impulse techniques

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Abstract

In this note we define the new activation functions, based on the well-known hyperbolic tangent and half–hyperbolic tangent activation functions. We consider the Hausdorff distance between the "double step" function $\sigma^*(t)$ (resp. function $\sigma^{**}(t)$) and the new classes of activation functions. The results have independent significance in the study of issues related to neural networks and impulse techniques. Numerical examples, illustrating our results are presented using programming environment Mathematica.

1. Introduction

The typical examples of impulse functions from antenna feeder technique has the following shape (see, Figures 1.1-1.2):

$$\sigma^*(t) = \begin{cases} 1, t \in [1, +\infty) \\ 0, t \in [-1, 1) \\ -1, t \in (-\infty, -1) \end{cases} \tag{1.1}$$

$$\sigma^{**}(t) = \begin{cases} 1, t \in [-\infty, -1) \cup (1, +\infty) \\ 0, t \in [-1, 1]. \end{cases} \tag{1.2}$$

In [3] the following basic problems are considered – approximation of functions and point sets by algebraic and trigonometric polynomials in Hausdorff metric [2] as well as their applications in the field of antenna-feeder technique, analysis and synthesis of antenna patterns and filters, noise minimization by suitable approximation of impulse functions.
The polynomial Hausdorff approximations of the signals of type (1.1)-(1.2) are visualized on Figure 1.3-1.4.
These polynomials play an important role in approximation of antenna factor for scanning of directed chart.
Evidently the task of great difficulty is to determine the coefficients of the polynomials and values of the best Hausdorff approximation.
For other results, see [4]–[14], [16].

2. Main results

Definition 2.1. *[1], [2] The Hausdorff distance (the H–distance) [1] $\rho(f,g)$ between two interval functions f,g on $\Omega \subseteq \mathbb{R}$, is the distance between their completed graphs $F(f)$ and $F(g)$ considered as closed subsets of $\Omega \times \mathbb{R}$. More precisely,*

$$\rho(f,g) = \max\{ \sup_{A \in F(f)} \inf_{B \in F(g)} \|A - B\|, \sup_{B \in F(g)} \inf_{A \in F(f)} \|A - B\|\}, \tag{2.1}$$

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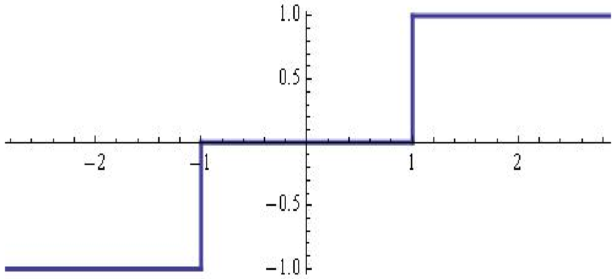

Figure 1.1: The signal of "double step" $\sigma^*(t)$ – type.

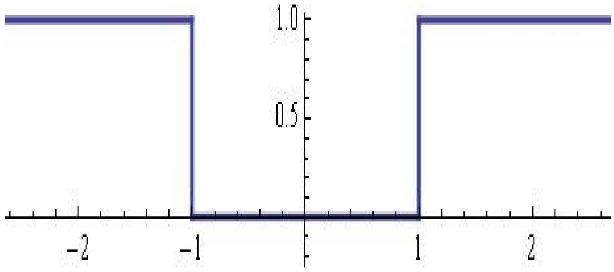

Figure 1.2: The signal of $\sigma^{**}(t)$ – type.

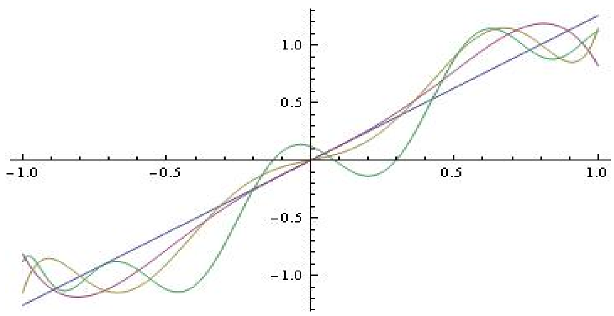

Figure 1.3: The polynomial approximation of the $\sigma^*(t)$ – type function.

