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Korovkin Theorem via Statistical e-Modular Convergence of Double Sequences

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Abstract

The main purpose of the present paper is to obtain an abstract version of the Korovkin type theorem via the concept of statistical e-convergence in modular spaces for double sequences of positive linear operators. After proving this theorem, we give an application showing that the new result is stronger than classical ones. Also, we study an extension to non-positive operators.

Keywords: Statistical e-modular convergence, double sequence, abstract Korovkin theorem

1. INTRODUCTION AND PRELIMINARIES

The Pringsheim convergence is well known convergence method for double sequences. Let $\mathbb{N}$ denote the set of all natural numbers. A double sequence $x=(x_{i,j})$ is said to be convergent in Pringsheim's sense if, for every $\varepsilon>0$, there exists $M=M(\varepsilon)\in\mathbb{N}$ such that $|x_{i,j}-Z|<\varepsilon$ whenever $i,j>M$. In this case the Pringsheim limit of x is denoted by $P-\lim x=Z$ and Z is called the Pringsheim limit of x (see [1]). In addition to the Pringsheim convergence, Boos et al. [2,3] introduced and investigated the following notion of e-convergence of double sequences, which is stronger method than Pringsheim's:

A double sequence $x=(x_{i,j})$ is e-convergent to a number L if

$$\forall\varepsilon>0 \quad \exists j_0\in\mathbb{N} \quad \forall j>j_0 \quad \exists i_j\in\mathbb{N} \quad \forall i\geq i_j : |x_{i,j}-Z|<\varepsilon .$$

Then, we write $e-\lim x_{i,j}=Z$. Recently, the statistical e-convergence has been introduced in [4] hereinbelow:

Let $B\subseteq\mathbb{N}$. Then the natural density of B is given by

$$\delta(B):=\lim_j\frac{1}{j}\left|\{k\leq j:k\in B\}\right|$$

provided that the limit on the right-hand side exists, where $|A|$ denotes the cardinality of the set A. Then a sequence $x=(x_{i,j})$ is called statistically e-convergent to the number Z if for every $\varepsilon>0$,

$$\delta\left(\left\{j:\delta\left(\left\{i:|x_{i,j}-Z|\geq\varepsilon\right\}\right)=0\right\}\right)=1.$$

In that case, we write $st_e-\lim_{i,j}x_{i,j}=Z$. Clearly, if a double sequence $x=(x_{i,j})$ is e-convergent then it is statistically e-convergent, too. But, the converse of this implication may not be true. Namely, if the sequence statistically e-convergent then, it does not need to be e-convergent. Also, a double sequence which is statistically e-

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convergent need not to be statistical convergent (see also [4]).

Many researchers studied some versions of Korovkin type theorem by using different type of convergence methods after Bardaro and Mantellini's work[5] on modular spaces and they get interesting results [6-11]. In this paper, we study generalized version of the Korovkin type approximation theorem for the operators $T_{i,j}$, $i,j \in \mathbb{N}$, are acting on an abstract modular function space via statistical e-modular convergence. Then, we give an application showing that our result is stronger than classical ones. We also study an extension to non-positive operators.

Now we recall some well known notations and properties of modular spaces.

Assume that G be a locally compact Hausdorff topological space given via a uniform structure $\mathrm{U} \subset 2^{G\times G}$ that generated the topology of G (see, [12]). Let B be the σ-algebra of all Borel subsets of G and $\mu : \mathrm{B} \to \mathbb{R}$ is a positive σ-finite regular measure. Let the space of all real valued μ-measurable functions on G with identification up to sets of measure μ zero denoted by $L^0(G)$, $C_b(G)$ be the space of all continuous real valued and bounded functions on G and $C_c(G)$ be the subspace of $C_b(G)$ of all functions with compact support on G. In that case, a functional $\rho : L^0(G) \to [0,\infty]$ is a modular on $L^0(G)$ if the following conditions are provided:

(i) $\rho(f)=0$ iff $f=0$ μ-almost everywhere on G,

(ii) $\rho(-f)=\rho(f)$ for every $f \in L^0(G)$,

(iii) $\rho(\alpha f+\beta g) \le \rho(f)+\rho(g)$ for every $f,g \in L^0(G)$ and for any $\alpha,\beta \ge 0$ with $\alpha+\beta=1$.

If there is a constant $N \ge 1$ such that the inequality

$$\rho(\alpha f+\beta g) \le N\alpha\rho(Nf)+N\beta\rho(Ng)$$

holds for every $f,g \in L^0(G)$, $\alpha,\beta \ge 0$ with $\alpha+\beta=1$ then we say that a modular ρ is N-quasi convex. Note that if $N=1$, then ρ is called convex. Furthermore, if there exists a constant $N \ge 1$ such that

$$\rho(\alpha f) \le N\alpha\rho(Nf)$$

holds for every $f \in L^0(G)$ and $\alpha \in (0,1]$ then a modular ρ is called N-quasi semiconvex.

