

Reduction of non-variational bi-Hamiltonian system of shallow-water waves propagation via symmetry approach

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Abstract

In this paper, non-variational bi-Hamiltonian system of shallow-water waves propagation is considered. Lie point generators are calculated and one dimensional optimal system of its subalgebras up to conjugacy classes are reported. Then similarity variables are computed by using these conjugacy classes which are further utilized for the reduction of considered system. Then, a transformation is used to convert the system from non-variational to variational system, thus standard Lagrangian is computed. Noether operators are calculated by using Noether approach and local conserved quantity is discussed for the new fourth order system of partial differential equations (PDEs). Further, inverse transformation is applied to get the corresponding local conserved quantity for the considered non-variational problem. Moreover, this local conservation law with the help of double reduction theorem is utilized to reduce the system.

1. Introduction

Exact solutions of a system of nonlinear PDEs have lot of importance in the solution of initial or boundary value problems. It also helps to see the exact inner picture of the phenomena. For higher order nonlinear system of PDEs, one does not have enough methods to find the exact solutions. Symmetry approach is one of the powerful tools for analyzing such systems. Lie group theory is also one of the most interesting and active field of research for the solution of nonlinear PDEs. It is a systematic technique to construct a class of exact solutions or reduce the considered differential equation into one of its simplest form. It also provides applicable approach to find the closed form solution of differential equation. There are lot of efficient techniques available for the computation of exact solutions of differential equations. Some well known techniques are separation of variables, traveling wave solution, self-similar solutions, solution by using ansatz and exponential self-similar solutions. These approaches are the particular case of Lie group method (see [15]).

Reduction of a differential equation by using Noether approach is one of the aspect of Lie theory. Conserved quantities play a vital role in the theory of differential equations. Solution and reduction of PDEs are some of the aspects where conserved quantities have its significant importance. On the derivation of conservation laws active research efforts have been made in the last few decades. One of the elegant and systematic approach is developed by Noether [12]. Direct method [2], characteristics approach [20] and the partial Noether approach [10] are the approaches which have been used frequently in literature. Computing utilities for the calculation of conservation laws are also practised in past (see [4, 6, 7, 8, 9, 21, 22]).

A first systematic method for finding the conservation laws [12] of the variational problems is given by Emmy Noether. She found a link between the symmetries and conservation laws [12]. She concluded that each symmetry corresponds to a conserved quantity. This approach is valid for the variational problems and thus depends on the existence of standard Lagrangian. There is a big class of PDEs and system of PDEs which do not possess standard Lagrangian, thus Noether approach is not applicable. In this article, we will apply Noether approach to the non-variational system of PDEs. For this we will use the following steps:

- (i): Convert the non-variational system of PDEs into variational problem by using suitable transformation.
- (ii): Standard Lagrangian is calculated and Noether approach is applied in new coordinates.

(iii): Noether operators and corresponding conservation laws are computed.

(iv): Use an inverse transformation to convert the conserved quantities into the coordinates of original problem. The bi-Hamiltonian Boussinesq system [1]:

$$u_t - \Omega v_x = 0, \quad v_t - \Omega(u_{xxx} + 8uu_x) = 0, \quad (1.1)$$

describes shallow-water wave propagation in both directions, where Ω is a real constant.

In the past, lot of work has been done on the different forms of shallow-water waves propagation by means of Lie symmetry analysis. In [14], M. Pandey discussed the one dimensional shallow water equation by using Lie approach. Exact solutions of one dimensional axis symmetric flow of shallow water equations involving bores was computed with the help of point symmetries in [17]. Plebanski's second heavenly equation and 2 + 1-dimensional hamiltonian integrable system and is discussed in [11, 23] by means of symmetries. The group analysis and an infinite aggregate of non-degenerate solutions of the equations of one-dimensional model of shallow water with a straight bottom is discussed in [5].

The pattern of this paper is as follow. In Section 2, basic operators are discussed. The Lie point generators and reduction by using similarity variables for the system (1.1) are discussed in Section 3. Local conserved quantity and double reduction for the considered system is computed in Section 4. At last conclusion is provided.

2. Basic operators

Let (t, x) be independent variables and (u, v) be dependent variables. Consider the third order system of PDEs with two independent and dependent variables i.e.

$$E_1(t, x, u, v, u_t, v_t, \dots) = 0, \quad E_2(t, x, u, v, u_t, v_t, \dots) = 0. \quad (2.1)$$

(1): A vector field:

$$Y = \tau(t, x, u, v) \frac{\partial}{\partial t} + \xi(t, x, u, v) \frac{\partial}{\partial x} + \phi(t, x, u, v) \frac{\partial}{\partial u} + \eta(t, x, u, v) \frac{\partial}{\partial v}$$

is a Lie point symmetry generator of (2.1) if it satisfies the invariance condition i.e.

$$Y^{[3]}E_1 = 0, \quad Y^{[3]}E_2 = 0, \quad \text{whenever } E_1 = 0, \quad E_2 = 0, \quad (2.2)$$

where $Y^{[3]}$ is the third prolongation [13].

