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RESEARCH ARTICLE

Generalized Helices in the Euclidean 3-space Through Flow-frame

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ABSTRACT

A representation of time-dependent rotation of a usual Frenet flow is called flow-frame; the angle of rotation is exactly the current parameter. In this paper, we investigate three types of helices in the Euclidean 3-space through flow-frame and give their geometric description with flow-frame apparatus. Then, we introduce the spherical images of a curve by translating flow-frame vectors to the center of the unit sphere in the Euclidean 3-space $\mathbb{R}^3$. Besides, we examine the relationships between a generalized helix and its spherical images.

Keywords: *Flow-frame, general helix, spherical indicatrix, Lancret's theorem.*
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1. Introduction

In the Euclidean 3-space $\mathbb{R}^3$, a curve of constant slope or general helix is defined by the property that the tangent line makes a constant angle with a fixed direction called the axis of the general helix (see [15, 20]). A classical result stated by M. A. Lancret in 1802 and first demonstrated by B de Saint Venant in 1845 ([21]): "A necessary and sufficient condition that the curve be a general helix is that the ratio of curvature to torsion be constant."

In a similar way, slant helices are defined by the property that their principal normal makes a constant angle with a fixed direction. The term slant helix was first introduced by Izumiya and Takeuchi ([12]); however, slant helices have been studied in different space forms, as well. (see [1, 14, 16, 17, 18]).

The Lancret theorem was revisited and solved by Barros ([3]) in 3-dimensional real space forms, where he used Killing vector fields along a curve. Besides, Lancret theorem for general helices in 3-dimensional Lorentzian space forms was presented in ([4]). Also see ([11]) for Lancret-type theorem for null generalized helices in the Lorentz-Minkowski spaces $\mathbb{L}^n$. Moreover, many researchers have introduced the concept of helices in Lie groups by using the fixed invariant directions ([9, 19, 22]).

The flow-frame with the flow-curvature of a curve, which is a new frame involves the time-dependent rotation of the usual Frenet flow ([5, 6, 7, 8]).

In this note, we deal with three types of generalized helices according to flow-frame in the 3-dimensional Euclidean space. We introduce some necessary and sufficient conditions for these generalized helices. Also, we present the spherical indicatrices of a curve by translating new frame's vector fields to the center of unit sphere (for details, see [2, 10, 13]). Furthermore, we show that the spherical image of a curve with flow-frame is a circle if and only if the curve is a generalized helix of the first, second or third kind. Also, we give some differential equations to determine the relationships between the generalized helices and their spherical images.

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2. Preliminaries

Let $\mathbb{R}^3$ be the three-dimensional Euclidean space equipped with the inner product $\langle a, b\rangle = a_1b_1 + a_2b_2 + a_3b_3$, where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3) \in \mathbb{R}^3$. The norm of vector a is given by $||a|| = \sqrt{\langle a, a\rangle}$ and a vector product is given by

$$a \times b = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix},$$

where $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$.
Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ be a regular curve in $\mathbb{R}^3$ defined on a real interval $I = (\alpha, \beta)$, that has at least four continuous derivatives. The arc-length of a curve γ, measured from $\gamma(t_0)$, $t_0 \in I$ is

$$s(t) = \int_{t_0}^{t} ||\dot{\gamma}(\rho)||d\rho.$$

Throughout in this paper, we denote the arc-length by s.
The sphere of radius $r > 0$ and with center in the origin in the space $\mathbb{R}^3$ is defined by

$$S^2 = \left\{q = (q_1, q_2, q_3) \in \mathbb{R}^3 : \langle q, q\rangle = r^2\right\}.$$

Denote by $\{T, N, B\}$ the standart Frenet frame along the curve γ where $T(s)$ is the tangent, $N(s)$ is the principal normal and $B(s)$ is the binormal vector and the pair $(curvature, torsion) = (\kappa, \tau)$. Then, the Frenet equations are given by the following relations:

$$\begin{pmatrix} T'(s) \\ N'(s) \\ B'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0(t) \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}.$$

Here, curvature functions are defined by $\kappa = \kappa(s) = ||T'||$ and $\tau(s) = -\langle N, B'\rangle$.

The flow-frame with the flow-curvature of bi-regular curve γ is a new frame involving the time-dependent rotation of the usual Frenet flow, which is expressed as follows with the rotation $R(t)$:

$$\begin{pmatrix} T(t) \\ F_2(t) \\ F_3(t) \end{pmatrix} := \begin{pmatrix} 1 & 0_2(h) \\ 0_2(v) & R(t) \end{pmatrix} \begin{pmatrix} T(t) \\ N(t) \\ B(t) \end{pmatrix}, 0_2(h) := \begin{pmatrix} 0 & 0 \end{pmatrix}, 0_2(v) := \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Here

$$R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SO(2).$$

Then, the moving equation yields

$$\begin{pmatrix} T'(t) \\ F_2'(t) \\ F_3'(t) \end{pmatrix} = ||\gamma'(t)|| \begin{pmatrix} 0 & \kappa_2(t) & \kappa_3(t) \\ -\kappa_2(t) & 0 & \kappa_4(t) \\ -\kappa_3(t) & -\kappa_4(t) & 0 \end{pmatrix} \begin{pmatrix} T(t) \\ F_2(t) \\ F_3(t) \end{pmatrix}. \tag{2.1}$$

With a simple computation, we obtain

$$\kappa_2(t) = \kappa(t)\cos t, \quad \kappa_3(t) = \kappa(t)\sin t, \quad \kappa_4(t) = \tau(t) - \frac{1}{||\gamma'(t)||}.$$

3. Generalized helices according to flow-frame

Definition 3.1. Let γ be a bi-regular curve with the flow-frame $\{T, F_2, F_3\}$. The curve γ is called the generalized helix of the first, second, or third kind with axis ξ if there exists a unit vector field ξ such that $\langle T, \xi\rangle = const$, $\langle F_2, \xi\rangle = const$, or $\langle F_3, \xi\rangle = const$, respectively.

Theorem 3.1. *A unit speed bi-regular curve γ with $\kappa_3(s) \neq 0$ is a generalized helix of the first kind if and only if*

$$\eta_1(s) = \left(\frac{(H_1\kappa_2 + \kappa_3)\sqrt{1+H_1^2}}{(H_1' - \kappa_4(1+H_1^2))} \right)(s)$$

is a constant function, where $H_1(s) = \frac{\kappa_2(s)}{\kappa_3(s)}$.

