Journal of Mathematical Sciences and Modelling, 1 (1) (2018) 45-55

Journal of Mathematical Sciences and Modelling

Journal Homepage: www.dergipark.gov.tr/jmsm

Existence of solutions for nonlocal boundary value problem for Caputo nonlinear fractional differential inclusion

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Article Info

Keywords: *Existence, Fractional differential inclusion, Fixed point theorem, Nonlocal boundary*
2010 AMS: *34A60, 34B15*
Received: *12 April 2018*
Accepted: *8 May 2018*
Available online: *27 May 2018*

Abstract

This paper deals with the existence of solutions for nonlinear fractional differential inclusions supplemented with three-point boundary conditions. First, we investigate it for L^1-Caratheodory convex-compact valued multifunction. Then, we investigate it for nonconvex-compact valued multifunction via some conditions. Two illustrative examples are presented at the end of the paper to illustrate the validity of our results.

1. Introduction

The concept of fractional calculus has played an important role in improving the work based on integer-order (classical) calculus in several diverse disciplines of science and engineering. In fact, quantum calculus has a rich history and the details of this basic notions, results and methods can be found in the text [2, 26, 37] and papers [10, 22]. The nonlocal nature of a fractional order differential operator, which take into account hereditary properties of various material and processes, has helped to improve the mathematical modeling of many natural phenomena and physical processes, see for example [4, 5, 21]. The increasing interest of fractional differential equations and inclusions are motivated by their applications in various fields of science such as physics chemistry, biology, economics, fluid mechanics, control theory, etc, we refer the reader to [9, 17, 30] and the references therein. Realistic problems arising from economics, optimal control, stochastic analysis can be modelled as differential inclusion. So much attention has been paid by many authors to study this kind of problems, see [4, 5, 36]. On the other hand boundary value problems with local and nonlocal boundary conditions constitute a very interesting and important class of problems. They include two, three and multipoint boundary value problems. The existence and multiplicity of positive solutions for such problems have received a great deal of attentinos. To identify a few, we refer the reader to [8, 11, 13, 18, 19, 20, 24, 25, 27, 28, 29, 31, 32, 33, 34].

In this paper, we are interested in the existence of solutions for the Caputo fractional differential inclusion

$$^{c}D^{\alpha}u(t) \in F\left(t,u(t),u'(t)\right), \quad t \in J=[0,1], \tag{1.1}$$

subject to three-point boundary conditions

$$\begin{cases} \beta u(0)+\gamma u(1)=u(\eta), \\ u(0)=\int_0^{\eta} u(s)\,ds, \\ \beta^{c}D^{p}u(0)+\gamma^{c}D^{p}u(1)={}^{c}D^{p}u(\eta), \end{cases} \tag{1.2}$$

where $2<\alpha\leq 3,\ 1<p\leq 2,\ 0<\eta<1,\ \beta,\gamma\in\mathbb{R}^{+},\ f\in C\left([0,1]\times\mathbb{R}^2,\mathbb{R}\right)$ and $^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order α.

The current paper is organized as follows. In section 2, we introduce some definitions and preliminary results that will be used in the remainder of the paper. In section 3, we present existence results for the problem $(1.1)-(1.2)$ when the right-hand side is convex-compact as well as nonconvex-compact values. In the first result we use the fixed-point theorem (Lemma 2.12) for multivalued maps (see [3]) while in the second result we prove the existence by applying a fixed-point theorem for contraction multivalued maps due to Covitz and Nadler and we give two examples to illustrate our results.

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2. Preliminaries

In this section, we introduce some necessary definitions and lemmas of fractional calculus to facilitate the analysis of the problem (1.1) − (1.2). These details can be found in the recent literature; see [1, 12, 16] and the references therein.

Definition 2.1. *Let $\alpha > 0$, $n-1 < \alpha < n$, $n = [\alpha]+1$ and $u \in C([0,\infty),\mathbb{R})$. The Caputo derivative of fractional order α for the function u is defined by*

$$^{c}D^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}(t-s)^{n-\alpha-1}u^{(n)}(s)\,ds.$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. *The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0,\infty) \to \mathbb{R}$ is given by*

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-s)^{\alpha-1}u(s)\,ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Euler gamma function, provided that the right side is pointwise defined on $(0,\infty)$.

Lemma 2.3. *[22] Let $\alpha, \beta \geq 0$ and $u \in L^{p}(0,1)$, $0 \leq p \leq +\infty$. Then the next formulas hold.*
(i) $\left(I^{\beta}I^{\alpha}u\right)(t) = I^{\alpha+\beta}u(t)$,
(ii) $\left(D^{\beta}I^{\alpha}u\right)(t) = I^{\alpha-\beta}u(t)$,
(iii) $(D^{\alpha}I^{\alpha}u)(t) = u(t)$.

Lemma 2.4. *[26] Let $\alpha > 0$, $n-1 < \alpha < n$ and the function $g : [0,T] \to \mathbb{R}$ be continuous for each $T > 0$. Then, the general solution of the fractional differential equation $^{c}D^{\alpha}g(t) = 0$ is given by*

$$g(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha]+1$.

Lemma 2.5. *[4] Assume that $u \in C[0,1] \cap L^{1}(0,1)$ with a Caputo fractional derivative of order $\alpha > 0$ that belongs to $u \in C^{n}[0,1]$, then*

$$I^{\alpha\,c}D^{\alpha}u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha]+1$.