(2): If $u \rightarrow U_x$ and $v \rightarrow V_x$ in (2.1) then the given third order system becomes fourth order system in U, V variables, i.e.

$$G_1(t, x, U, V, U_t, V_t, \dots) = 0, \quad G_2(t, x, U, V, U_t, V_t, \dots) = 0. \quad (2.3)$$

(3): The Euler operator is:

$$\frac{\delta}{\delta U} = \frac{\partial}{\partial U} - D_t \frac{\partial}{\partial U_t} - D_x \frac{\partial}{\partial U_x} + D_t^2 \frac{\partial}{\partial U_{tt}} + D_x^2 \frac{\partial}{\partial U_{xx}} + \dots, \quad (2.4)$$

$$\frac{\delta}{\delta V} = \frac{\partial}{\partial V} - D_t \frac{\partial}{\partial V_t} - D_x \frac{\partial}{\partial V_x} + D_t^2 \frac{\partial}{\partial V_{tt}} + D_x^2 \frac{\partial}{\partial V_{xx}} + \dots, \quad (2.5)$$

where

$$D_t = \frac{\partial}{\partial t} + U_t \frac{\partial}{\partial U} + V_t \frac{\partial}{\partial V} + U_{tt} \frac{\partial}{\partial U_t} + V_{tt} \frac{\partial}{\partial V_t} + U_{tx} \frac{\partial}{\partial U_x} + \dots \quad (2.6)$$

and

$$D_x = \frac{\partial}{\partial x} + U_x \frac{\partial}{\partial U} + V_x \frac{\partial}{\partial V} + U_{xx} \frac{\partial}{\partial U_x} + V_{xx} \frac{\partial}{\partial V_x} + U_{tx} \frac{\partial}{\partial U_t} + \dots \quad (2.7)$$

are known as the total derivative operators.

(4): The generalized operator is defined by

$$X = \tau \frac{\partial}{\partial t} + \xi \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial U} + \eta \frac{\partial}{\partial V} + \phi^x \frac{\partial}{\partial U_x} + \eta^x \frac{\partial}{\partial V_x} + \phi^t \frac{\partial}{\partial U_t} + \dots \quad (2.8)$$

(5): A standard Lagrangian $L = L(t, x, U, V, U_t, V_t, \dots) \in A$ (space of differential functions) and satisfies:

$$\frac{\delta L}{\delta U} = 0 \quad \text{and} \quad \frac{\delta L}{\delta V} = 0.$$

(6): The generalized operator is known as Noether operator associated with a standard Lagrangian L if it satisfies:

$$X^{[2]}L + L(\tau_t + \tau_U U_t + \tau_V V_t + \xi_x + \xi_U U_x + \xi_V V_x) = B_t^1 + B_U^1 U_t + B_V^1 V_t + B_x^2 + B_U^2 U_x + B_V^2 V_x. \quad (2.9)$$

In Eq.(2.9), B^i 's are known as the gauge terms, while $X^{[2]}$ is the second prolongation of the generator X .

(7): The equation:

$$(T^1)_t + (T^2)_x = 0 \tag{2.10}$$

evaluated on the solution space given by (2.1) is known as the conservation laws for Eq. (2.1). The vector $T = (T^1, T^2)$ is a conserved vector where T^1, T^2 are its components.

The conserved vectors of the system (2.1) associated with a Noether operator X can be determined from the formula:

$$T^i = B^i - N^i(L) \quad i = 1, 2, \tag{2.11}$$

where

$$\begin{aligned} N^1 = & \tau + (\phi - \tau U_t - \xi U_x) \frac{\delta}{\delta U_t} + (\eta - \tau V_t - \xi V_x) \frac{\delta}{\delta V_t} - D_t(\phi - \tau U_t - \xi U_x) \frac{\delta}{\delta U_{tt}} \\ & - D_t(\eta - \tau V_t - \xi V_x) \frac{\delta}{\delta V_{tt}} - D_x(\phi - \tau U_t - \xi U_x) \frac{\delta}{\delta U_{tx}} + \dots, \end{aligned} \tag{2.12}$$

$$\begin{aligned} N^2 = & \xi + (\phi - \tau U_t - \xi U_x) \frac{\delta}{\delta U_x} + (\eta - \tau V_t - \xi V_x) \frac{\delta}{\delta V_x} - D_x(\phi - \tau U_t - \xi U_x) \frac{\delta}{\delta U_{xx}} \\ & - D_x(\eta - \tau V_t - \xi V_x) \frac{\delta}{\delta V_{xx}} - D_t(\phi - \tau U_t - \xi U_x) \frac{\delta}{\delta U_{tx}} + \dots \end{aligned} \tag{2.13}$$

Following statements are taken from [3, 18, 19].