Proof. If γ is a generalized helix of the first kind then there exists a unit vector field ξ such that $\langle T, \xi \rangle = \cos\theta$, where θ is a constant. Since $\xi' = 0$, then using equation 2.1 we get

$$\kappa_2\langle F_2, \xi\rangle + \kappa_3\langle F_3, \xi\rangle = 0.$$

Hence,

$$\langle F_3, \xi\rangle = -\frac{\kappa_2}{\kappa_3}\langle F_2, \xi\rangle = -H_1\langle F_2, \xi\rangle, \tag{3.1}$$

where $H_1 = \frac{\kappa_2}{\kappa_3}$. Therefore, $\langle F_3', \xi\rangle = -H_1'\langle F_2, \xi\rangle - H_1\langle F_2', \xi\rangle$. From equation 2.1 $\langle -\kappa_3 T - \kappa_4 F_2, \xi\rangle = -H_1'\langle F_2, \xi\rangle - H_1\langle -\kappa_2 T + \kappa_4 F_3, \xi\rangle$. Using $\langle T, \xi\rangle = \cos\theta$ and equation 3.1 we get

$$\langle F_2, \xi\rangle(H_1' - \kappa_4(1+H_1^2)) = \cos\theta(H_1\kappa_2 + \kappa_3).$$

It means that

$$\langle F_2, \xi\rangle = \frac{\cos\theta(H_1\kappa_2 + \kappa_3)}{H_1' - \kappa_4(1+H_1^2)}. \tag{3.2}$$

In combination with 3.1, we get

$$\xi = \left(T + \frac{(H_1\kappa_2 + \kappa_3)}{(H_1' - \kappa_4(1+H_1^2))}F_2 - \frac{H_1(H_1\kappa_2 + \kappa_3)}{(H_1' - \kappa_4(1+H_1^2)}F_3 \right)\cos\theta.$$

Since $|\xi| = 1$,

$$\left(1 + \frac{(H_1\kappa_2 + \kappa_3)^2}{(H_1' - \kappa_4(1+H_1^2))^2} + \frac{H_1^2(H_1\kappa_2 + \kappa_3)^2}{(H_1' - \kappa_4(1+H_1^2)^2}\right) = \frac{1}{\cos^2\theta}.$$

Thus, we obtain that

$$\frac{(H_1\kappa_2 + \kappa_3)\sqrt{1+H_1^2}}{(H_1' - \kappa_4(1+H_1^2))} = \tan\theta. \tag{3.3}$$

Moreover, 3.3 implies

$$\xi = \left(\cos\theta T + \frac{1}{\sqrt{1+H_1^2}}\sin\theta F_2 - \frac{H_1}{\sqrt{1+H_1^2}}\sin\theta F_3 \right). \tag{3.4}$$

Conversely, take ξ given by 3.4 and suppose 3.3 fulfilled. Then

$$(a) = \langle T, \xi\rangle = \cos\theta; (b) = \langle F_2, \xi\rangle = \frac{1}{\sqrt{1+H_1^2}}\sin\theta; (c) = \langle F_3, \xi\rangle = -\frac{H_1}{\sqrt{1+H_1^2}}\sin\theta.$$

The derivative of (a) yields $\langle T', \xi\rangle + \langle T, \xi'\rangle = 0$. Using 2.1, we obtain

$$\kappa_2\langle F_2, \xi\rangle + \kappa_3\langle F_3, \xi\rangle + \langle T, \xi'\rangle = 0$$

or

$$\kappa_3\left(H_1\langle F_2, \xi\rangle + \langle F_3, \xi\rangle\right) + \langle T, \xi'\rangle = 0.$$

From 3.1, it follows that $\langle T, \xi'\rangle = 0$.
The derivative of (b) yields

$$\begin{aligned}
\langle F_2, \xi'\rangle =& \frac{d}{ds}\langle F_2, \xi\rangle - \langle F_2', \xi\rangle \\
=& \frac{d}{ds}\left(\frac{1}{\sqrt{1+H_1^2}}\sin\theta\right) + \kappa_2\langle T, \xi\rangle - \kappa_4\langle F_3, \xi\rangle \\
=& -\frac{H_1'H_1}{(1+H_1^2)^{3/2}}\sin\theta + \kappa_2\langle T, \xi\rangle - \kappa_4\langle F_3, \xi\rangle.
\end{aligned}$$

We can express H_1' from 3.2, then using (a), (b) and (c), we get

$$\begin{aligned}\langle F_2, \xi' \rangle &= -\frac{H_1}{(1+H_1^2)^{3/2}} \sin\theta \left(\frac{\cos\theta}{\sin\theta} (H_1\kappa_2 + \kappa_3) \sqrt{1+H_1^2} + \kappa_4 \left(1+H_1^2\right) \right) \\ &+ \kappa_2 \cos\theta + \frac{\kappa_4 H_1}{\sqrt{1+H_1^2}} \sin\theta \\ &= \left(-\frac{H_1 (H_1\kappa_2 + \kappa_3)}{1+H_1^2} \cos\theta - \frac{H_1\kappa_4}{\sqrt{1+H_1^2}} \sin\theta \right) + \kappa_2 \cos\theta + \frac{\kappa_4 H_1}{\sqrt{1+H_1^2}} \sin\theta \\ &= 0.\end{aligned}$$

In a similar way, the derivative of (c) yields

$$\begin{aligned}\langle F_3, \xi' \rangle =& \frac{d}{ds} \langle F_3, \xi \rangle - \langle F_3', \xi \rangle \\ =& \frac{d}{ds} \left(-\frac{H_1}{\sqrt{1+H_1^2}} \sin\theta \right) + \kappa_3 \langle T, \xi \rangle + \kappa_4 \langle F_2, \xi \rangle \\ =& -\frac{H_1'}{(1+H_1^2)^{3/2}} \sin\theta + \kappa_3 \langle T, \xi \rangle + \kappa_4 \langle F_2, \xi \rangle.\end{aligned}$$

Again, we can express H_1' from 3.2, then using (a) and (b), we get

$$\begin{aligned}\langle F_3, \xi' \rangle &= -\frac{1}{(1+H_1^2)^{3/2}} \sin\theta \left(\frac{\cos\theta}{\sin\theta} (H_1\kappa_2 + \kappa_3) \sqrt{1+H_1^2} + \kappa_4 \left(1+H_1^2\right) \right) \\ &+ \kappa_3 \cos\theta + \frac{\kappa_4}{\sqrt{1+H_1^2}} \sin\theta \\ &= \left(-\frac{(H_1\kappa_2 + \kappa_3)}{1+H_1^2} \cos\theta - \frac{\kappa_4}{\sqrt{1+H_1^2}} \sin\theta \right) + \kappa_3 \cos\theta + \frac{\kappa_4}{\sqrt{1+H_1^2}} \sin\theta \\ &= 0.\end{aligned}$$

Since $\langle T, \xi' \rangle = 0$, $\langle F_2, \xi' \rangle = 0$, and $\langle F_3, \xi' \rangle = 0$, we have $\xi' = 0$. This completes the proof. □