We will present notations, definitions and preliminary facts from multivalued analysis which are used throughout this paper. Here $(C[0,1],\mathbb{R})$ denotes the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm $\|u\| = sup\{|u(t)| : for\,all\, t \in [0,1]\}$, $L^{1}([0,1],\mathbb{R})$, the Banach space of measurable functions $u : [0,1] \to \mathbb{R}$ which are Lebesgue integrable, normed by $\|u\|_{L^1} = \int_0^1 |u(t)|\,dt$, and $AC^{i}([0,1],\mathbb{R})$ the space of $i-times$ differentiable functions $u : [0,1] \to \mathbb{R}$ whose $i-th$ derivative $u^{(i)}$ is absolutely continuous.
Let (X,d) be a metric space induced from the normed space $(X,\|\cdot\|)$. We denote

$$P_0(X) = \{A \in P(X) : A \neq \phi\},$$

$$P_b(X) = \{A \in P_0(X) : A\, is\, bounded\},$$

$$P_{cl}(X) = \{A \in P_0(X) : A\, is\, closed\},$$

$$P_{cp,cv}(X) = \{A \in P_0(X) : A\, is\, compact\ and\ convex\},$$

$$P_{b,cl}(X) = \{A \in P_0(X) : A\, is\, closed\ and\ bounded\}.$$

Definition 2.6. *A multivalued maps $G : X \to P(X)$.*
(1) *is convex (closed) valued if $G(X)$ is convex (closed) for all $u \in X$,*
(2) *is bounded on bounded sets if $G(B) = \bigcup_{u\in B} G(u)$ is bounded in X for all $B \in P_b(X)$ i.e. $\sup_{u\in B}\{sup\{|v|, v \in G(u)\}\} < \infty$,*
(3) *is called upper semicontinuous (u.s.c) on X if for each $u_0 \in X$, the set $G(u_0)$ is a nonempty closed subset of X and if for each open set N of X containing $G(u_0)$ there exists an open neighborhood N_0 of u_0 such that $G(N_0) \subseteq N$,*
(4) *is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(X)$,*
(5)*has a fixed point if there is $u \in X$ such that $u \in G(X)$. The fixed point set of the multi-valued operator G will be denote by Fix G.*

Remark 2.7. *It is well known that, if the multi-valued map G is completely continuous with nonempty compact values, then G is u.s.c if and only if G has closed graph i.e., $u_n \to u$, $v_n \to v$, $v_n \in G(u_n)$ imply $v \in G(u)$.*

Definition 2.8. *A multi-valued maps $G : [0,1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$ the function*

$$t \longmapsto d(y, G(t)) = inf\{\|y-z\| : z \in G(t)\},$$

is measurable.

Definition 2.9. *A multi-valued maps $F : [0,1] \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is said to be Caratheodory if,*
(i) $t \longmapsto F(t,u,v)$ for all $u, v \in \mathbb{R}$,
(ii) $t \longmapsto F(t,u,v)$ is upper semi-continuous for almost all $t \in [0,1]$. Further a Caratheodory function is called L^1- Caratheodory,
(iii) for each $\rho > 0$, there exists $\phi_\rho \in L^1([0,1], \mathbb{R}^+)$ such that $\|F(t,u,v)\| = sup\{|v|, v \in F(t,u,v)\} \leq \phi_\rho(t)$, for all $|u|, |v| < \rho$.

Definition 2.10. *Let Y be a nonempty closed subset of a Banach spase E and $G : Y \to P_{cl}(E)$ be a multivalued operator with nonempty closed values. G is said to be lower semicontinuous (l.s.c) if the set $\{x \in X : G(x) \cap U \neq \phi\}$ is open for any open set U in E.*

For $y \in (C[0,1], \mathbb{R})$, define the set of selection of F by

$$S_{F,u} = \left\{v \in AC([0,1], \mathbb{R}), v \in F\left(t, u(t), u'(t)\right),\ for\ almost\ all\ t \in [0,1]\right\}.$$

For $P(X) = 2^X$, consider the Pompeiu-Hausdorff metric (see[**?**])
$H_d : 2^{\mathbb{X}} \times 2^{\mathbb{X}} \to [0,\infty)$ given by

$$H_d(A,B) = max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where $d(a,B) = \inf_{b\in B} d(a,b)$ and $d(b,A) = \inf_{a\in A} d(a,b)$. Then $\left(P_{b,cl}(X), H_d\right)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space see [7].

Let Y be a nonempty closed subset of a Banach spase E and $G : Y \to P_{cl}(E)$ be a multivalued operator with nonempty closed values.
G is said to be lower semicontinuous (l.s.c) if the set $\{x \in X : G(x) \cap U \neq \phi\}$ is open for any open set U in E.
G has a fixed point if there is $x \in Y$ such that $x \in G(x)$. For more details on the multi-valued maps, see the book of Aubin and Celina [14], Demling [15], Gorniewicz [16] and Hu and Papageorgiou [35].