The modular space $L_\rho(G)$ with modular ρ, given by

$$L_\rho(G) := \left\{ f \in L^0(G) : \lim_{\lambda \to 0^+} \rho(\lambda f)=0 \right\}$$

and the space of the finite elements of $L_\rho(G)$, given by

$$E_\rho(G) := \left\{ f \in L_\rho(G) : \rho(\lambda f) < \infty \text{ for all } \lambda > 0 \right\}$$

Also, note that if ρ is N-quasi semiconvex, then the space $\left\{ f \in L^0(G) : \rho(\lambda f) < \infty \text{ for some } \lambda > 0 \right\}$ coincides with $L_\rho(G)$.

We will need the following notions in this paper.

A modular ρ is monotone if $\rho(f) \le \rho(g)$ for $|f| \le |g|$. A modular ρ is called finite if $\chi_A \in L_\rho(G)$ whenever $A \in \mathrm{B}$ with $\mu(A)<\infty$. A modular ρ is strongly finite if $\chi_A \in E_\rho(G)$ for all $A \in \mathrm{B}$ such that $\mu(A)<\infty$ and a modular ρ is said to be absolutely continuous if there exists an $\alpha>0$ such that: for every $f \in L^0(G)$ with $\rho(f)<\infty$, the following conditions hold:

∘ for each $\varepsilon>0$ there exists a set $A \in \mathrm{B}$ such that $\mu(A)<\infty$ and $\rho(\alpha f \chi_{G\setminus A}) \le \varepsilon$,

∘ for every $\varepsilon>0$ there is a $\delta>0$ with $\rho(\alpha f \chi_B) \le \varepsilon$ for every $B \in \mathrm{B}$ with $\mu(B)<\delta$.

If a modular ρ is monotone and finite, then $C(G) \subset L_\rho(G)$. If ρ is monotone and strongly finite, then $C(G) \subset E_\rho(G)$. Also, if ρ is monotone, strongly finite and absolutely continuous, $C_c(G)=L_\rho(G)$ with regard to the modular convergence in the ordinary sense (for details and properties see also [13-15]).

Now we introduce the statistical e-modular and e-strong convergence of double sequences.

1.1 Definition: Let $\left(f_{i,j}\right)$ be a double function sequence that its terms belong to $L_\rho\left(G\right)$. Then, $\left(f_{i,j}\right)$ is said to be statistically e-modularly convergent to $f \in L_\rho\left(G\right)$ iff

$$st_e - \lim_{i,j} \rho\left(\lambda_0\left(f_{i,j} - f\right)\right) = 0$$

for some $\lambda_0 > 0$.

Also, $\left(f_{i,j}\right)$ is statistically e-strongly convergent to f iff

$$st_e - \lim_{i,j} \rho\left(\lambda\left(f_{i,j} - f\right)\right) = 0$$

for every $\lambda > 0$.

We note that if there exists a constant $M > 0$ such that $\rho\left(2f\right) \le M\rho\left(f\right)$ for every $f \in L^0\left(G\right)$ (see [16]) then it is said that the modular ρ satisfies a Δ_2-condition. These two convergence methods are equivalent if and only if the modular satisfies the Δ_2-condition.

Let we introduce the statistical e-superior limit and e-inferior limit of double sequences. For any real double sequence $x = \left(x_{i,j}\right)$, the statistical e-superior limit of x is

$$st_e - \limsup_{i,j} x_{i,j} = \begin{cases} \sup B_x, & \text{if } B_x \ne \varnothing, \\ -\infty & \text{if } B_x = \varnothing, \end{cases}$$

where
$B_x := \left\{b \in \mathbb{R} : \delta\left(\left\{j : \delta\left(\left\{i : x_{i,j} > b\right\}\right) \ne 0\right\}\right) = 1\right\}$ and $\varnothing$ denotes the empty set. Concordantly, in general, by $\delta\left(K\right) \ne 0$ we mean either $\delta\left(K\right) > 0$ or K fails to have the natural density. Similarly, the statistical inferior limit of x is

$$st_e - \liminf_{i,j} x_{i,j} = \begin{cases} \inf A_x, & \text{if } A_x \ne \varnothing, \\ \infty & \text{if } A_x = \varnothing, \end{cases}$$

where
$A_x := \left\{a \in \mathbb{R} : \delta\left(\left\{j : \delta\left(\left\{i : x_{i,j} < a\right\}\right) \ne 0\right\}\right) = 1\right\}$. For any real double sequence $x = \left(x_{i,j}\right)$,

$$st_e - \liminf_{i,j} x_{i,j} \le st_e - \limsup_{i,j} x_{i,j}$$

and also that, for any double sequence $x = \left(x_{i,j}\right)$ satisfying

$$\delta\left(\left\{j : \delta\left(\left\{i : \left|x_{i,j}\right| > M\right\}\right) = 0\right\}\right) = 1$$

for some $M > 0$,

$$st_e - \lim_{i,j} x_{i,j} = Z$$

iff $st_e - \liminf_{i,j} x_{i,j} = st_e - \limsup_{i,j} x_{i,j} = Z.$

2. KOROVKIN TYPE APPROXIMATION VIA STATISTICAL E-CONVERGENCE

Here, we prove a Korovkin type approximation theorem with respect to an abstract finite set of test functions $e_0, e_1, ..., e_k$ via the statistical e-convergence.