Theorem 3.2. *Let γ be a unit speed bi-regular curve with flow-frame, then γ is a generalized helix of the first kind if and only if*

$$\det\left(T', T'', T'''\right) = 0.$$

Proof. Suppose that η_1 is constant. Then, the following equalities are satisfied

$$\begin{aligned}&T' = \kappa_2 F_2 + \kappa_3 F_3, \\ &T'' = \left(-\kappa_2^2 - \kappa_3^2\right) T + \left(\kappa_2' - \kappa_3\kappa_4\right) F_2 + \left(\kappa_3' + \kappa_2\kappa_4\right) F_3, \\ &T''' = -3\left(\kappa_2\kappa_2' + \kappa_3\kappa_3'\right) T + \left(\kappa_2'' - 2\kappa_3'\kappa_4 - \kappa_3\kappa_4' - \kappa_2\left(\kappa_2^2 + \kappa_3^2 + \kappa_4^2\right)\right) F_2 \\ &\;+ \left(\kappa_3'' + 2\kappa_2'\kappa_4 + \kappa_2\kappa_4' - \kappa_3\left(\kappa_2^2 + \kappa_3^2 + \kappa_4^2\right)\right) F_3.\end{aligned}$$

So, we get

$$\det\left(T', T'', T'''\right) = \det \begin{pmatrix} 0 & \kappa_2 & \kappa_3 \\ \left(-\kappa_2^2 - \kappa_3^2\right) & \left(\kappa_2' - \kappa_3\kappa_4\right) & \left(\kappa_3' + \kappa_2\kappa_4\right) \\ -3\left(\kappa_2\kappa_2' + \kappa_3\kappa_3'\right) & \begin{array}{c}\kappa_2'' - 2\kappa_3'\kappa_4 - \kappa_3\kappa_4' \\ -\kappa_2\left(\kappa_2^2 + \kappa_3^2 + \kappa_4^2\right)\end{array} & \begin{array}{c}\kappa_3'' + 2\kappa_2'\kappa_4 + \kappa_2\kappa_4' \\ -\kappa_3\left(\kappa_2^2 + \kappa_3^2 + \kappa_4^2\right)\end{array} \end{pmatrix}.$$

Therefore, we can calculate that

$$\det\left(T', T'', T'''\right) = \eta_1' \left(1/\eta_1^2\right) \left(\kappa_2^2 + \kappa_3^2\right)^{5/2}.$$

Since γ is a curve with flow-frame and η_1 is constant, we have

$$\det\left(T', T'', T'''\right) = 0.$$

Conversely, assume that $\det\left(T', T'', T'''\right) = 0$. Since η_1' is zero, it follows that η_1 is constant. □

Theorem 3.3. *A unit speed bi-regular curve γ with $\kappa_4(s) \neq 0$ is a generalized helix of the second kind if and only if*

$$\eta_2(s) = \left(\frac{(H_2\kappa_2 + \kappa_4)\sqrt{1 + H_2^2}}{(H_2' + \kappa_3(1 + H_2^2))}\right)(s)$$

is a constant function, where $H_2(s) = \frac{\kappa_2(s)}{\kappa_4(s)}$.

Proof. If γ is a generalized helix of the second kind then there exists a unit vector field ξ such that $\langle F_2, \xi\rangle = \cos\theta$, where θ is a constant. Since $\xi' = 0$, then using equation 2.1 we get

$$-\kappa_2\langle T, \xi\rangle + \kappa_4\langle F_3, \xi\rangle = 0.$$

Hence,

$$\langle F_3, \xi\rangle = \frac{\kappa_2}{\kappa_4}\langle T, \xi\rangle = H_2\langle T, \xi\rangle, \tag{3.5}$$

where $H_2 = \frac{\kappa_2}{\kappa_4}$. Therefore, $\langle F_3', \xi\rangle = H_2'\langle T, \xi\rangle + H_2\langle T', \xi\rangle$. From equation 2.1 $\langle -\kappa_3 T - \kappa_4 F_2, \xi\rangle = H_2'\langle T, \xi\rangle + H_2\langle \kappa_2 F_2 + \kappa_3 F_3, \xi\rangle$. Using $\langle F_2, \xi\rangle = \cos\theta$ and equation 3.5 we get

$$\langle T, \xi\rangle(H_2' + \kappa_3(1 + H_2^2)) = -\cos\theta(H_2\kappa_2 + \kappa_4).$$

It means that

$$\langle T, \xi\rangle = \frac{\cos\theta(H_2\kappa_2 + \kappa_4)}{H_2' + \kappa_3(1 + H_2^2)}. \tag{3.6}$$

In combination with 3.5, we get

$$\xi = \left(-\frac{(H_2\kappa_2 + \kappa_4)}{(H_2' + \kappa_3(1 + H_2^2))}T + F_2 - \frac{H_2(H_2\kappa_2 + \kappa_4)}{(H_2' + \kappa_3(1 + H_2^2))}F_3\right)\cos\theta.$$

Since $|\xi| = 1$,

$$\left(\frac{(H_2\kappa_2 + \kappa_4)^2}{(H_2' + \kappa_3(1 + H_2^2))^2} + 1 + \frac{H_2^2(H_2\kappa_2 + \kappa_4)^2}{(H_2' + \kappa_3(1 + H_2^2)^2}\right) = \frac{1}{\cos^2\theta}.$$

Thus, we obtain that

$$\frac{(H_2\kappa_2 + \kappa_4)\sqrt{1 + H_2^2}}{(H_2' + \kappa_3(1 + H_2^2))} = \tan\theta. \tag{3.7}$$

Moreover, 3.7 implies

$$\xi = \left(-\frac{1}{\sqrt{1 + H_2^2}}\sin\theta T + \cos\theta F_2 - \frac{H_2}{\sqrt{1 + H_2^2}}\sin\theta F_3\right). \tag{3.8}$$

Conversely, take the vector field ξ given by 3.8 and suppose 3.7 fulfilled. Then, by the same process as in the proof of Theorem 3.1, we can check that ξ is a constant and $\langle F_2, \xi\rangle = \cos\theta$. □

Theorem 3.4. *Let γ be a unit speed bi-regular curve with flow-frame, then γ is a generalized helix of the second kind if and only if*

$$\det\left(F_2', F_2'', F_2'''\right) = 0.$$

Proof. Suppose that η_2 is constant. We can calculate that

$$\det\left(F_2', F_2'', F_2'''\right) = \eta_2'\left(1/\eta_2^2\right)\left(\kappa_2^2 + \kappa_4^2\right)^{5/2}.$$

Since γ is a curve with flow-frame and η_2 is constant, we have

$$\det\left(F_2', F_2'', F_2'''\right) = 0.$$

Conversely, assume that $\det\left(F_2', F_2'', F_2'''\right) = 0$. Since η_2' is zero, then it is clear that η_2 is constant. □

Theorem 3.5. *A unit speed bi-regular curve γ with $\kappa_4(s) \neq 0$ is a generalized helix of the third kind if and only if*

$$\eta_3(s) = \left(\frac{(H_3\kappa_3 + \kappa_4)\sqrt{1+H_3^2}}{(H_3' - \kappa_2(1+H_3^2))} \right)(s)$$

is a constant function, where $H_3(s) = \frac{\kappa_3(s)}{\kappa_4(s)}$.