Lemma 2.11. *[1] Let X be a Banach space. $F : [0,1] \times \mathbb{X} \to P_{cp,cv}(X)$ an L^1-Caratheodory multifunction and Θ a linear continuous mapping from $L^1([0,1], X)$ to $C([0,1], X)$. Then the operator $(\Theta \circ S_F)(u) = \Theta\left(S_{F,u}\right)$ is a closed graph operator in $C([0,1], X) \times C([0,1], X)$.*

Lemma 2.12. *[3] Let E be a Banach space. C a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to P_{cp,cv}(C)$ is an upper semi-continuous compact map, where $P_{cp,cv}(C)$ denotes the family of nonempty, compact convex subset of C. Then either F has a fixed point in $\overline{U}$ or there exist $u \in \partial U$ and $\lambda \in (0,1)$ such that $u \in \lambda F(U)$.*

Lemma 2.13. *[12] A multifunction $F : X \to C(X)$ is called a contraction whenever there exists $\gamma \in (0,1)$ such that $H_d(N(u), N(v)) \leq \gamma d(u,v)$ for all $u, v \in X$*

Lemma 2.14. *(Covitz-Nadler) Let (X,d) be a complete metric space. If $N : X \to P_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

3. Existence results

Let $X = \{u : u, u' \in C([0,1], \mathbb{R})\}$ endowed with the norm defined by $\|u\| = \sup_{t\in[0,1]} |u(t)| + \sup_{t\in[0,1]} |u'(t)|$ such that $\|u\| < \infty$. Then $(X, \|.\|)$ is a Banach space.

Lemma 3.1. *Let $y \in C([0,1], \mathbb{R})$. Then the integral solution of the linear fractional differential equation*

$$^cD^\alpha u(t) = y(t) \quad t \in [0,1],\ \alpha \in (2,3], \tag{3.1}$$

subject to three-point boundary conditions

$$\beta u(0) + \gamma u(1) = u(\eta), \quad \beta \geq 0,\ \gamma \geq 0, \tag{3.2}$$

$$u(0) = \int_0^\eta u(s)\,ds, \quad \eta \in (0,1), \tag{3.3}$$

$$\beta^c D^p u(0) + \gamma^c D^p u(1) =^c D^p u(\eta), \quad p \in (1,2], \tag{3.4}$$

is given by

$$\begin{aligned} u(t) \quad = \quad & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\,d\tau\right) ds - \frac{\Lambda_1(t)}{Q_1(1-\eta)}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\,d\tau\right) ds \\ & - \frac{\Lambda_2(t) M_1}{6(1-\eta) Q_1}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds\right] \\ & + \frac{\Lambda_1(t)}{Q_1(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds\right], \end{aligned} \tag{3.5}$$

where

$$\Lambda_1(t) = (\beta+\gamma-1)\left(\eta^2+2(1-\eta)t\right), \; M_1 = \frac{\Gamma(3-p)}{\gamma-\eta^{2-p}}$$

$$\Lambda_2(t) = \left(\eta^3(\beta+\gamma-1)+3\left(\gamma-\eta^2\right)(1-\eta)\right)\left(\eta^2+2(1-\eta)t\right) - Q_1\left(\eta^3+3(1-\eta)t^2\right),$$

and

$$Q_1 = 2(1-\eta)(\gamma-\eta)+\eta^2(\beta+\gamma-1) \neq 0.$$

Proof. In view of Lemma 2.3 and Lemma 2.5, the solution of equation (3.1) can be written as

$$u(t) = I^{\alpha}y(t)+c_0+c_1t+c_2t^2 = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds+c_0+c_1t+c_2t^2, \tag{3.6}$$

where $c_0, c_1, c_2 \in \mathbb{R}$ are arbitrary constants.
Differentiating both sides of (3.6) and applying Definition 2.1, Lemma 2.3 and Lemma 2.5, we obtain

$${}^cD^pu(t) = I^{\alpha-p}y(t)+c_2\frac{2t^{2-p}}{\Gamma(3-p)} = \int_0^t \frac{(t-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds+\frac{2t^{2-p}}{\Gamma(3-p)}c_2, \tag{3.7}$$

where $\alpha \in (2,3]$ and $p \in (1,2]$.
Integrating both sides of (3.6), we obtain

$$\int_0^\eta u(t)\,dt = \int_0^\eta \left(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\right)dt+c_0\eta+\frac{1}{2}c_1\eta^2+\frac{1}{3}c_2\eta^3. \tag{3.8}$$

By using the boundary condition (3.2) in (3.6), we obtain

$$c_0(\beta+\gamma-1)+c_1(\gamma-\eta)+c_2\left(\gamma-\eta^2\right) = \int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds \tag{3.9}$$

By using the boundary condition (3.3) in (3.6) and (3.8), we obtain

$$(1-\eta)c_0-\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\frac{1}{2}c_1\eta^2-\frac{1}{3}\eta^3 = 0. \tag{3.10}$$

By using the boundary condition (3.4) in (3.7), we obtain

$$c_2 = \frac{M_1}{2}\left(\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds-\gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds\right). \tag{3.11}$$