Let $\mathrm{T} = \left(T_{i,j}\right)$ be a double sequence of positive linear operators from D into $L^0\left(G\right)$ with $C_b\left(G\right) \subset D \subset L^0\left(G\right)$. Let ρ be monotone and finite modular on $L^0\left(G\right)$. Suggest that the double sequence T, together with modular ρ, satisfies the following property:

there exists a subset $X_{\mathrm{T}} \subset D \cap L_\rho\left(G\right)$ with $C_b\left(G\right) \subset X$ such that the inequality

$$st_e - \limsup_{i,j} \rho\left(\lambda T_{i,j} h\right) \le R\rho\left(\lambda h\right) \tag{2.1}$$

holds for every $h \in X_{\mathrm{T}}$, $\lambda > 0$ and for a positive constant R.

Set $e_0\left(v\right) \equiv 1$ for all $v \in G$, let e_r, $r = 1, 2, ..., k$ and a_r, $r = 0, 1, 2, ..., k$, be functions in $C_b\left(G\right)$. Put

$$P_u\left(v\right) = \sum_{r=0}^{k} a_r\left(u\right) e_r\left(v\right),\ u, v \in G, \tag{2.2}$$

and suppose that $P_u\left(v\right)$, $u, v \in G$, satisfies the following properties:

(P.1) $P_u\left(u\right) = 0$, for all $u \in G$, (P.2) for every neighbourhood $U \in \mathrm{U}$ there exists a positive number η with $P_u(v) \geq \eta$ whenever $u, v \in G$, $(u,v) \notin \mathrm{U}$ (see for examples [17]).

Now, we can give our main theorem of this paper.

2.1.Theorem: Let ρ be a monotone, strongly finite, absolutely continuous and N-quasi semiconvex modular. Suppose that e_r and a_r, $r = 0,1,2,...,k$, satisfy properties (P.1) and (P.2). Let $\mathrm{T} = (T_{i,j})$ be a double sequence of positive linear operators from D into $L^0(G)$ satisfying (2.1). If

$$st_e - \lim_{i,j} \rho\left(\lambda\left(T_{i,j}e_r - e_r\right)\right) = 0 \qquad (2.1.1)$$

for every $\lambda > 0$, $r = 0,1,2,...,k$ in $L_\rho(G)$, then for every $f \in D \cap L_\rho(G)$, with $f - C_b(G) \subset X_{\mathrm{T}}$,

$$st_e - \lim_{i,j} \rho\left(\lambda_0\left(T_{i,j}f - f\right)\right) = 0 \qquad (2.1.2)$$

for some $\lambda_0 > 0$ in $L_\rho(G)$.

Proof: We first claim that, for every $f \in C_c(G)$,

$$st_e - \lim_{i,j} \rho\left(\gamma\left(T_{i,j}f - f\right)\right) = 0 \qquad (2.1.3)$$

for every $\gamma > 0$.

To see this assume that $f \in C_c(G)$. Then, since G is furnished with the uniformity U, f is bounded and uniformly continuous on G. By the uniform continuity of f, choose $\varepsilon \in (0,1]$, there is a set $U \in \mathrm{U}$ such that $|f(u) - f(v)| \leq \varepsilon$ whenever $u, v \in G$, $(u,v) \in \mathrm{U}$.

For all $u, v \in G$ let $P_u(v)$ be as in (2.2), and $\eta > 0$ satisfy condition (P.2). Then for $u, v \in G$, $(u,v) \notin \mathrm{U}$, we have $|f(u) - f(v)| \leq \frac{2M}{\eta} P_u(v)$ where $M := \sup|f(v)|$. Therefore, in any case we get $|f(u) - f(v)| \leq \varepsilon + \frac{2M}{\eta} P_u(v)$ for all $u, v \in G$, namely,

$$-\varepsilon - \frac{2M}{\eta} P_u(v) \leq f(u) - f(v) \leq \varepsilon + \frac{2M}{\eta} P_u(v)$$
(2.1.4)

Since $T_{i,j}$ is linear and positive, by applying it to (2.1.4) for every $i, j \in \mathbb{N}$ we have

$$-\varepsilon T_{i,j}(e_0;u) - \frac{2M}{\eta} T_{i,j}(P_u;u)$$
$$\leq f(u) T_{i,j}(e_0;u) - T_{i,j}(f;u)$$
$$\leq \varepsilon T_{i,j}(e_0;u) + \frac{2M}{\eta} T_{i,j}(P_u;u)$$