(8): For nonlocal variable ω such that $T^1 = \omega_x$ and $T^2 = -\omega_t$. While using similarity variables one can have $T^r = \omega_s$ and $T^s = -\omega_r$, where

$$T^r = \frac{T^1 D_t(r) + T^2 D_x(r)}{D_t(r) D_x(s) - D_x(r) D_t(s)}, \quad T^s = \frac{T^1 D_t(s) + T^2 D_x(s)}{D_t(r) D_x(s) - D_x(r) D_t(s)}, \tag{2.14}$$

so that $D_r T^r + D_s T^s = 0$.

Theorem 2.1. *If system (2.1) admits a nontrivial conserved form that has at least one associated symmetry in every reduction can be reduced to a non linear system of 2^{nd} order ordinary differential equations namely $T^r = K$, while T^r is defined in (2.14).*

3. Lie Point Symmetries

In this section, we will apply standard method for finding infinitesimal generators for (1.1). Using (1.1) in (2.2) one can have an over-determined system of linear PDEs that corresponds the following Lie point symmetries [13]:

$$Y_1 = t \frac{\partial}{\partial t} + \frac{x}{2} \frac{\partial}{\partial x} - u \frac{\partial}{\partial u} - \frac{3v}{2} \frac{\partial}{\partial v}, \quad Y_2 = \frac{\partial}{\partial t}, \quad Y_3 = \frac{\partial}{\partial x}, \quad Y_4 = \frac{\partial}{\partial v}. \tag{3.1}$$

After rescaling

$$\mathbf{Y}_1 = \frac{2}{3} Y_1, \quad \mathbf{Y}_2 = Y_2, \quad \mathbf{Y}_3 = Y_3 \quad \text{and} \quad \mathbf{Y}_4 = Y_4$$

By the definition of a subalgebra, one can easily conclude that there are infinite number of a one-dimensional subalgebras of

$$L = \{\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3, \mathbf{Y}_4\}.$$

The greatest or best representative of these algebras is called optimal system of one-dimensional subalgebra. In view of all these we can write an arbitrary element from L i.e.

$$\mathbf{Y} = a\mathbf{Y}_1 + b\mathbf{Y}_2 + c\mathbf{Y}_3 + d\mathbf{Y}_4, \tag{3.2}$$

where a, b, c and d are real constants. Before going to write down optimal system for the one-dimensional algebra it should be noted that \mathcal{L}_i and \mathcal{L}_j are said to be equivalent conjugacy classes if

$$\mathcal{L}_i = Ad Y_i(\mathcal{L}_j),$$

where $Y_i \in L_1$ and

$$Ad \left[\exp(tY_i) \right] Y_j = Y_j - t[Y_i, Y_j] + \frac{t^2}{2} [Y_i, [Y_i, Y_j]] - \dots$$

Now our next task is to simplify (3.2) by applying a carefully chosen adjoint transformation [16] that gives:

$$\mathcal{L}_1 = \langle \mathbf{Y}_1 \rangle, \quad \mathcal{L}_2 = \langle \mathbf{Y}_2 \rangle, \quad \mathcal{L}_3 = \langle \mathbf{Y}_3 \rangle, \quad \mathcal{L}_4 = \langle \mathbf{Y}_4 \rangle,$$

$$\mathcal{L}_5 = \langle \mathbf{Y}_4 + \varepsilon \mathbf{Y}_3 \rangle, \quad \mathcal{L}_6 = \langle \mathbf{Y}_2 + \varepsilon \mathbf{Y}_3 \rangle, \quad \mathcal{L}_7 = \langle \mathbf{Y}_4 + \varepsilon \mathbf{Y}_2 + \alpha \mathbf{Y}_3 \rangle,$$

where $\varepsilon = \pm 1$ and $\alpha \neq 0$.