Solving the above system of the equations (3.9), (3.10) and (3.11) for c_0, c_1, c_2, we get

$$c_2 = \frac{M_1}{2}\left(I^{\alpha-p}y(\eta)-\gamma I^{\alpha-p}y(1)\right) = \frac{M_1}{2}\left(\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds-\gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds\right),$$

$$\begin{aligned} c_0 \;=\; & -\frac{2\eta^2(\beta+\gamma-1)}{2(1-\eta)Q_1}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds+\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds \\ & -\frac{\left(\eta^2\left[\eta^3(\beta+\gamma-1)+3\left(\gamma-\eta^2\right)(1-\eta)\right]-\eta^3Q_1\right)M_1}{2(1-\eta)Q_1}\left[I^{\alpha-p}y(\eta)-\gamma I^{\alpha-p}y(1)\right] \\ & +\frac{\eta^2}{Q_1}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\right], \end{aligned}$$

and

$$\begin{aligned} c_1 \;=\; & \frac{-2(\beta+\gamma-1)}{Q_1}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds-\frac{\left(\eta^3(\beta+\gamma-1)+3\left(\gamma-\eta^2\right)(1-\eta)\right)M_1}{3Q_1}\left[I^{\alpha-p}y(\eta)-\gamma I^{\alpha-p}y(1)\right] \\ & +\frac{2(1-\eta)}{Q_1}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\right], \end{aligned}$$

where

$$I^{\alpha-p}y(\eta)-\gamma I^{\alpha-p}y(1)=\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds-\gamma\int_0^{1}\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds.$$

Substituting the values of constants c_0, c_1 and c_2 in (3.6), we get (3.5). The proof is complete. □

Throughout the paper, we let

$$M=\frac{\Gamma(3-p)}{|\gamma-\eta^{2-p}|}\neq 0,\ |\beta+\gamma-1|\neq 0,\ \left|\gamma-\eta^2\right|\neq 0,\ \ Q=\left|2(1-\eta)(\gamma-\eta)+\eta^2|\beta+\gamma-1|\right|\neq 0,$$

$$A(t)=|\beta+\gamma-1|\left(\eta^2+2(1-\eta)t\right),$$

and

$$B(t)=\left(\eta^3|\beta+\gamma-1|+3\left|\gamma-\eta^2\right|(1-\eta)\right)\left(\eta^2+2(1-\eta)t\right)-Q\left(\eta^3+3(1-\eta)t^2\right).$$

The following inequalities hold:

$$|A(t)|\leq|\beta+\gamma-1|\left(\eta^2+2(1-\eta)\right)=A_1,$$

$$|B(t)|\leq\left|\left(\eta^3|\beta+\gamma-1|+3\left|\gamma-\eta^2\right|(1-\eta)\right)\left(\eta^2+2(1-\eta)\right)-Q\left(\eta^3+3(1-\eta)\right)\right|=B_1,$$

$$\left|A'(t)\right|\leq 2|\beta+\gamma-1|(1-\eta)=A_1',$$

and

$$\left|B'(t)\right|\leq 2(1-\eta)\left|\left(\eta^3|\beta+\gamma-1|+3\left|\gamma-\eta^2\right|(1-\eta)\right)-3Q\right|=B_1'.$$

To simplify the proofs in the forthcoming theorems, we establish the bounds for the integrals arising in the sequel.

Lemma 3.2. *For $y\in C([0,1],\mathbb{R})$, we have*

$$\left|\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds\right|\leq\frac{\eta^{\alpha+1}}{\Gamma(\alpha+2)}\|y\|.$$

Proof. Obviously,

$$\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}d\tau=\left[-\frac{(s-\tau)^{\alpha}}{\Gamma(\alpha)}\right]_0^{s}=\frac{s^{\alpha}}{\alpha\Gamma(\alpha)}=\frac{s^{\alpha}}{\Gamma(\alpha+1)}.$$

Hence

$$\left|\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds\right|\leq\|y\|\int_0^{\eta}\frac{s^{\alpha}}{\Gamma(\alpha+1)}ds=\frac{\eta^{\alpha+1}}{\Gamma(\alpha+2)}\|y\|.$$

□

For the sake of brevity, we set

$$\triangle_1=\frac{\eta^{\alpha+1}}{(1-\eta)\Gamma(\alpha+2)}+\frac{A_1\eta^{\alpha+1}}{Q(1-\eta)\Gamma(\alpha+2)}+\frac{MB_1\left(\eta^{\alpha-p}+\gamma\right)}{(1-\eta)Q\Gamma(\alpha-p+1)}+\frac{A_1\left(\eta^{\alpha}+\gamma\right)}{Q|\beta+\gamma-1|\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha+1)}, \tag{3.12}$$

and

$$\triangle_2=\frac{A_1'\eta^{\alpha+1}}{Q(1-\eta)\Gamma(\alpha+2)}+\frac{MB_1'\left(\eta^{\alpha-p}+\gamma\right)}{(1-\eta)Q\Gamma(\alpha-p+1)}+\frac{A_1'\left(\eta^{\alpha}+\gamma\right)}{Q|\beta+\gamma-1|\Gamma(\alpha+1)}+\frac{1}{\Gamma(\alpha)}. \tag{3.13}$$

An element $u\in AC^2([0,1,\mathbb{R}])$ is called a solution of the problem (1.1) whenever it satisfies the integral boundary conditions and there exists a function $y\in S_{F,u}$ such that

$$\begin{aligned}u(t) &= \int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds+\frac{1}{1-\eta}\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds-\frac{B(t)M}{6(1-\eta)Q}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds-\gamma\int_0^{1}\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds\right]\\ &\quad+\frac{A(t)}{Q|\beta+\gamma-1|}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\gamma\int_0^{1}\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds.\right]-\frac{A(t)}{Q(1-\eta)}\int_0^{\eta}\left(\int_0^{s}\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds.\end{aligned} \tag{3.14}$$

for all $t\in J$.
For investigation of the problem (1.1) – (1.2) we provide two different methods.