Proof. If γ is a generalized helix of the third kind then there exists a unit vector field ξ such that $\langle F_3, \xi \rangle = \cos\theta$, where θ is a constant. Since $\xi' = 0$, then using equation 2.1 we get

$$-\kappa_3\langle T, \xi\rangle - \kappa_4\langle F_2, \xi\rangle = 0.$$

Hence,

$$\langle F_2, \xi\rangle = -\frac{\kappa_3}{\kappa_4}\langle T, \xi\rangle = -H_3\langle T, \xi\rangle, \tag{3.9}$$

where $H_3 = \frac{\kappa_3}{\kappa_4}$. Therefore, $\langle F_2', \xi\rangle = -H_3'\langle T, \xi\rangle - H_3\langle T', \xi\rangle$. From equation 2.1 $\langle -\kappa_2 T + \kappa_4 F_3, \xi\rangle = -H_3'\langle T, \xi\rangle - H_3\langle \kappa_2 F_2 + \kappa_3 F_3, \xi\rangle$. Using $\langle F_3, \xi\rangle = \cos\theta$ and equation 3.9 we get

$$\langle T, \xi\rangle(H_3' - \kappa_2(1+H_3^2)) = -\cos\theta(H_3\kappa_3 + \kappa_4).$$

It means that

$$\langle T, \xi\rangle = -\frac{\cos\theta(H_3\kappa_3 + \kappa_4)}{H_3' - \kappa_2(1+H_3^2)}. \tag{3.10}$$

In combination with 3.9, we get

$$\xi = \left(-\frac{(H_3\kappa_3 + \kappa_4)}{(H_3' - \kappa_2(1+H_3^2))}T + \frac{H_3(H_3\kappa_3 + \kappa_4)}{(H_3' - \kappa_2(1+H_3^2)}F_2 + F_3 \right)\cos\theta.$$

Since $|\xi| = 1$,

$$\left(\frac{(H_3\kappa_3 + \kappa_4)^2}{(H_3' - \kappa_2(1+H_3^2))^2} + \frac{H_3^2(H_3\kappa_3 + \kappa_4)^2}{(H_3' - \kappa_2(1+H_3^2)^2} + 1 \right) = \frac{1}{\cos^2\theta}.$$

Thus, we obtain that

$$\frac{(H_3\kappa_3 + \kappa_4)\sqrt{1+H_3^2}}{(H_3' - \kappa_2(1+H_3^2))} = \tan\theta. \tag{3.11}$$

Moreover, 3.11 implies

$$\xi = \left(-\frac{1}{\sqrt{1+H_3^2}}\sin\theta T + \frac{H_3}{\sqrt{1+H_3^2}}\sin\theta F_2 + \cos\theta F_3 \right). \tag{3.12}$$

Conversely, take the vector field ξ given by 3.12 and suppose 3.11 fulfilled. Then, by the same process as in the proof of Theorem 3.1, we can check that ξ is a constant and $\langle F_3, \xi\rangle = \cos\theta$. □

Theorem 3.6. *Let γ be a unit speed bi-regular curve with flow-frame, then γ is a generalized helix of the third kind if and only if*

$$\det(F_3', F_3'', F_3''') = 0.$$

Proof. Suppose that η_3 is constant. We can calculate that

$$\det(F_3', F_3'', F_3''') = \eta_3'\left(1/\eta_3^2\right)\left(\kappa_3^2 + \kappa_4^2\right)^{5/2}.$$

Since γ is a curve with flow-frame and η_3 is constant, we have

$$\det(F_3', F_3'', F_3''') = 0.$$

Conversely, assume that $\det(F_3', F_3'', F_3''') = 0$. Since η_3' is zero, it follows that η_3 is constant. □

4. New spherical images of a bi-regular curve

Let γ be a unit speed bi-regular curve in the Euclidean 3-space with $\{T, F_2, F_3\}$. Translating new frame's vector fields to the center of a unit sphere, generate new spherical images. If we translate the unit tangent vector along a curve γ, we obtain $\gamma_T = T$ on the unit sphere. The curve γ_T is called the spherical indicatrix of T, in other words, tangent indicatrix of the curve γ. Similarly, one can consider the F_2 indicatrix $\gamma_{F_2} = F_2$ and the F_3 indicatrix $\gamma_{F_3} = F_3$.
In this section, we introduce a representation of spherical indicatrices of a bi-regular curve with flow-frame in the Euclidean 3-space $\mathbb{R}^3$ and then investigate the relationships between the helices and their spherical indicatrices.
D denotes the covariant differentiation of $\mathbb{R}^3$.

4.1. *Tangent indicatrix of a bi-regular curve*

Give a unit speed bi-regular space curve γ with flow-frame and its tangent indicatrix $\gamma_T(s_T) = T(s)$ with the natural representation s_T. If the Serret-Frenet frame of γ_T is $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, then we have the following formula:

$$\begin{pmatrix} \mathbf{T}'(s_T) \\ \mathbf{N}'(s_T) \\ \mathbf{B}'(s_T) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_T & 0 \\ -\kappa_T & 0 & \tau_T \\ 0 & -\tau_T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s_T) \\ \mathbf{N}(s_T) \\ \mathbf{B}(s_T) \end{pmatrix} \tag{4.1}$$

where

$$\begin{cases} \mathbf{T} = \dfrac{H_1 F_2 + F_3}{\sqrt{1 + H_1^2}}, \\ \mathbf{N} = \dfrac{f_1}{\sqrt{1+f_1^2}} \left(\dfrac{H_1 F_3 - F_2}{\sqrt{1+H_1^2}} - \dfrac{T}{f_1} \right), \\ \mathbf{B} = \dfrac{1}{\sqrt{1+f_1^2}} \left(\dfrac{H_1 F_3 - F_2}{\sqrt{1+H_1^2}} + f_1 T \right), \end{cases} \tag{4.2}$$

and

$$s_T = \int \sqrt{\kappa_2^2 + \kappa_3^2}\, ds + c, \quad \kappa_T = \sqrt{1 + f_1^2}, \quad \tau_T = -\sigma_1 \sqrt{1 + f_1^2}, \tag{4.3}$$

where

$$f_1 = \frac{1}{\eta_1}, \tag{4.4}$$

and

$$\sigma_1 = \frac{f_1'}{\sqrt{\kappa_2^2 + \kappa_3^2}\, (1 + f_1^2)^{3/2}}. \tag{4.5}$$

Here, κ_T and τ_T are the curvature and the torsion of the curve γ_T, respectively. Therefore, we have

$$\frac{\tau_T}{\kappa_T} = -\sigma_1. \tag{4.6}$$

Theorem 4.1. *Give a bi-regular unit speed curve γ with flow-frame $\{T, F_2, F_3\}$ and the curvatures $\kappa_2, \kappa_3, \kappa_4$. γ is a generalized helix of the first kind if and only if γ_T is a circle.*

Proof. Suppose that γ is a generalized helix of the first kind. From 4.3 the curvature and the torsion of γ_T

$$\kappa_T = \sqrt{1 + f_1^2}, \quad \tau_T = -\sigma_1 \sqrt{1 + f_1^2}$$

respectively. Since η_1 is a constant function, from 4.4 f_1 is also a constant function which leads to $\sigma_1 = 0$. Therefore, κ_T is a non-zero constant and $\tau_T = 0$. Hence, γ_T is a circle.
Conversely, assume that γ_T is a circle. Then it is obvious. □