3.0.1. Reduction by calculating similarity variables

< \mathbf{Y}_1 >

This operator has the following characteristic system

$$\frac{dt}{t} = \frac{dx}{\left(\frac{x}{2}\right)} = \frac{du}{-u} = \frac{dv}{\left(\frac{-3v}{2}\right)}. \quad (3.3)$$

(3.3) can be separated into the following linear equations

$$(i): \frac{dt}{t} = \frac{dx}{\left(\frac{x}{2}\right)}, \quad (ii): \frac{dt}{t} = \frac{du}{-u}, \quad (iii): \frac{dx}{\left(\frac{x}{2}\right)} = \frac{dv}{\left(\frac{-3v}{2}\right)}. \quad (3.4)$$

Integrating (3.4 (i)), (3.4 (ii)) and (3.4 (iii)) one by one we get the following results respectively:

$$\xi = \frac{t}{x^2}, \quad u = \frac{P(\xi)}{t}, \quad v = \frac{Q(\xi)}{x^3}.$$

Substituting the obtained invariants in (1.1) we obtain the following system of ordinary differential equation:

$$\begin{aligned} \frac{dP}{d\xi} + 2\Omega\xi^2 \frac{dQ}{d\xi} - \frac{P}{\xi} + 3\Omega\xi Q &= 0, \\ 8\Omega\xi^3 \frac{d^3P}{d\xi^3} + 36\Omega\xi^2 \frac{d^2P}{d\xi^2} + 8(3\xi + 2P)\Omega \frac{dP}{d\xi} - \xi \frac{dQ}{d\xi} &= 0. \end{aligned} \quad (3.5)$$

Similarly, working on the same line we discuss each class one by one.

< \mathbf{Y}_2 >

$$\xi = x, \quad u = P(\xi), \quad v = Q(\xi),$$

system after using similarity variables we have:

$$\frac{dQ}{d\xi} = 0, \quad \frac{d^3P}{d\xi^3} + 8P \frac{dP}{d\xi} = 0. \quad (3.6)$$

< \mathbf{Y}_3 >

For this case, one can have constant solution.

< \mathbf{Y}_4 >

It does not give any new solution.

< $\mathbf{Y}_4 + \varepsilon\mathbf{Y}_3$ >

$$\xi = t, \quad u = P(\xi), \quad v = \frac{x}{\varepsilon} + Q(\xi).$$

System (1.1) after using similarity variables yields the following solution:

$$v = \frac{x}{\varepsilon} + c_1, \quad u = \frac{\Omega t}{\varepsilon} + c_2. \quad (3.7)$$

< $\mathbf{Y}_2 + \varepsilon\mathbf{Y}_3$ >

Similarity variables:

$$\xi = \varepsilon t - x, \quad u = P(\xi), \quad \text{and } v = Q(\xi)$$

converts the system (1.1) into:

$$\frac{dP}{d\xi} + \Omega \frac{dQ}{d\xi} = 0, \quad \Omega^2 \frac{d^3P}{d\xi^3} + (\varepsilon + 8\Omega^2 P) \frac{dP}{d\xi} = 0. \quad (3.8)$$

< $\mathbf{Y}_4 + \varepsilon\mathbf{Y}_2 + \alpha\mathbf{Y}_3$ >

$$\xi = \varepsilon \alpha t - \varepsilon x, \quad u = P(\xi), \quad v = \frac{t}{\varepsilon} + Q(\xi).$$

System after using similarity variables we have:

$$\alpha \frac{dP}{d\xi} + \Omega \varepsilon \frac{dQ}{d\xi} = 0, \quad \Omega \varepsilon^3 \frac{d^3P}{d\xi^3} - 8\Omega P \frac{dP}{d\xi} - \alpha \frac{dQ}{d\xi} - \frac{1}{\varepsilon} = 0. \quad (3.9)$$

4. Noether operators and conservation laws

In this section, we will calculate the conserved quantities of the system of shallow-water waves propagation.

4.1. System of shallow-water waves propagation

It should be noted that system (1.1) belongs to the class of non-variational problems and hence does not possess a standard Lagrangian. In order to make system (1.1) variational let us take $u = U_x$ and $v = V_x$. The system (1.1) becomes

$$U_{tx} - \Omega V_{xx} = 0, \quad V_{tx} - \Omega \left(U_{xxx} + 8U_x U_{xx} \right) = 0. \tag{4.1}$$

The standard Lagrangian for system (4.1) is:

$$L = \frac{\Omega}{2} U_{xx}^2 + \frac{\Omega}{2} V_x^2 + \frac{4}{3} \Omega U_x^3 - \frac{1}{2} U_t V_x - \frac{1}{2} U_x V_t. \tag{4.2}$$