Theorem 3.3. . *Suppose that* $F : J\times\mathbb{R}\times\mathbb{R}\to P_{cp,cv}(\mathbb{R})$ *is* L^1-*Caratheodory multifunction and there exist a bounded continuous nondecreasing map*
$\psi : [0,\infty)\to(0,\infty)$ *and a continuous function* $p : J\to(0,\infty)$ *such that*
$\|F(t,u(t)),u'(t)\| = sup\{|v| : v\in F(t,u(t),u'(t))\le p(t)\psi(\|u\|)\}$, *for all* $t\in J$ *and* $u\in X$. *Then the inclusion problem* (1.1) − (1.2) *has at least one solution.*

Proof. Define the operator

$$N(u)=\left\{\begin{array}{c} h\in X,: \text{there exists } y\in S_{F,u} \text{ such that } h(t)=\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds \\ -\frac{B(t)M}{6(1-\eta)Q}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y(s)\,ds\right] \\ +\frac{A(t)}{Q|\beta+\gamma-1|}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds\right] \\ +\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{}y(\tau)\,d\tau\right)ds-\frac{A(t)}{Q(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y(\tau)\,d\tau\right)ds, t\in J \end{array}\right\}$$

We show that the operator N has a fixed point. First, we show that N maps bounded sets of X into bounded sets. Suppose that $r>0$ and $B_r=\{u\in X : \|u\|\le r\}$. Let $u\in B_r$ and $h\in N(u)$. Choose $v\in S_{F,u}$ such that $h(t)$ defined above for almost all $t\in J$. Thus

$$\begin{aligned}|h(t)| &\le \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds+\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y(\tau)|\,d\tau\right)ds+\frac{|A(t)|}{Q(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y(\tau)|\,d\tau\right)ds \\ &\quad+\frac{|B(t)|M}{12(1-\eta)Q}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y(s)|\,ds+\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y(s)|\,ds\right] \\ &\quad+\frac{(\gamma-\eta)|A(t)|}{2Q(\beta+\gamma-1)}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds+\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds\right] \\ &\le \Delta_1\|p\|_\infty\psi(\|u\|),\end{aligned}$$

and

$$\begin{aligned}|h'(t)| &\le \int_0^t\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}|y(s)|\,ds+\frac{|A'(t)|}{Q(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y(\tau)|\,d\tau\right)ds \\ &\quad+\frac{|B'(t)|M}{12(1-\eta)Q}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y(s)|\,ds+\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y(s)|\,ds\right] \\ &\quad+\frac{(\gamma-\eta)|A'(t)|}{2Q(\beta+\gamma-1)}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds+\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds\right] \\ &\le \Delta_2\|p\|_\infty\psi(\|u\|),\end{aligned}$$

for all $t\in J$, where $\|p\|_\infty=\sup_{t\in J}|p(t)|$.
Hence,

$$\|h\|=\max_{t\in J}|h(t)|+\max_{t\in J}|h'(t)|\le(\Delta_1+\Delta_2)\|p\|_\infty\psi(\|u\|)$$

Now, we show that N maps bounded sets into equicontinuous subsets of X. Let $u\in B_r$ and $t_1,t_2\in J$ with $t_1<t_2$. Then we have

$$\begin{aligned}|h(t_2)-h(t_1)| &\le \int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds+\int_0^{t_1}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds \\ &\quad+\frac{|B(t_2)-B(t_1)|M}{12(1-\eta)Q}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y(s)|\,ds+\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y(s)|\,ds\right] \\ &\quad+\frac{(\gamma-\eta)|A(t_2)-A(t_1)|}{2Q(\beta+\gamma-1)}\left[\int_0^\eta\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds+\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds\right] \\ &\quad+\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y(\tau)|\,d\tau\right)ds+\frac{|A(t_2)-A(t_1)|}{Q(1-\eta)}\int_0^\eta\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y(\tau)|\,d\tau\right)ds\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^{t_1} \frac{(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\|p\|_\infty \psi(\|u\|)\right]ds+\int_{t_1}^{t_2}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\|p\|_\infty \psi(\|u\|)\right]ds\\
&+\frac{|B(t_2)-B(t_1)|M}{12(1-\eta)Q}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left[\|p\|_\infty \psi(\|u\|)\right]ds+\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left[\|p\|_\infty \psi(\|u\|)\right]ds\right]\\
&+\frac{(\gamma-\eta)|A(t_2)-A(t_1)|}{2Q(\beta+\gamma-1)}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\|p\|_\infty \psi(\|u\|)\right]ds+\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\|p\|_\infty \psi(\|u\|)\right]ds\right]\\
&+\frac{1}{1-\eta}\int_0^{\eta}\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\left[\|p\|_\infty \psi(\|u\|)\right]d\tau\right)ds+\frac{|A(t_2)-A(t_1)|}{Q(1-\eta)}\int_0^{\eta}\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\|p\|_\infty \psi(\|u\|)\,d\tau\right)ds,
\end{aligned}
$$