Corollary 4.1. *Give a bi-regular unit speed curve γ with flow-frame $\{T, F_2, F_3\}$ and the curvatures $\kappa_2, \kappa_3, \kappa_4$. γ is a generalized helix of the first kind if and only if the $\mathbf{T}$ and the $\mathbf{N}$ vector field of γ_T satisfy the following equations:*

$$(i) D_{\mathbf{T}}^2 \mathbf{T} + \kappa_T^2 \mathbf{T} = 0,$$
$$(ii) D_{\mathbf{T}}^2 \mathbf{N} + \kappa_T^2 \mathbf{N} = 0.$$

Theorem 4.2. *Give a bi-regular unit speed curve γ with flow-frame $\{T, F_2, F_3\}$. We denote the curvature and the torsion of γ_T by κ_T and τ_T, respectively. γ_T is a generalized helix of the second kind if and only if*

$$\delta_T(s) = \left(\frac{\kappa_T^2}{(\kappa_T^2 + \tau_T^2)^{3/2}} \left(\frac{\tau_T}{\kappa_T} \right)' \right)(s)$$

is a constant function.

Theorem 4.3. *Give a bi-regular unit speed curve γ with flow-frame $\{T, F_2, F_3\}$. We denote the curvature and the torsion of γ_T by κ_T and τ_T, respectively. γ_T is a generalized helix of the second kind if and only if the curve $\beta : I \subset \mathbb{R} \to \mathbb{R}^2$, $\beta(s) = (\beta_1(s), \beta_2(s))$ is a circle, where $\beta_1(s) = \int \kappa_T(s)ds$ and $\beta_2(s) = \int \tau_T(s)ds$.*

Proof. We can calculate that the curvature of the curve β

$$\kappa_\beta = \frac{\beta_1'\beta_2'' - \beta_1''\beta_2'}{((\beta_1')^2 + (\beta_2')^2)^{3/2}} = \frac{\kappa_T^2}{(\kappa_T^2 + \tau_T^2)^{3/2}} \left(\frac{\tau_T}{\kappa_T} \right)' = \delta_T(s).$$

Thus, $\kappa_\beta = \delta_T = constant$. This completes the proof. □

4.2. *F_2 indicatrix of a bi-regular curve*

Give a unit speed bi-regular space curve γ with flow-frame and its F_2 indicatrix $\gamma_{F_2}(s_{F_2}) = F_2(s)$ with the natural representation s_{F_2}. If the Serret-Frenet frame of γ_{F_2} is $\{\mathcal{T}, \mathcal{N}, \mathcal{B}\}$, then we have the following formula:

$$\begin{pmatrix} \mathcal{T}'(s_{F_2}) \\ \mathcal{N}'(s_{F_2}) \\ \mathcal{B}'(s_{F_2}) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_{F_2} & 0 \\ -\kappa_{F_2} & 0 & \tau_{F_2} \\ 0 & -\tau_{F_2} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{T}(s_{F_2}) \\ \mathcal{N}(s_{F_2}) \\ \mathcal{B}(s_{F_2}) \end{pmatrix} \tag{4.7}$$

where

$$\begin{cases} \mathcal{T} = \dfrac{-H_2 T + F_3}{\sqrt{1 + H_2^2}}, \\ \mathcal{N} = \dfrac{f_2}{\sqrt{1 + f_2^2}} \left(\dfrac{H_2 F_3 + T}{\sqrt{1 + H_2^2}} - \dfrac{F_2}{f_2} \right), \\ \mathcal{B} = \dfrac{1}{\sqrt{1 + f_2^2}} \left(\dfrac{H_2 F_3 + T}{\sqrt{1 + H_2^2}} + f_2 F_2 \right), \end{cases} \tag{4.8}$$

and

$$s_{F_2} = \int \sqrt{\kappa_2^2 + \kappa_4^2} ds + c, \quad \kappa_{F_2} = \sqrt{1 + f_2^2}, \quad \tau_{F_2} = -\sigma_2 \sqrt{1 + f_2^2}, \tag{4.9}$$

where

$$f_2 = \frac{1}{\eta_2}, \tag{4.10}$$

and

$$\sigma_2 = \frac{f_2'}{\sqrt{\kappa_2^2 + \kappa_4^2}\,(1 + f_2^2)^{3/2}}. \tag{4.11}$$

Here, κ_{F_2} and τ_{F_2} are the curvature and the torsion of the curve γ_{F_2}, respectively. Therefore, we have

$$\frac{\tau_{F_2}}{\kappa_{F_2}} = -\sigma_2. \tag{4.12}$$

Theorem 4.4. *Give a bi-regular unit speed curve γ with flow-frame $\{T, F_2, F_3\}$ and the curvatures $\kappa_2, \kappa_3, \kappa_4$. γ is a generalized helix of the second kind if and only if γ_{F_2} is a circle.*

Proof. Suppose that γ is a generalized helix of the second kind. From 4.9 the curvature and the torsion of γ_{F_2}

$$\kappa_{F_2} = \sqrt{1 + f_2^2}, \quad \tau_{F_2} = -\sigma_2 \sqrt{1 + f_2^2}$$

respectively. Since η_2 is a constant function, from 4.10 f_2 is also a constant function which leads to $\sigma_2 = 0$. Therefore, κ_{F_2} is a non-zero constant and $\tau_{F_2} = 0$. Hence, γ_{F_2} is a circle.
Conversely, assume that γ_{F_2} is a circle. Then it is obvious. □

Corollary 4.2. *Give a bi-regular unit speed curve γ with flow-frame $\{T, F_2, F_3\}$ and the curvatures $\kappa_2, \kappa_3, \kappa_4$. γ is a generalized helix of the second kind if and only if the $\mathcal{T}$ and the $\mathcal{N}$ vector field of γ_{F_2} satisfy the following equations:*

$$(i) D^2_{\mathcal{T}}\mathcal{T} + \kappa^2_{F_2}\mathcal{T} = 0,$$
$$(ii) D^2_{\mathcal{T}}\mathcal{N} + \kappa^2_{F_2}\mathcal{N} = 0.$$

Theorem 4.5. *Give a bi-regular unit speed curve γ with flow-frame $\{T, F_2, F_3\}$. We denote the curvature and the torsion of γ_{F_2} by κ_{F_2} and τ_{F_2}, respectively. γ_{F_2} is a generalized helix of the second kind if and only if*

$$\delta_{F_2}(s) = \left(\frac{\kappa^2_{F_2}}{(\kappa^2_{F_2} + \tau^2_{F_2})^{3/2}}\left(\frac{\tau_{F_2}}{\kappa_{F_2}}\right)'\right)(s)$$

is a constant function.