Hence

$$\begin{aligned}\left|T_{i,j}(f;u) - f(u)\right| &\leq \left|T_{i,j}(f;u) - f(u)T_{i,j}(e_0;u)\right| \\ &\quad + |f(u)|\left|T_{i,j}(e_0;u) - e_0(u)\right| \\ &\leq \varepsilon T_{i,j}(e_0;u) + \frac{2M}{\eta} T_{i,j}(P_u;u) \\ &\quad + \mathrm{M}\left|T_{i,j}(e_0;u) - e_0(u)\right| \\ &\leq \varepsilon + (\varepsilon + M)\left|T_{i,j}(e_0;u) - e_0(u)\right| \\ &\quad + \frac{2M}{\eta}\sum_{r=0}^{k} a_r(u)\left|T_{i,j}(e_r;u) - e_r(u)\right|\end{aligned}$$

Let $\gamma > 0$. Now for each $r = 0,1,2,...,k$ and $u \in G$, choose $M_0 > 0$ such that $|a_r(u)| \leq M_0$ the last inequality gives that

$$\gamma\left|T_{i,j}(f;u) - f(u)\right| \leq \gamma\varepsilon + K\gamma\sum_{r=0}^{k}\left|T_{i,j}(e_r;u) - e_r(u)\right|$$

where $K := \varepsilon + M + \left(\frac{2M}{\eta}\right)M_0$. By applying the modular ρ to the above inequality and using the monotonicity of ρ, we get

$$\rho\left(\gamma\left(T_{i,j}f - f\right)\right) \leq \rho\left(\gamma\varepsilon + K\gamma\sum_{r=0}^{k}\left(T_{i,j}e_r - e_r\right)\right).$$

Thus, we can see that

$$\begin{aligned}\rho\left(\gamma\left(T_{i,j}f - f\right)\right) &\leq \rho\left((k+2)\gamma\varepsilon\right) \\ &\quad + \sum_{r=0}^{k}\rho\left((k+2)K\gamma\left(T_{i,j}e_r - e_r\right)\right).\end{aligned}$$

Because of ρ is strongly finite and N-quasi semiconvex, we get,

$$\begin{aligned}\rho\left(\gamma\left(T_{i,j}f - f\right)\right) &\leq N\varepsilon\rho\left((k+2)\gamma N\right) \\ &\quad + \sum_{r=0}^{k}\rho\left((k+2)K\gamma\left(T_{i,j}e_r - e_r\right)\right).\end{aligned}$$
(2.1.5)

For a given $\varepsilon^*>0$, choose an $\varepsilon\in(0,1]$ such that $N\varepsilon\rho((k+2)\gamma N)<\varepsilon^*$. Now we define the followings:

$$S_\gamma:=\left\{j:\delta\left(\left\{i:\rho\left(\gamma\left(T_{i,j}f-f\right)\right)\geq\varepsilon^*\right\}\right)=0\right\}$$

$$S_{\gamma,r}:=\left\{j:\delta\left(\left\{i:\rho\left((k+2)K\gamma\left(T_{i,j}e_r-e_r\right)\right)\geq\frac{\varepsilon^*-N\varepsilon\rho((k+2)\gamma N)}{k+1}\right\}\right)=0\right\},$$

where $r=0,1,2,...,k$. Then, by hypothesis (2.1.1) we get $\delta(S_{\gamma,r})=1$, $r=0,1,2,...,k$. If we take $S_{\gamma,k+1}=\bigcap_{r=0}^{k}S_{\gamma,r}$, we have $\delta(S_{\gamma,k+1})=1$. For each $j\in S_{\gamma,k+1}$ we define

$$S_{\gamma,r}^{j}=\left\{i:\rho\left((k+2)K\gamma\left(T_{i,j}e_r-e_r\right)\right)\geq\frac{\varepsilon^*-N\varepsilon\rho((k+2)\gamma N)}{k+1}\right\},$$

$r=0,1,2,...,k$. From the inequality (2.1.5) for each $j\in S_{\gamma,k+1}$

$$\left\{i:\rho\left(\gamma\left(T_{i,j}f-f\right)\right)\geq\varepsilon^*\right\}\subseteq\bigcup_{r=0}^{k}S_{\gamma,r}^{j}.$$

Hence $\delta(S_{\gamma,r}^{j})=0$, we obtain

$$\delta\left(\left\{i:\rho\left(\gamma\left(T_{i,j}f-f\right)\right)\geq\varepsilon^*\right\}\right)=0.$$

This implies that $S_{\gamma,k+1}\subseteq S_\gamma$. So, $\delta(S_\gamma)=1$, which proves our claim (2.1.3). Now, let $f\in D\cap L_\rho(G)$ with $f-C_b(G)\subset X_{\mathrm{T}}$. It is known from ([14], [18]) that there exists a sequence $(g_{k,l})\subset C_c(G)$ such that $\rho(3\lambda_0^*f)<\infty$ and $P-\lim_{k,l}\rho\left(3\lambda_0^*\left(g_{k,l}-f\right)\right)=0$ for some $\lambda_0^*>0$. That is to say, for every $\varepsilon>0$, there is a positive number $k_0=k_0(\varepsilon)$ with

$$\rho\left(3\lambda_0^*\left(g_{k,l}-f\right)\right)<\varepsilon \text{ for every } k,l\geq k_0. \quad (2.1.6)$$