It is seen that $|(h)(t_2)-(h)(t_1)|\to 0$, as $t_2\to t_1$. Also, we have

$$
\begin{aligned}
\left|h'(t_2)-h'(t_1)\right| &\leq \int_0^{t_2}\frac{(t_2-s)^{\alpha-2}}{\Gamma(\alpha-1)}|y(s)|\,ds+\int_0^{t_1}\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}|y(s)|\,ds\\
&+\frac{|B'(t_2)-B'(t_1)|M}{12(1-\eta)Q}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y(s)|\,ds+\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y(s)|\,ds\right]\\
&+\frac{(\gamma-\eta)|A'(t_2)-A'(t_1)|}{2Q(\beta+\gamma-1)}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds+\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|y(s)|\,ds\right]\\
&+\frac{|A'(t_2)-A'(t_1)|}{Q(1-\eta)}\int_0^{\eta}\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y(\tau)|\,d\tau\right)ds\\
&\leq \int_0^{t_1}\frac{(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-1}}{\Gamma(\alpha-1)}\left[\|p\|_\infty \psi(\|u\|)\right]ds+\int_{t_1}^{t_2}\frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}\left[\|p\|_\infty \psi(\|u\|)\right]ds\\
&+\frac{|B'(t_2)-B'(t_1)|M}{12(1-\eta)Q}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left[\|p\|_\infty \psi(\|u\|)\right]ds+\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}\left[\|p\|_\infty \psi(\|u\|)\right]ds\right]\\
&+\frac{(\gamma-\eta)|A'(t_2)-A'(t_1)|}{2Q(\beta+\gamma-1)}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\|p\|_\infty \psi(\|u\|)\right]ds+\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}\left[\|p\|_\infty \psi(\|u\|)\right]ds\right]\\
&+\frac{|A'(t_2)-A'(t_1)|}{Q(1-\eta)}\int_0^{\eta}\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}\|p\|_\infty \psi(\|u\|)\,d\tau\right)ds.
\end{aligned}
$$

Again, we see that $|(h)(t_2)-(h)(t_1)|\to 0$, as $t_2\to t_1$. Also, we have $\|(h)(t_2)-(h)(t_1)\|\to 0$, as $t_2\to t_1$. Thus N is equicontinuous and so N is relatively compact on B_r. Consequently the Ascoli-Arzela theorem implies that N is compact on B_r.

Now, we show that N has a closed graph. Let $u_n\to u_0,\ h_n\in N(u_n)$ for all n and $h\to h_0$. We prove that $h_0\in N(u_0)$. For each n choose $y_n\in S_{F,u_n}$ such that, for all $t\in J$,

$$
\begin{aligned}
h_n(t) &= \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y_n(s)\,ds+\frac{1}{1-\eta}\int_0^{\eta}\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y_n(\tau)\,d\tau\right)ds-\frac{A(t)}{Q(1-\eta)}\int_0^{\eta}\left(\int_0^s\frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}y_n(\tau)\,d\tau\right)ds\\
&-\frac{B(t)M}{12(1-\eta)Q}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y_n(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}y_n(s)\,ds\right]\\
&+\frac{(\gamma-\eta)A(t)}{2Q(\beta+\gamma-1)}\left[\int_0^{\eta}\frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}y_n(s)\,ds-\gamma\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y_n(s)\,ds\right].
\end{aligned}
$$

Consider the continuous linear operator $\theta : L^1(J,\mathbb{R}) \to X$ defined by

$$\begin{aligned}\theta(y)(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\,d\tau\right) ds - \frac{A(t)}{Q(1-\eta)}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\,d\tau\right) ds \\ &\quad - \frac{B(t)M}{12(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds\right] \\ &\quad + \frac{(\gamma-\eta)A(t)}{2Q(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds\right].\end{aligned}$$

By using Lemma 2.13, $\theta \circ S_F$ is closed graph operator. Since $u_n \to u$ and $h_n \in \theta\left(S_{F,u_n}\right)$ for all $n \in \mathbb{N}$, there exist $y_0 \in S_{F,,u_0}$ such that

$$\begin{aligned}h_0(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y_0(s)\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_0(\tau)\,d\tau\right) ds - \frac{A(t)}{Q(1-\eta)}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_0(\tau)\,d\tau\right) ds \\ &\quad - \frac{B(t)M}{12(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_0(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_0(s)\,ds\right] \\ &\quad + \frac{(\gamma-\eta)A(t)}{2Q(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y_0(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y_0(s)\,ds\right].\end{aligned}$$

Thus N has a closed graph.
Now we show that $N(u)$ is convex for all $u \in X$. Let $h_1, h_2 \in N(u)$ and $w \in [0,1]$. Choose $y_1, y_2 \in S_{F,u}$. Then

$$\begin{aligned}wh_1(t) - (1-w)h_2(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} [wy_1(s) - (1-w)y_2(s)]\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} [y_1(\tau) - (1-w)y_2(\tau)]\,d\tau\right) ds \\ &\quad - \frac{A(t)}{Q(1-\eta)}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} [y_1(\tau) - (1-w)y_2(\tau)]\,d\tau\right) ds \\ &\quad - \frac{B(t)M}{12(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} [wy_1(s) - (1-w)y_2(s)]\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} [wy_1(s) - (1-w)y_2(s)]\,ds\right] \\ &\quad + \frac{(\gamma-\eta)A(t)}{2Q(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} [wy_1(s) - (1-w)y_2(s)]\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} [wy_1(s) - (1-w)y_2(s)]\,ds\right]\end{aligned}$$

for all $t \in J$. Since F has convex values, $S_{F,u}$ is convex and so $wh_1(t) - (1-w)h_2(t) \in N(u)$.
If there exists $\lambda \in (0,1)$ such that $u \in \lambda N(u)$ then there exists $y \in S_{F,u}$ such that