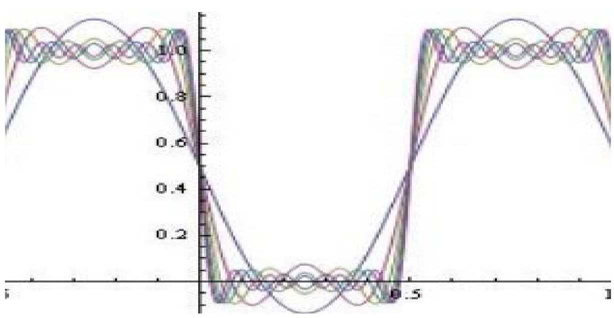

Figure 1.4: The polynomial approximation of the $\sigma^{**}(t)$ – type function.

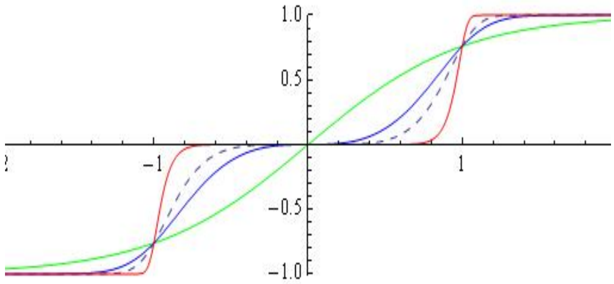

Figure 2.1: The family of activation functions $\varphi^*(t)$ for $m=1$ (green); $m=3$ (blue); $m=5$ (dashed); $m=15$ (red); ($\beta=1$ is fixed).

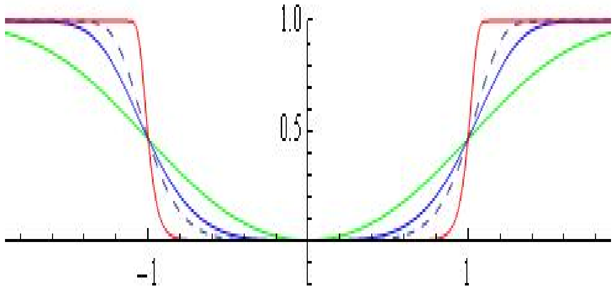

Figure 2.2: The family of activation functions $\varphi^{**}(t)$ for $m=2$ (green); $m=4$ (blue); $m=6$ (dashed); $m=20$ (red); ($\beta=1$ is fixed).

wherein $\|.\|$ *is any norm in* $\mathbb{R}^2$*, e. g. the maximum norm* $\|(t,x)\| = \max\{|t|,|x|\}$*; hence the distance between the points* $A=(t_A,x_A)$*,* $B=(t_B,x_B)$ *in* $\mathbb{R}^2$ *is* $\|A-B\| = max(|t_A-t_B|,|x_A-x_B|)$.

Definition 2.2. *Define the following "new activation function" as*

$$\varphi^*(t) = 1 - \frac{2e^{-\beta t^m}}{e^{\beta t^m}+e^{-\beta t^m}}, \tag{2.2}$$

where m is an odd parameter.

Definition 2.3. *Define the following "new half–activation function as:*

$$\varphi^{**}(t) = \frac{1-e^{-\beta t^m}}{1+e^{-\beta t^m}}, \tag{2.3}$$

where m is an even parameter.

The Hausdorff distance $d = d(\varphi^*(t),\sigma^*(t)$ between the activation function $\varphi^*(t)$ and the function $\sigma^*(t)$ can be calculating by solving nonlinear equation:

$$\varphi^*(1-d) = d. \tag{2.4}$$

Analogously, for the Hausdorff distance $d = d(\varphi^{**}(t),\sigma^{**}(t)$ between the activation function $\varphi^{**}(t)$ and the function $\sigma^{**}(t)$ we have

$$\varphi^{**}(1-d) = d. \tag{2.5}$$

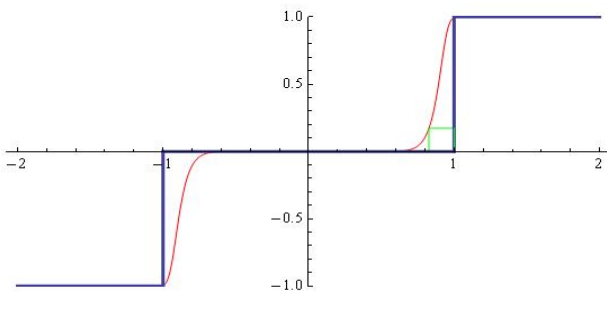

Figure 2.3: The activation functions $\varphi^*(t)$ for $\beta=3$; $m=15$; Hausdorff distance $d=0.172738$.