For all $i,j\in\mathbb{N}$, since the operators $T_{i,j}$ are linear and positive, we have

$$\begin{aligned}\lambda_0^*\left|T_{i,j}(f;u)-f(u)\right|\leq{}&\lambda_0^*\left|T_{i,j}\left(f-g_{k_0,k_0};u\right)\right|\\&+\lambda_0^*\left|T_{i,j}\left(g_{k_0,k_0};u\right)-g_{k_0,k_0}(u)\right|\\&+\lambda_0^*\left|g_{k_0,k_0}(u)-f(u)\right|\end{aligned}$$

holds for every $u\in G$. Now, applying modular ρ in the last inequality and using the monotonicity of ρ, we get

$$\begin{aligned}\rho\left(\lambda_0^*\left(T_{i,j}f-f\right)\right)\leq{}&\rho\left(3\lambda_0^*T_{i,j}\left(f-g_{k_0,k_0}\right)\right)\\&+\rho\left(3\lambda_0^*\left(T_{i,j}g_{k_0,k_0}-g_{k_0,k_0}\right)\right)\\&+\rho\left(3\lambda_0^*\left(g_{k_0,k_0}-f\right)\right).\end{aligned}$$

Hence, we have

$$\begin{aligned}\rho\left(\lambda_0^*\left(T_{i,j}f-f\right)\right)\leq{}&\varepsilon+\rho\left(3\lambda_0^*T_{i,j}\left(f-g_{k_0,k_0}\right)\right)\\&+\rho\left(3\lambda_0^*\left(T_{i,j}g_{k_0,k_0}-g_{k_0,k_0}\right)\right).\end{aligned}$$

By property (2.1) and also by using $g_{k_0,k_0}\in C_c(G)$ and $f-g_{k_0,k_0}\in X_{\mathrm{T}}$, we obtain

$$\begin{aligned}&st_e-\limsup_{i,j}\rho\left(\lambda_0^*\left(T_{i,j}f-f\right)\right)\leq\varepsilon+R\rho\left(3\lambda_0^*\left(f-g_{k_0,k_0}\right)\right)\\&+st_e-\limsup_{i,j}\rho\left(3\lambda_0^*\left(T_{i,j}g_{k_0,k_0}-g_{k_0,k_0}\right)\right)\\&\leq\varepsilon(1+R)+st_e-\limsup_{i,j}\rho\left(3\lambda_0^*\left(T_{i,j}g_{k_0,k_0}-g_{k_0,k_0}\right)\right)\end{aligned}$$

also, by (2.1.3)

$$\begin{aligned}0&=st_e-\lim_{i,j}\rho\left(3\lambda_0^*\left(T_{i,j}g_{k_0,k_0}-g_{k_0,k_0}\right)\right)\\&=st_e-\limsup_{i,j}\rho\left(3\lambda_0^*\left(T_{i,j}g_{k_0,k_0}-g_{k_0,k_0}\right)\right)\end{aligned}$$

which gives

$$0\leq st_e-\limsup_{i,j}\rho\left(\lambda_0^*\left(T_{i,j}f-f\right)\right)\leq\varepsilon(1+R).$$

From arbitrariness of $\varepsilon>0$, it follows that

$$st_e-\limsup_{i,j}\rho\left(\lambda_0^*\left(T_{i,j}f-f\right)\right)=0.$$

Furthermore,

$$st_e-\lim_{i,j}\rho\left(\lambda_0^*\left(T_{i,j}f-f\right)\right)=0,$$

this completes the proof.

2.2.Remark: In general, it is not possible to get statistical e-strong convergence unless the modular ρ satisfies the Δ_2-condition in 2.1.Theorem.

If we take the e-limit instead of the statistical e-limit, then the condition (2.1) reduces to

$$e-\limsup_{i,j} \rho\left(\lambda T_{i,j}h\right) \leq R\rho\left(\lambda h\right) \qquad (2.2.1)$$

for every $h \in X_{\mathrm{T}}$, $\lambda > 0$ and for an absolute positive constant R. In that case, the following result immediately ensue from our 2.1.Theorem.

2.3.Corollary: Let ρ be a monotone, absolutely continuous, N-quasi semiconvex and strongly finite modular. Suppose that e_r and a_r, $r = 0,1,2,...,k$, satisfy properties (P.1) and (P.2). Let $\mathrm{T} = \left(T_{i,j}\right)$ be a double sequence of positive linear operators satisfying (2.2.1). If $\left(T_{i,j}e_r\right)$ is e-strongly convergent to e_r, $r = 0,1,2,...,k$, in $L_\rho\left(G\right)$ then $\left(T_{i,j}f\right)$ is e-modularly convergent to f in $L_\rho\left(G\right)$ where f is any function in $D \cap L_\rho\left(G\right)$ with $f - C_b\left(G\right) \subset X_{\mathrm{T}}$. Now, we give an application showing that in general, our result is stronger.