$$\begin{aligned}u(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\,d\tau\right) ds - \frac{A(t)}{Q(1-\eta)}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y(\tau)\,d\tau\right) ds \\ &\quad - \frac{B(t)M}{12(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y(s)\,ds\right] \\ &\quad + \frac{(\gamma-\eta)A(t)}{2Q(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds\right],\end{aligned}$$

for almost all $t \in J$. Choose $L > 0$ such that $\frac{L}{(\Delta_1+\Delta_2)\|p\|_\infty \psi(\|u\|)} > 1$ for all $u \in X$. Thus $\|u\| < L$. Now, put $U = \{u \in X : \|u\| < L+1\}$. Note there are no $u \in \partial U$ and $\lambda \in (0,1)$ such that $u \in \lambda N(u)$ and the operator $N : \overline{U} \to P_{cp,cv}\left(\overline{U}\right)$ is upper semi-continuous because it is completely continuous. Now, by using Lemma 2.12, N has fixed point in $\overline{U}$ which is solution of the inclusion problem (1.1). This complete the proof. □

We provide another result about the existence of solutions for the problem $(1.1)-(1.2)$ by changing the assumptions of convex values for multifunction.

Theorem 3.4. *Let $m \in C\left(J, \mathbb{R}^+\right)$ be such that $\|m\|_\infty (\Delta_1 + \Delta_2) < 1$.*
Suppose that $F : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to P_{cp}(\mathbb{R})$ is an integrable bounded multifunction such that the map $t \longmapsto F(t,u,v,w)$ is measurable and $H_d\left(F\left(t,u_1,u_2,u_3\right), F\left(t,v_1,v_2,v_3\right)\right) \le m(t)\left(|u_1-v_1| + |u_2-v_2| + |u_3-v_3|\right)$ for almost all $t \in J$ and $u,v,w,u_1,u_2,u_3,v_1,v_2,v_3 \in \mathbb{R}$. Then the problem $(1.1)-(1.2)$ has a solution.

Proof. Note that, the multivalued map $t \longmapsto F(t,u(t),v(t),w(t))$ is measurable and closed for all $u \in X$. Hence, it has a measurable selection and so the set $S_{F,u}$ is nonempty. Now, consider the operator $N : X \to 2^X$ defined by

$$N(u) = \left\{h \in X : \textit{there exists } v \in S_{F,u} \textit{ such that } h(t) = u(t),\ t \in J\right\},$$

where $u(t)$ defined by (3.5), for all $t \in J$.
First, we show that $N(u)$ is a closed subset of X for all $u \in X$. Let $u \in X$ and $\{u_n\}_{n\geq 1}$ be a sequence in $N(u)$ with $u_n \to u$ for each n, such that

$$\begin{aligned} u_n(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y_n(s)\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_n(\tau)\,d\tau\right) ds - \frac{A(t)}{Q(1-\eta)}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_n(\tau)\,d\tau\right) ds \\ &\quad - \frac{B(t)M}{12(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_n(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_n(s)\,ds\right] \\ &\quad + \frac{(\gamma-\eta)A(t)}{2Q(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y_n(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y_n(s)\,ds\right], \end{aligned}$$

for almost all $t \in J$. Since F has compact values, $\{y_n\}_{n\geq 1}$ has a subsequence which converges to some $y \in L^1(J,\mathbb{R})$. It is easy to check that $y \in S_{F,u}$ and $u_n(t) \to u(t)$ for all $t \in J$. This implies that $u \in N(u)$. Thus the multifunction N has closed values.
Now, we show that N is a contractive multifunction with constant
$l = \|m\|_\infty(\Delta_1+\Delta_2) < 1$.
Let u, $v \in X$ and $h_1 \in N(v)$. Choose $y_1 \in S_{F,,v}$ such that

$$\begin{aligned} h_1(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y_1(s)\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_1(\tau)\,d\tau\right) ds - \frac{A(t)}{Q(1-\eta)}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_1(\tau)\,d\tau\right) ds \\ &\quad - \frac{B(t)M}{12(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_1(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_1(s)\,ds\right] \\ &\quad + \frac{(\gamma-\eta)A(t)}{2Q(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y_1(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y_1(s)\,ds\right], \end{aligned}$$

for almost all $t \in J$. Since
$H_d(F(t,u(t),u'(t)),F(t,v(t),v'(t))) \leq m(t)(|u(t)-v(t)|+|u'(t)-v'(t)|)$ for almost all $t \in J$ there exists $w \in F(t,u(t),u'(t))$ such that

$$|y_1(t)-w| \leq m(t)\left(|u(t)-v(t)|+\left|u'(t)-v'(t)\right|\right),$$

for almost all $t \in J$.
Define the multifunction $U : J \to 2^{\mathbb{R}}$ by

$$U(t) = \left\{w \in \mathbb{R} : |y_1(t)-w| \leq m(t)\left(|u(t)-v(t)|+\left|u'(t)-v'(t)\right|\right)\ \textit{for almost all } t \in J\right\}.$$