Theorem 4.6. *Give a bi-regular unit speed curve γ with flow-frame $\{T, F_2, F_3\}$. We denote the curvature and the torsion of γ_{F_2} by κ_{F_2} and τ_{F_2}, respectively. γ_{F_2} is a generalized helix of the second kind if and only if the curve $\beta : I \subset \mathbb{R} \to \mathbb{R}^2$, $\beta(s) = (\beta_1(s), \beta_2(s))$ is a circle, where $\beta_1(s) = \int \kappa_{F_2}(s)ds$ and $\beta_2(s) = \int \tau_{F_2}(s)ds$.*

Proof. We can calculate that the curvature of the curve β

$$\kappa_\beta = \frac{\beta_1'\beta_2'' - \beta_1''\beta_2'}{((\beta_1')^2 + (\beta_2')^2)^{3/2}} = \frac{\kappa^2_{F_2}}{(\kappa^2_{F_2} + \tau^2_{F_2})^{3/2}}\left(\frac{\tau_{F_2}}{\kappa_{F_2}}\right)' = \delta_{F_2}(s).$$

Thus, $\kappa_\beta = \delta_{F_2} = constant$. This completes the proof. □

4.3. F_3 indicatrix of a bi-regular curve

Give a unit speed bi-regular space curve γ with flow frame and its F_3 indicatrix $\gamma_{F_3}(s_{F_3}) = F_3(s)$ with the natural representation s_{F_3}. If the Serret-Frenet frame of γ_{F_3} is $\{\mathbb{T}, \mathbb{N}, \mathbb{B}\}$, then we have the following formula:

$$\begin{pmatrix} \mathbb{T}'(s_{F_3}) \\ \mathbb{N}'(s_{F_3}) \\ \mathbb{B}'(s_{F_3}) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_{F_3} & 0 \\ -\kappa_{F_3} & 0 & \tau_{F_3} \\ 0 & -\tau_{F_3} & 0 \end{pmatrix} \begin{pmatrix} \mathbb{T}(s_{F_3}) \\ \mathbb{N}(s_{F_3}) \\ \mathbb{B}(s_{F_3}) \end{pmatrix} \tag{4.13}$$

where

$$\begin{cases} \mathbb{T} = \dfrac{-H_3T - F_2}{\sqrt{1 + H_3^2}}, \\ \mathbb{N} = \dfrac{f_3}{\sqrt{1 + f_3^2}}\left(\dfrac{T - H_3F_2}{\sqrt{1 + H_3^2}} - \dfrac{F_3}{f_3}\right), \\ \mathbb{B} = \dfrac{1}{\sqrt{1 + f_3^2}}\left(\dfrac{T - H_3F_2}{\sqrt{1 + H_3^2}} + f_3F_3\right), \end{cases} \tag{4.14}$$

and

$$s_{F_3} = \int \sqrt{\kappa_3^2 + \kappa_4^2}ds + c, \quad \kappa_{F_3} = \sqrt{1 + f_3^2}, \quad \tau_{F_3} = -\sigma_3\sqrt{1 + f_3^2}, \tag{4.15}$$

where

$$f_3 = \frac{1}{\eta_3}, \tag{4.16}$$

and

$$\sigma_3 = \frac{f_3'}{\sqrt{\kappa_3^2 + \kappa_4^2}\,(1 + f_3^2)^{3/2}}. \tag{4.17}$$

Here, κ_{F_3} and τ_{F_3} are the curvature and torsion of the curve γ_{F_3}, respectively. Therefore, we have

$$\frac{\tau_{F_3}}{\kappa_{F_3}} = -\sigma_3. \tag{4.18}$$

Theorem 4.7. *Give a bi-regular unit speed curve γ with flow-frame $\{T, F_2, F_3\}$ and the curvatures $\kappa_2, \kappa_3, \kappa_4$. γ is a generalized helix of the third kind if and only if γ_{F_3} is a circle.*

Proof. Suppose that γ is a generalized helix of the third kind. From 4.15 the curvature and the torsion of γ_{F_3}

$$\kappa_{F_3} = \sqrt{1+f_3^2}, \quad \tau_{F_3} = -\sigma_3\sqrt{1+f_3^2}$$

respectively. Since η_3 is a constant function, from 4.16 f_3 is also a constant function which leads to $\sigma_3 = 0$. Therefore, κ_{F_3} is a non-zero constant and $\tau_{F_3} = 0$. Hence, γ_{F_3} is a circle.
Conversely, assume that γ_{F_3} is a circle. Then it is obvious. □

Corollary 4.3. *Give a bi-regular unit speed curve γ with flow-frame $\{T, F_2, F_3\}$ and the curvatures $\kappa_2, \kappa_3, \kappa_4$. γ is a generalized helix of the third kind if and only if the $\mathbb{T}$ and the $\mathbb{B}$ vector field of γ_{F_3} satisfy the following equations:*

$$(i) D_{\mathbb{T}}^2\mathbb{T} + \kappa_{F_3}^2\mathbb{T} = 0,$$
$$(ii) D_{\mathbb{T}}^2\mathbb{B} + \kappa_{F_3}^2\mathbb{B} = 0.$$

Theorem 4.8. *Give a bi-regular unit speed curve γ with flow-frame $\{T, F_2, F_3\}$. We denote the curvature and the torsion of γ_{F_3} by κ_{F_3} and τ_{F_3}, respectively. γ_{F_3} is a generalized helix of the second kind if and only if*

$$\delta_{F_3}(s) = \left(\frac{\kappa_{F_3}^2}{(\kappa_{F_3}^2 + \tau_{F_3}^2)^{3/2}} \left(\frac{\tau_{F_3}}{\kappa_{F_3}}\right)' \right)(s)$$

is a constant function.

Theorem 4.9. *Give a bi-regular unit speed curve γ with flow-frame $\{T, F_2, F_3\}$. We denote the curvature and the torsion of γ_{F_3} by κ_{F_3} and τ_{F_3}, respectively. γ_{F_3} is a generalized helix of the second kind if and only if the curve $\beta : I \subset \mathbb{R} \to \mathbb{R}^2$, $\beta(s) = (\beta_1(s), \beta_2(s))$ is a circle, where $\beta_1(s) = \int \kappa_{F_3}(s)ds$ and $\beta_2(s) = \int \tau_{F_3}(s)ds$.*

Proof. We can calculate that the curvature of the curve β

$$\kappa_\beta = \frac{\beta_1'\beta_2'' - \beta_1''\beta_2'}{((\beta_1')^2 + (\beta_2')^2)^{3/2}} = \frac{\kappa_{F_3}^2}{(\kappa_{F_3}^2 + \tau_{F_3}^2)^{3/2}} \left(\frac{\tau_{F_3}}{\kappa_{F_3}}\right)' = \delta_{F_3}(s).$$

Thus $\kappa_\beta = \delta_{F_3} = constant$. This completes the proof. □

5. Examples

In this section, we give some examples how to find a regular curve's flow-frame and illustrate the new spherical images.