Using Eq.(4.2) in Eq.(2.9) and separating with respect to derivatives of U and V one gets:

$$(i) : \xi_x = 0, \quad (ii) : \xi_t = 0, \quad (iii) : \xi_U = 0, \quad (iv) : \xi_V = 0, \quad (v) : \eta_U = 0, \tag{4.3}$$

$$(i) : \tau_V = 0, \quad (ii) : \tau_U = 0, \quad (iii) : \tau_x = 0, \quad (iv) : \tau_t = 0, \quad (v) : \eta_V = 0, \tag{4.4}$$

$$(i) : \phi_U = 0, \quad (ii) : \phi_V = 0, \quad (iii) : \phi_x = 0, \quad (iv) : \phi_t = 0, \tag{4.5}$$

$$(i) : B_U^1 = -\frac{1}{2} \eta_x, \quad (ii) : B_V^1 = 0, \quad (iii) : B_U^2 = -\frac{1}{2} \eta_t, \quad (iv) : B_V^2 = \Omega \eta_x \tag{4.6}$$

and

$$B_t^1 + B_x^2 = 0. \tag{4.7}$$

The solution of Eqs. (4.3)-(4.7) is

$$\tau = c_1, \quad \xi = c_2, \quad \phi = a, \quad \eta = b + c_3 x, \tag{4.8}$$

$$B^1 = -\frac{1}{2} c_3 U + \lambda(t, x), \quad B^2 = -\frac{1}{2} b_t U - \frac{1}{2} a_t V + c_3 \Omega V + \delta(t, x). \tag{4.9}$$

where

$$\lambda_t + \delta_x = 0, \tag{4.10}$$

while a and b are any arbitrary functions of t . Without loss of generality one can take $\lambda = 0 = \delta$. Thus the Noether operators for the system (4.1) will be:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = x \frac{\partial}{\partial V}, \quad X_{(a,b)} = a \frac{\partial}{\partial U} + b \frac{\partial}{\partial V}. \tag{4.11}$$

Using (2.11) and after applying the inverse transformation, i.e. $U \rightarrow \int u dx, V \rightarrow \int v dx$ we can get the corresponding conserved vectors for the system (1.1). One can easily verify that only X_2 leads to local conserved quantity:

$$T_2^1 = -uv, \quad T_2^2 = \frac{\Omega}{2} v^2 - \frac{\Omega}{2} u_x^2 + \frac{8\Omega}{3} u^3 + \Omega u u_{xx}. \tag{4.12}$$

4.2. Double reduction by Noether approach

In this section, we will apply double reduction theory [3, 18, 19] for the reduction of the system (1.1). One can see that Y_1 and Y_2 are the associated Lie point symmetries thus using a linear transformation:

$$Y = \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial x},$$

the corresponding similarity variables will be:

$$s = t, \quad r = \beta t - x, \quad u = u(r), \quad v = v(r). \tag{4.13}$$

Using nonlocal conservation law (4.12) in (2.14) we have

$$T^r = -\beta uv - \frac{\Omega v^2}{2} + \frac{\Omega u_r^2}{2} - \frac{8\Omega u^3}{3} - \Omega u u_{rr}. \tag{4.14}$$

Using Theorem (2.1) we get:

$$-\Omega u \frac{d^2 u}{dr^2} + \frac{\Omega}{2} \left(\frac{du}{dr} \right)^2 - \frac{8\Omega u^3}{3} - \beta uv - \frac{\Omega v^2}{2} = K, \tag{4.15}$$

where K is a real constant.

So system (1.1) after reduction converts to:

$$\frac{dv}{dr} = -\beta \Omega \frac{du}{dr}, \quad -\Omega u \frac{d^2 u}{dr^2} + \frac{\Omega}{2} \left(\frac{du}{dr} \right)^2 - \frac{8\Omega u^3}{3} - \beta uv - \frac{\Omega v^2}{2} = K. \tag{4.16}$$

4.3. Conclusion

In this paper, we presented a complete classification of one dimensional subalgebra of the symmetry algebra of the system of shallow-water waves propagation up to conjugacy classes. These conjugacy classes were further used to reduce the considered system (1.1) into system of ordinary differential equation.

Further, Noether approach was applied to the non-variational third order systems of PDEs (1.1). In order to convert the considered system into variational problem the transformation $u = U_x$ and $v = V_x$ was applied. Moreover, local conserved quantity for new fourth order systems in U, V variables was computed by using Noether approach. Then an inverse transformation $U \rightarrow \int u dx, V \rightarrow \int v dx$ was used to obtain the local conserved quantity for the main problem. The obtained local conservation law with corresponding Lie point symmetries was used to get double reduction for the system (1.1).

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