It is easy to chek that the multifunction $U(\cdot)\cap F(\cdot),u(\cdot),u'(\cdot)$ is measurable. Thus, we can choose $y_2 \in S_{F,u}$ such that

$$|y_1(t)-y_2(t)| \leq m(t)\left(|u(t)-v(t)|+\left|u'(t)-v'(t)\right|\right),$$

for almost all $t \in J$. Now, consider $h_2 \in N(u)$ which is defined by

$$\begin{aligned} h_2(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y_2(s)\,ds + \frac{1}{1-\eta}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_2(\tau)\,d\tau\right) ds - \frac{A(t)}{Q(1-\eta)}\int_0^\eta \left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)} y_2(\tau)\,d\tau\right) ds \\ &\quad - \frac{B(t)M}{12(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_2(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)} y_2(s)\,ds\right] \\ &\quad + \frac{(\gamma-\eta)A(t)}{2Q(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)} y_2(s)\,ds - \gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y_2(s)\,ds\right]. \end{aligned}$$

Hence, we get

$$\begin{aligned} |h_1(t)-h_2(t)| &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|y_1(s)-y_2(s)|ds+\frac{1}{1-\eta}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y_1(\tau)-y_2(\tau)|d\tau\right)ds \\ &+\frac{|A(t)|}{Q(1-\eta)}\int_0^\eta\left(\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)}|y_1(\tau)-y_2(\tau)|d\tau\right)ds \\ &+\frac{|B(t)|M}{12(1-\eta)Q}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y_1(s)-y_2(s)|ds+\gamma\int_0^1 \frac{(1-s)^{\alpha-p-1}}{\Gamma(\alpha-p)}|y_1(s)-y_2(s)|ds\right] \\ &+\frac{(\gamma-\eta)|A(t)|}{2Q(\beta+\gamma-1)}\left[\int_0^\eta \frac{(\eta-s)^{\alpha-1}}{\Gamma(\alpha)}|y_1(s)-y_2(s)|ds+\gamma\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|y_1(s)-y_2(s)|ds\right] \\ &\le \Delta_1\|m\|_\infty\|u-v\|, \end{aligned}$$

and so $\|h_1-h_2\|\le(\triangle_1+\triangle_2)\|m\|_\infty\|u-v\|=l\|u-v\|$. This implies that the multifunction N is a contraction which closed values. Thus, by using the result of Covitz and Nadler, N has a fixed point which is solution for the inclusion problem.

□

We construct two examples to illustrate the applicability of the results presented.

Example 3.5. *Consider the problem*

$$^cD^3u(t)\in F(t,u,v),\quad t\in[0,1], \tag{3.15}$$

subject to the three-point boundary conditions

$$\begin{cases} \frac{1}{100}u(0)+\frac{1}{10}u(1)=u\left(\frac{1}{2}\right), \\ u(0)=\int_0^{0,5}u(s)ds, \\ \frac{1}{100}{}^cD^{\frac{3}{2}}u(0)+\frac{1}{10}{}^cD^{\frac{3}{2}}u(1)={}^cD^{\frac{3}{2}}u\left(\frac{1}{2}\right) \end{cases}, \tag{3.16}$$

where $\eta=0,5,\beta=0,01,\gamma=0,1,p=1,5$ and $F(t,u,v):[0,1]\times\mathbb{R}^2\to2^{\mathbb{R}}$ multivalued map given by

$$u\longmapsto F(t,u,v)=\left(\left(\frac{3+t^2}{4}\right)\left(\frac{|u|}{1+u}+sin(v)\right),\frac{|u|^3}{2\left(1+|u|^3\right)}+5t^3+4\right),\ u,v\in\mathbb{R}$$

verifying (H_1).
Obviously, for $f\in F$, we have

$$|f|=max\left(\left(\frac{3+t^2}{4}\right)\left(\frac{|u|}{1+|u|}+sin(v)\right),\frac{|u|^3}{2\left(1+|u|^3\right)}+5t^3+4\right)\le\frac{19}{2},\ u,v\in\mathbb{R}.$$

Thus

$$\|F(t,u)\|=sup\{|f|:f\in F(t,u,v)\}\le\frac{19}{2},\ u,v\in\mathbb{R},$$

where $p(t)=1$ and $\psi(t)=\frac{19}{2}$, then one can check that the assumptions of Theorem 3.3 hold. and so the problem (3.15) – (3.16) has at least one solution.

Example 3.6. *Consider the problem (3.15) – (3.16), where $\eta=0,5,\beta=0,01,\gamma=0,1,p=1,5$ and $F(t,u,v):[0,1]\times\mathbb{R}^2\to2^{\mathbb{R}}$ multivalued map given by*

$$u\longmapsto F(t,u,v)=\left(0,\frac{t|u|}{2(1+|u|)}+\frac{|v|^3}{2\left(1+|v|^3\right)}\right),\ u,v\in\mathbb{R}.$$

Obviously,

$$H_d(F(t,u_1,u_2),F(t,v_1,v_2))\le\left(\frac{t}{2}+\frac{1}{2}\right)\sum_{i=1}^2|u_i-v_i|,\ u,v\in\mathbb{R},\ t\in[0,1].$$

If $m(t)=\frac{t}{2}+\frac{1}{2}$ for all $t\in[0,1]$ $H_d(F(t,u_1,u_2),F(t,v_1,v_2))\le m(t)\sum_{i=1}^2|u_i-v_i|$.
On the other hand, it can be easily found that $M=1,4597546147$, $Q=\frac{9}{400}$, $\Delta_1\cong0,4141664514$ and $\Delta_2\cong0?9758011659$.
Finally, since $\|m\|_\infty(\triangle_1+\triangle_2)\cong0,143492<1$, thus all assumptions of Theorem 3.4 are satisfied. Hence, The inclusion problem has at least one solution.

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