2.4.Example: Let us consider $G = \left[0,1\right]^2 = \left[0,1\right] \times \left[0,1\right] \subset \mathbb{R}^2$ and let $\varphi : \left[0,\infty\right) \to \left[0,\infty\right)$ is a continuous function with φ is convex, $\varphi\left(0\right) = 0$, $\varphi\left(x\right) > 0$ for any $x > 0$ and $\lim_{x\to\infty} \varphi\left(x\right) = \infty$. Then, the functional ρ^φ defined by

$$\rho^\varphi\left(f\right) := \int_0^1\int_0^1 \varphi\left(\left|f\left(x,y\right)\right|\right)dxdy \text{ for } f \in L^0\left(G\right),$$

is a convex modular on $L^0\left(G\right)$ and

$$L^\varphi\left(G\right) := \left\{f \in L^0\left(G\right) : \rho^\varphi\left(\lambda f\right) < +\infty \text{ for some } \lambda > 0\right\}$$

is the Orlicz space generated by φ.

For every $\left(x,y\right) \in G$, let $e_0\left(x,y\right) = a_3\left(x,y\right) = 1$, $e_1\left(x,y\right) = x$, $e_2\left(x,y\right) = y$, $e_3\left(x,y\right) = a_0\left(x,y\right) = x^2 + y^2$, $a_1\left(x,y\right) = -2x$, $a_2\left(x,y\right) = -2y$.

For every $i,j \in \mathbb{N}$, $u_1,u_2 \in \left[0,1\right]$, let $K_{i,j}\left(u_1,u_2\right) = \left(i+1\right)\left(j+1\right)u_1^i u_2^j$ and for $f \in C\left(G\right)$ and $x,y \in \left[0,1\right]$ set

$$M_{i,j}\left(f;x,y\right) = \int_0^1\int_0^1 K_{i,j}\left(u_1,u_2\right) f\left(u_1x,u_2y\right) du_1du_2.$$

Then we get

$$\int_0^1\int_0^1 K_{i,j}\left(u_1,u_2\right) du_1du_2$$

$$= \left(i+1\right)\left(\int_0^1 u_1^i du_1\right)\left(j+1\right)\left(\int_0^1 u_2^j du_2\right) = 1,$$

and hence, $M_{i,j}\left(e_0;x,y\right) = e_0\left(x,y\right) = 1$. Also, we know from [19] that

$$\left|M_{i,j}\left(e_1;x,y\right) - e_1\left(x,y\right)\right| \leq \frac{1}{i+2},$$

$$\left|M_{i,j}\left(e_2;x,y\right) - e_2\left(x,y\right)\right| \leq \frac{1}{j+2},$$

$$\left|M_{i,j}\left(e_1^2;x,y\right) - e_1^2\left(x,y\right)\right| \leq \frac{2}{i+3},$$

$$\left|M_{i,j}\left(e_2^2;x,y\right) - e_2^2\left(x,y\right)\right| \leq \frac{2}{j+3},$$

and for each $i,j \geq 2$, $f \in L^\varphi\left(G\right)$ we get $\rho^\varphi\left(M_{i,j}f\right) \leq 32\rho^\varphi\left(f\right)$. Now, we define the following double sequence of positive linear operators $\mathrm{T} = \left(T_{i,j}\right)$ on $L^\varphi\left(G\right)$ by using the operators $\mathrm{M} = \left(M_{i,j}\right)$:

$T_{i,j}\left(f;x,y\right) = s_{i,j}M_{i,j}\left(f;x,y\right)$, for $f \in L^\varphi\left(G\right)$, $x,y \in \left[0,1\right]$ and $i,j \in \mathbb{N}$, where $\left(s_{i,j}\right)$ is given by

$$s_{i,j} = \begin{cases} 2, & i \leq j, \\ 0, & i > j \text{ and } i \text{ is square}, \\ 1, & i > j \text{ and } i \text{ is not square}. \end{cases}$$

Observe now that $st_e - \lim_{i,j} s_{i,j} = 1$. However, $e - \lim_{i,j} s_{i,j}$, $P - \lim_{i,j} s_{i,j}$ and $st - \lim_{i,j} s_{i,j}$ do not exist. Then, it can be easily seen that, for every $h \in X_{\mathrm{T}} = L^\varphi\left(G\right)$, $\lambda > 0$ and for positive constant R_0 that

$$st_e - \limsup_{i,j} \rho^\varphi\left(\lambda T_{i,j}h\right) \leq R_0\rho^\varphi\left(\lambda h\right).$$

Now, observe that

$$T_{i,j}\left(e_0;x,y\right) - e_0\left(x,y\right) = s_{i,j} - 1,$$

$$T_{i,j}\left(e_1;x,y\right)-e_1\left(x,y\right)\leq\frac{2}{i+2}+\left(s_{i,j}-1\right),$$

$$T_{i,j}\left(e_2;x,y\right)-e_2\left(x,y\right)\leq\frac{2}{j+2}+\left(s_{i,j}-1\right),$$

$$T_{i,j}\left(e_3;x,y\right)-e_3\left(x,y\right)$$
$$\leq 4\left(\frac{1}{i+3}+\frac{1}{j+3}\right)+2\left(s_{i,j}-1\right).$$