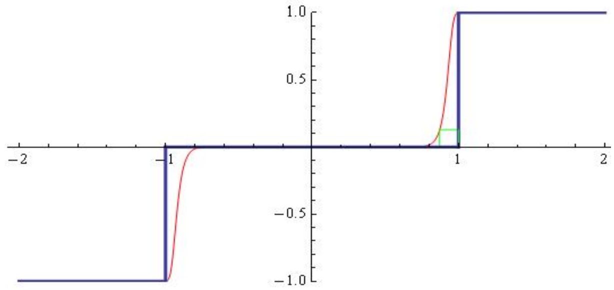

Figure 2.4: The activation functions $\varphi^{*}(t)$ for $\beta = 4$; $m = 25$; Hausdorff distance $d = 0.128338$.

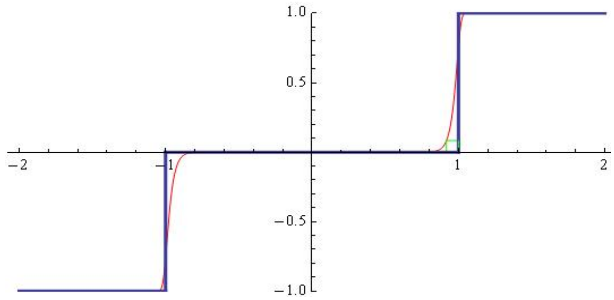

Figure 2.5: The activation functions $\varphi^{*}(t)$ for $\beta = 1$; $m = 29$; Hausdorff distance $d = 0.0824019$.

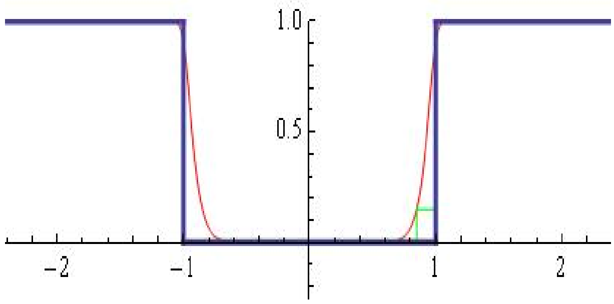

Figure 2.6: The activation functions $\varphi^{**}(t)$ for $\beta = 3$; $m = 14$; Hausdorff distance $d = 0.15085$.