Example 5.1. First, let us consider a unit speed circular helix of $\mathbb{R}^3$ by

$$\alpha = \alpha(s) = \left(cos\frac{s}{2}, \frac{\sqrt{2}}{2} sin\frac{s}{2} - \frac{\sqrt{6}}{4}s + 2, \frac{\sqrt{2}}{2} sin\frac{s}{2} + \frac{\sqrt{6}}{4}s + 3 \right). \tag{5.1}$$

One can calculate its Frenet-Serret apparatus as the following

$$\begin{cases} T = \left(\frac{-1}{2} sin\frac{s}{2}, \frac{\sqrt{2}}{4} cos\frac{s}{2} - \frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4} cos\frac{s}{2} + \frac{\sqrt{6}}{4} \right), \\ N = \left(-cos\frac{s}{2}, -\frac{\sqrt{2}}{2} sin\frac{s}{2}, -\frac{\sqrt{2}}{2} sin\frac{s}{2} \right), \\ B = \left(\frac{\sqrt{3}}{2} sin\frac{s}{2}, -\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} cos\frac{s}{2}, \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} cos\frac{s}{2} \right), \\ \kappa = \frac{1}{4}, \\ \tau = \frac{\sqrt{3}}{4}. \end{cases} \tag{5.2}$$

We plot the classical spherical images of α in Figure 1 to compare our new spherical images.

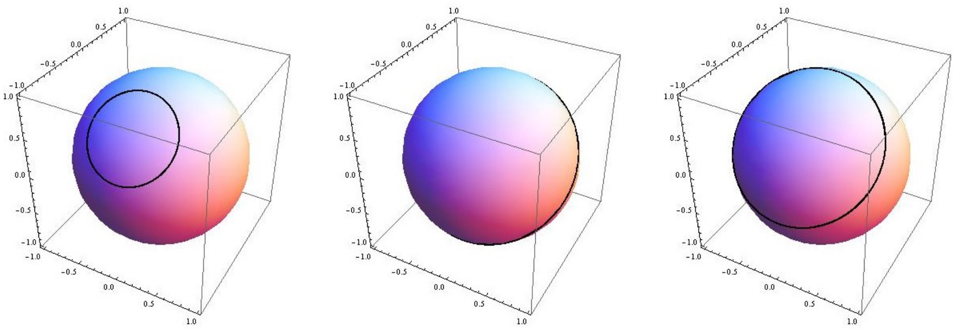

Figure 1. Spherical images of $\alpha = \alpha(s)$ with respect to Frenet-Serret frame.

Now we focus on the flow-frame. We can write the transformation matrix.

$$\begin{pmatrix} T(s) \\ F_2(s) \\ F_3(s) \end{pmatrix} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\, s & -sin\, s \\ 0 & sin\, s & cos\, s \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}. \tag{5.3}$$

One can obtain flow-frame of α as follows:

$$\begin{cases} T = \left(\frac{-1}{2} sin \frac{s}{2}, \frac{\sqrt{2}}{4} cos \frac{s}{2} - \frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4} cos \frac{s}{2} + \frac{\sqrt{6}}{4} \right), \\ F_2 = \begin{pmatrix} -cos\, s\, cos \frac{s}{2} - \frac{\sqrt{3}}{2} sin \frac{s}{2} sin\, s, \\ -\frac{\sqrt{2}}{2} sin \frac{s}{2} cos\, s + \frac{\sqrt{2}}{4} sin\, s + \frac{\sqrt{6}}{4} sin\, s\, cos \frac{s}{2}, \\ -\frac{\sqrt{2}}{2} sin \frac{s}{2} cos\, s - \frac{\sqrt{2}}{4} sin\, s + \frac{\sqrt{6}}{4} sin\, s\, cos \frac{s}{2} \end{pmatrix}, \\ F_3 = \begin{pmatrix} -sin\, s\, cos \frac{s}{2} + \frac{\sqrt{3}}{2} sin \frac{s}{2} cos\, s, \\ -\frac{\sqrt{2}}{2} sin \frac{s}{2} sin\, s - \frac{\sqrt{2}}{4} cos\, s - \frac{\sqrt{6}}{4} cos \frac{s}{2} cos\, s, \\ -\frac{\sqrt{2}}{2} sin \frac{s}{2} sin\, s + \frac{\sqrt{2}}{4} cos\, s - \frac{\sqrt{6}}{4} cos \frac{s}{2} cos\, s \end{pmatrix}, \\ \kappa_2 = \frac{1}{4} cos\, s, \\ \kappa_3 = \frac{1}{4} sin\, s, \\ \kappa_4 = \frac{\sqrt{3} - 4}{4}. \end{cases} \tag{5.4}$$

So, we can illustrate new spherical images, see Figure 2.

Example 5.2. Consider some other unit speed circular helix of $\mathbb{R}^3$ by

$$\beta = \beta(s) = \left(5cos \frac{s}{13}, 5sin \frac{s}{13}, \frac{12s}{13} \right). \tag{5.5}$$

One can calculate its Frenet-Serret apparatus as the following

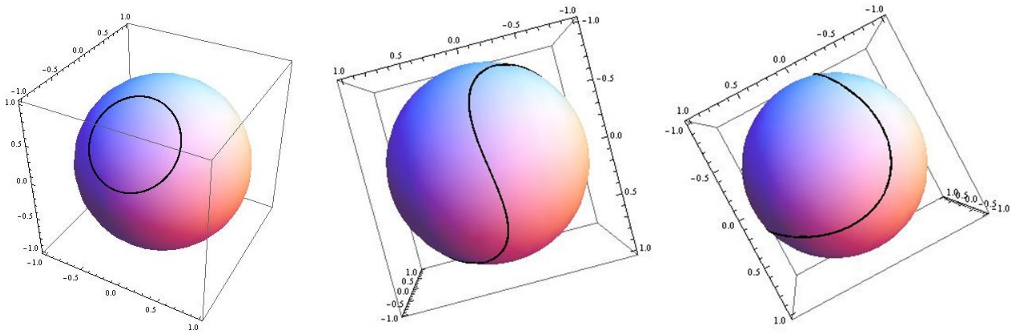

Figure 2. Tangent, F_2 and F_3 spherical images of $\alpha = \alpha(s)$.