Hence, we can see, for any $\lambda>0$, that

$$\rho^{\varphi}\left(\lambda\left(T_{i,j}e_0-e_0\right)\right)=\rho^{\varphi}\left(\lambda\left(s_{i,j}-1\right)\right)$$
$$=\int_0^1\int_0^1\varphi\left(\left|\lambda\left(s_{i,j}-1\right)\right|\right)dxdy=\varphi\left(\left|\lambda\left(s_{i,j}-1\right)\right|\right)$$
$$=\left|s_{i,j}-1\right|\varphi\left(\lambda\right)$$

(2.4.1)

because of the definition of $\left(s_{i,j}\right)$. Now, since

$st_e-\lim_{i,j}\left(s_{i,j}-1\right)=0$, we get

$$st_e-\lim_{i,j}\rho^{\varphi}\left(\lambda\left(T_{i,j}e_0-e_0\right)\right)=0.$$

Also, we have

$$\rho^{\varphi}\left(\lambda\left(T_{i,j}e_1-e_1\right)\right)\leq\rho^{\varphi}\left(\lambda\left(\frac{2}{i+2}+\left(s_{i,j}-1\right)\right)\right)$$
$$\leq\rho^{\varphi}\left(\frac{\lambda}{i+2}\right)+\rho^{\varphi}\left(2\lambda\left(s_{i,j}-1\right)\right)$$
$$=\varphi\left(\frac{\lambda}{i+2}\right)+\varphi\left(\left|2\lambda\left(s_{i,j}-1\right)\right|\right),$$

which implies, for any $\lambda>0$, that

$$\rho^{\varphi}\left(\lambda\left(T_{i,j}e_1-e_1\right)\right)\leq\varphi\left(\frac{\lambda}{i+2}\right)+\left|s_{i,j}-1\right|\varphi\left(2\lambda\right)$$

(2.4.2)

since φ is continuous, we have

$$e-\lim_{i,j}\varphi\left(\frac{\lambda}{i+2}\right)=\varphi\left(e-\lim_{i,j}\frac{\lambda}{i+2}\right)=\varphi\left(0\right)=0,$$

also considering $st_e-\lim_{i,j}\left(s_{i,j}-1\right)=0$, from the inequality (2.4.2), we get

$$st_e-\lim_{i,j}\rho^{\varphi}\left(\lambda\left(T_{i,j}e_1-e_1\right)\right)=0.$$

Similarly, we can write that

$$st_e-\lim_{i,j}\rho^{\varphi}\left(\lambda\left(T_{i,j}e_2-e_2\right)\right)=0.$$

Finally, since

$$\rho^{\varphi}\left(\lambda\left(T_{i,j}e_3-e_3\right)\right)$$
$$\leq\rho^{\varphi}\left(\lambda\left(4\left(\frac{1}{i+3}+\frac{1}{j+3}\right)+2\left(s_{i,j}-1\right)\right)\right)$$
$$\leq\rho^{\varphi}\left(8\lambda\left(\frac{1}{i+3}+\frac{1}{j+3}\right)\right)+\rho^{\varphi}\left(4\lambda\left(s_{i,j}-1\right)\right)$$
$$=\varphi\left(8\lambda\left(\frac{1}{i+3}+\frac{1}{j+3}\right)\right)+\varphi\left(\left|4\lambda\left(s_{i,j}-1\right)\right|\right),$$

which yields

$$\rho^{\varphi}\left(\lambda\left(T_{i,j}e_3-e_3\right)\right)\leq\varphi\left(\frac{16\lambda}{i+3}\right)+\varphi\left(\frac{16\lambda}{j+3}\right)$$
$$+\left|s_{i,j}-1\right|\varphi\left(4\lambda\right)$$

(2.4.3)

then, since φ is continuous

$$e-\lim_{i,j}\varphi\left(\frac{16\lambda}{i+3}\right)=0 \text{ and } e-\lim_{i,j}\varphi\left(\frac{16\lambda}{j+3}\right)=0,$$

it follows from the inequality (2.4.3) that

$$st_e-\lim_{i,j}\rho^{\varphi}\left(\lambda\left(T_{i,j}e_3-e_3\right)\right)=0.$$

So, our new operator $\mathrm{T}=\left(T_{i,j}\right)$ satisfies all conditions of 2.1.Theorem and therefore we obtain

$$st_e-\lim_{i,j}\rho^{\varphi}\left(\lambda\left(T_{i,j}f-f\right)\right)=0$$

for some $\lambda>0$, for any $f\in L^{\varphi}\left(G\right)$. However, $\left(T_{i,j}e_0\right)$ is not e-modularly convergent. Thus $\left(T_{i,j}\right)$ does not fulfil the 2.3.Corollary. Also, $\left(T_{i,j}e_0\right)$ is neither modulary convergent nor statistically modularly convergent. Hence, modular Korovkin theorem and statistical modular Korovkin theorem for double sequences do not satisfy.