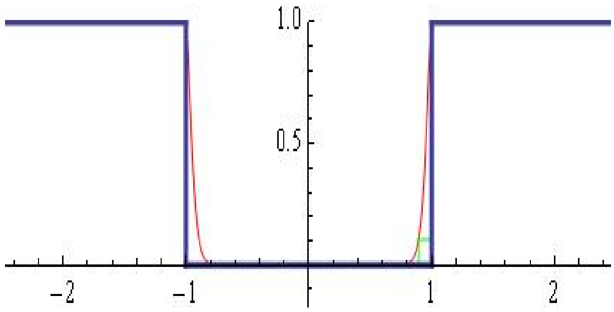

Figure 2.7: The activation functions $\varphi^{**}(t)$ for $\beta = 4$; $m = 26$; Hausdorff distance $d = 0.106529$.

```
ϕ1[θ_] := (1 - Exp[-β * (Pi * Cos[θ] + a)^m]) / (1 + Exp[-β * (Pi * Cos[θ] + a)^m])
Manipulate[PolarPlot[(1 - Exp[-β * (Pi * Cos[θ] + a)^m]) / (1 + Exp[-β * (Pi * Cos[θ] + a)^m]) ,
  {θ, -2 Pi, 2 Pi}], {β, 0.05, 10., Appearance → "Open"}, {m, 1, 30, Appearance → "Open"},
 {a, 0.1, 2 * Pi, Appearance → "Open"}]
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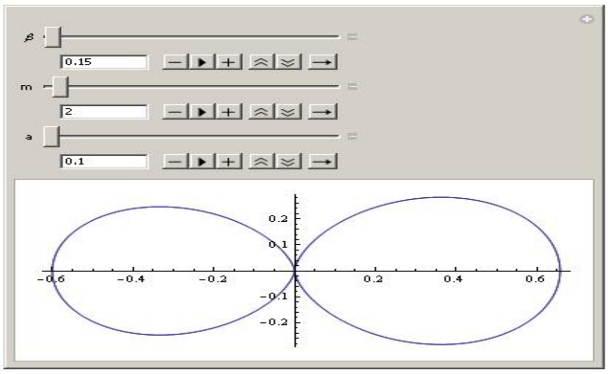

Figure 2.8: Typical emitting chart ($\varphi^{**}(\theta)$) for $\beta = 0.15$; $a = 0.1$; $m = 2$.

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ϕ1[θ_] := (1 - Exp[-β * (Pi * Cos[θ] + a)^m]) / (1 + Exp[-β * (Pi * Cos[θ] + a)^m])
Manipulate[PolarPlot[(1 - Exp[-β * (Pi * Cos[θ] + a)^m]) / (1 + Exp[-β * (Pi * Cos[θ] + a)^m]) ,
  {θ, -2 Pi, 2 Pi}], {β, 0.05, 10., Appearance → "Open"}, {m, 1, 30, Appearance → "Open"},
 {a, 0.1, 2 * Pi, Appearance → "Open"}]
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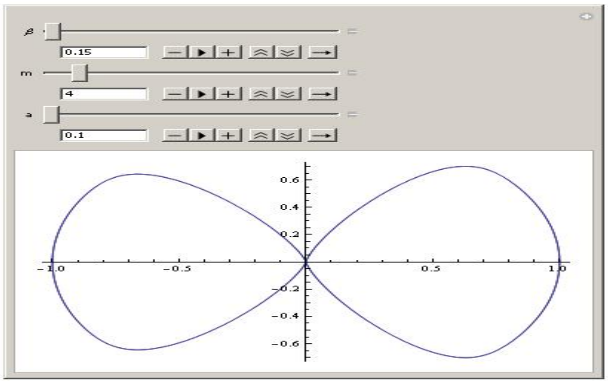

Figure 2.9: Typical emitting chart ($\varphi^{**}(\theta)$) for $\beta = 0.15$; $a = 0.1$; $m = 4$.

Some families of activation functions $\varphi^*(t)$ and $\varphi^{**}(t)$ are visualized on Figures 2.1-2.7.
After the substitution $t = kl\cos\theta + a$, where
– $k = \frac{2\pi}{\lambda}$, λ is the wave length;
– a is the phase difference;
– θ is the azimuthal angle;
– l is the distance between the emitters ($l = \frac{\lambda}{2}$ is fixed),
the activation function $\varphi^{**}(t)$ (or emitting chart of antenna factor can be written in the form

$$\varphi^{**}(\theta) = \frac{1 - e^{-\beta(\pi\cos\theta + a)^m}}{1 + e^{-\beta(\pi\theta + a)^m}}. \tag{2.6}$$

Typical emitting chart are visualized on Figures 2.8-2.9.
If $l = \lambda$ we have the chart - Figure 2.10.
Of course, the question of the practical realization of the activation functions which are generated as emitting charts remains open.
The mathematical apparatus proposed in the article can be successfully used for imitation and simulation of such charts.
I will explicitly say that the results have independent significance in the study of issues related to impulse technics [3], [15] and neural networks (see, [17]–[21]).
Remarks.

For the special case $m = 1$ we have the following
Theorem [16]. For the Hausdorff distance d between the *sgn* function and the half–hyperbolic function the following inequalities hold for $\beta \geq 5$:

$$d_l = \frac{1}{0.5(2+\beta)} < d < \frac{\ln(0.5(2+\beta))}{0.5(2+\beta)} = d_r.$$

Following the ideas given in [16], the reader may formulate the corresponding approximation problem for each number m.

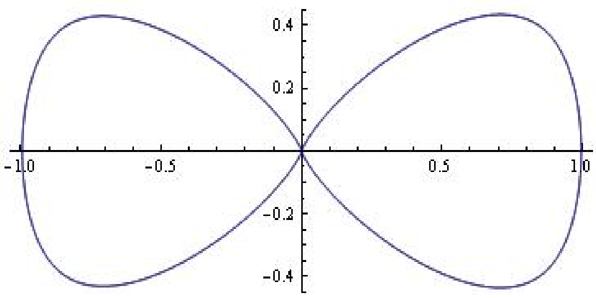

Figure 2.10: The emitting chart for $\beta = 0.0001$; $a = 0.01$; $m = 6$; $l = \lambda$.

3. Acknowledgements

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