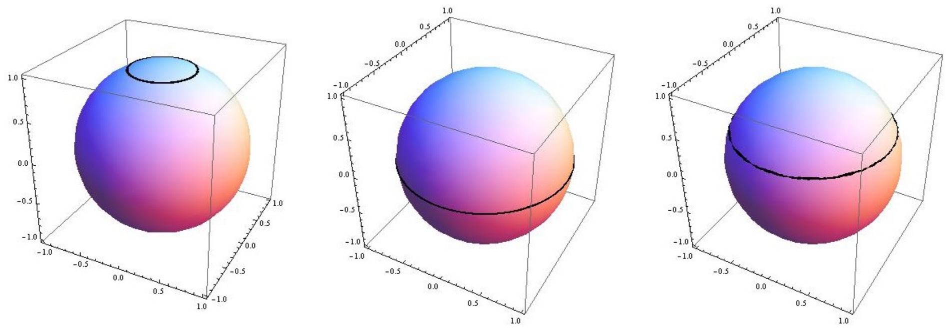

Figure 3. Spherical images of $\beta = \beta(s)$ with respect to Frenet-Serret frame.

$$\begin{cases} T = \left(-\dfrac{5}{13}sin\dfrac{s}{13}, \dfrac{5}{13}cos\dfrac{s}{13}, \dfrac{12}{13}\right), \\ N = \left(-cos\dfrac{s}{13}, -sin\dfrac{s}{13}, 0\right), \\ B = \left(\dfrac{12}{13}sin\dfrac{s}{13}, -\dfrac{12}{13}cos\dfrac{s}{13}, \dfrac{5}{13}\right), \\ \kappa = \dfrac{5}{169}, \\ \tau = \dfrac{12}{169}. \end{cases} \tag{5.6}$$

First, we plot the classical spherical images of β in Figure 3 to compare our new spherical images.

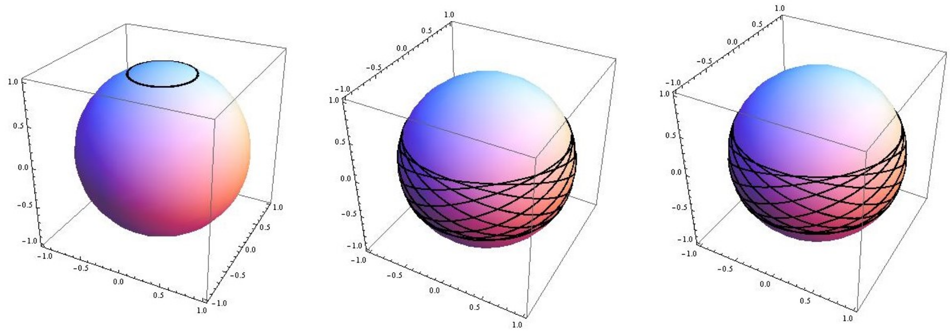

Figure 4. Tangent, F_2 and F_3 spherical images of $\beta = \beta(s)$.

By using the transformation matrix, we can obtain flow-frame of β as follows:

$$\begin{cases} T = \left(-\frac{5}{13}sin\frac{s}{13}, \frac{5}{13}cos\frac{s}{13}, \frac{12}{13}\right), \\ F_2 = \left(-cos\, s\, cos\frac{s}{13} - \frac{12}{13}sin\frac{s}{13}sin\, s, -sin\frac{s}{13}cos\, s + \frac{12}{13}sin\, s\, cos\frac{s}{13}, -\frac{5}{13}sin\, s\right), \\ F_3 = \left(-sin\, s\, cos\frac{s}{13} + \frac{12}{13}sin\frac{s}{13}cos\, s, -sin\, s\, sin\frac{s}{13} - \frac{12}{13}cos\, s\, cos\frac{s}{13}, \frac{5}{13}cos\, s\right), \\ \kappa_2 = \frac{5}{169}cos\, s, \\ \kappa_3 = \frac{5}{169}sin\, s, \\ \kappa_4 = -\frac{157}{169}. \end{cases} \tag{5.7}$$

So, we can illustrate new spherical images, see Figure 4.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Ali, A.T., López, R.: *Slant helices in Minkowski space E_1^3*. J. Korean Math. Soc. **48**(1), 159–167 (2011).
[2] Ali, A.T.: *New special curves and their spherical indicatrices*. Glob. J. Adv. Res. Class. Mod. Geom. **1**(2), 28–38 (2012).
[3] Barros, M.: *General helices and a theorem of Lancret*. Proc. Am. Math. Soc. **125**(5), 1503–1509 (1997).
[4] Barros, M., Ferrández, A., Lucas, P., Meroaño, M.A.: *General helices in the 3-dimensional Lorentzian space forms*. Rocky Mt. J. Math. **31**(2), 373–388 (2001).
[5] Crasmareanu, M.: *The flow-curvature of spacelike parametrized curves in the Lorentz plane*. Proc. Int. Geom. Cent. **15**(2), 101–109 (2022).
[6] Crasmareanu, M.: *The flow-curvature of plane parametrized curves*. Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat. **72**(2), 1–11 (2023).
[7] Crasmareanu, M.: *The flow-geodesic curvature and the flow-evolute of hyperbolic plane curves*. Int. Electron. J. Geom. **16**(2), 225–231 (2023).
[8] Crasmareanu, M.: *The flow-curvature of curves in a geometric surface*. Commun. Korean Math. Soc. **38**(4), 1261–1269 (2023).
[9] Çiftçi, Ü.: *A generalization of Lancert's theorem*. J. Geom. Phys. **59**(12), 1597–1603 (2009).
[10] Ekmekci, N., Okuyucu, O.Z., Yaylı, Y.: *Characterization of speherical helices in Euclidean 3-space*. Analele Stiint. ale Univ. Ovidius Constanta **22**(2), 99–108 (2014).
[11] Ferrández, A., Giménez, A., Lucas, P.: *Null generalized helices in Lorentz-Minkowski spaces*. J. Phys. A Math. Gen. **36**(39), 8243–8251 (2002).
[12] Izumiya, S., Takeuchi, N.: *New special curves and developable surfaces*. Turk. J. Math. **28**(2), 153–163 (2004).
[13] Kula, L., Yaylı, Y.: *On slant helix and its spherical indicatrix*. Appl. Math. Comput. **169**(1), 600–607 (2005).
[14] Kula, L., Ekmekci, N., Yaylı, Y., İlarslan K.: *Characterizations of slant helices in Euclidean 3-space*. Turk. J. Math. **34**, 261–273 (2010).
[15] Lancret, M.: *Mémoire sur les courbes á double courbure*. Mémoires présentés á l'Institut1. 416–454 (1806).
[16] Lucas, P., Ortega-Yagües, J.A.: *Slant helices in the Euclidean 3-space revisited*. Bull. Belg. Math. Soc. - Simon Stevin **23**(1), 133–150 (2016).
[17] Lucas, P., Ortega-Yagües, J.A.: *Slant helices in the three-dimensional sphere*. J. Korean Math. Soc. **54**(4), 1331–1343 (2017).
[18] Lucas, P., Ortega-Yagües, J.A.: *Slant helices: a new approximation*. Turk. J. Math. **43**(1), 473–485 (2019).
[19] Okuyucu, O.Z, Gök, İ., Yaylı, Y., Ekmekci, N.: *Slant helices in three dimensional Lie groups*. Appl. Math. Comput. **221**, 672–683 (2013).
[20] Scofield, P.D.: *Curves of constant precession*. Am. Math. Mon. **102**(6), 531–537 (1995).
[21] Struik, D.J.:Lectures on classical differential geometry. Dover Publications, Inc., New York (1988).
[22] Yampolsky, A., Opariy, A.: *Generalized helices in three-dimensional Lie groups*. Turk. J. Math. **43**, 1447–1455 (2019).

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