3. AN EXTENSION TO NON-POSITIVE OPERATORS

In this section, we relax the positivity condition of linear operators in the Korovkin theorem. In ([17], [19]) there are some positive answers. Following this approach, we give some positive answers also for statistical e-modular convergence and we prove a statistical e-modular Korovkin theorem.

Let $I \subset \mathbb{R}$ be a bounded interval, $C^2(I)$ (resp. $C_b^2(I)$) be the space of all functions defined on I, (resp. bounded and) continuous with their first and second derivatives,

$C_+ := \{ f \in C_b^2(I) : f \geq 0 \}$,
$C_+^2 := \{ f \in C_b^2(I) : f'' \geq 0 \}$.

Let e_r, $r = 1,2,...,k$ and a_r, $r = 0,1,2,...,k$, be functions in $C_b^2(I)$, $P_u(v)$, $u,v \in I$, be as in (2.1.4), and suppose that $P_u(v)$ satisfies the properties (P.1), (P.2) and

(P.3) there exists a positive real constant S_0 such that $P_u''(v) \geq S_0$ for all $u,v \in I$ (The second derivative is intended with respect to v).

Now, we prove the following Korovkin type approximation theorem for linear operators that not necessarly positive .

3.1.Theorem: Let ρ be as in 2.1.Theorem and e_r, a_r, $r = 0,1,2,...,k$ and $P_u(v)$, $u,v \in I$, satisfies the properties (P.1), (P.2) and (P.3). Assume that $\mathrm{T} = (T_{i,j})$ be a double sequence of linear operators and $T_{i,j}(C_+ \cap C_+^2) \subset C_+$ for all $i,j \in \mathbb{N}$. If $T_{i,j}e_r$ is statistically e-strongly convergent to e_r, $r = 0,1,2,...,k$, in $L_\rho(I)$ then $T_{i,j}f$ is statistically e-modularly convergent to f in $L_\rho(I)$ for every $f \in D \cap L_\rho(I)$ with $f - C_b(I) \subset X_{\mathrm{T}}$.

Proof: Let $f \in C_b^2(I)$. Since f is uniformly continuous and bounded on I, given $\varepsilon > 0$ with $0 < \varepsilon \leq 1$, there exists a $\delta > 0$ such that $|f(u) - f(v)| \leq \varepsilon$ for all $u,v \in I$, $|u - v| \leq \delta$. Let $P_u(v)$, $u,v \in I$, be as in (2.2) and let $\eta > 0$ be associated with δ, satisfying (P.2). As in 2.1.Theorem, for every $\beta \geq 1$ and $u,v \in I$, we have

$$-\varepsilon - \frac{2M}{\eta} P_u(v) \leq f(u) - f(v) \leq \varepsilon + \frac{2M}{\eta} P_u(v)$$

(3.1.1)

where $M := \sup_{v \in I} |f(v)|$. From (3.1.1) it follows that

$$h_{1,\beta}(v) := \varepsilon + \frac{2M\beta}{\eta} P_u(v) + f(v) - f(u) \geq 0,$$

(3.1.2)

$$h_{2,\beta}(v) := \varepsilon + \frac{2M\beta}{\eta} P_u(v) - f(v) + f(u) \geq 0.$$

(3.1.3)

Let H_0 satisfy (P3). For each $v \in I$, we get

$$h_{1,\beta}''(v) \geq \frac{2M\beta H_0}{\eta} + f''(v),$$

$$h_{2,\beta}''(v) \geq \frac{2M\beta H_0}{\eta} - f''(v).$$

Because of f'' is bounded on I, we can choose $\beta \geq 1$ in such a way that $h_{1,\beta}''(v) \geq 0$, $h_{2,\beta}''(v) \geq 0$ for each $v \in I$. Hence $h_{1,\beta}, h_{2,\beta} \in C_+ \cap C_+^2$ and then, by hypothesis

$T_{i,j}(h_{\kappa,\beta}; u) \geq 0$ for all $i,j \in \mathbb{N}$, $u \in I$ and $\kappa = 1,2.$ (3.1.4)

From (3.1.2)-(3.1.4) and the linearity of $T_{i,j}$, we get

$$\varepsilon T_{i,j}(e_0;u) + \frac{2M\beta}{\eta} T_{i,j}(P_u;u) + T_{i,j}(f;u) - f(u) T_{i,j}(e_0;u) \geq 0,$$

$$\varepsilon T_{i,j}(e_0;u) + \frac{2M\beta}{\eta} T_{i,j}(P_u;u) - T_{i,j}(f;u) + f(u) T_{i,j}(e_0;u) \geq 0,$$

thus,

$$-\varepsilon T_{i,j}(e_0;u) - \frac{2M\beta}{\eta} T_{i,j}(P_u;u) \leq f(u) T_{i,j}(e_0;u) - T_{i,j}(f;u) \leq \varepsilon T_{i,j}(e_0;u) + \frac{2M\beta}{\eta} T_{i,j}(P_u;u).$$

Similarly as in the proof of 2.1.Theorem, using the modular ρ and for $i,j \in \mathbb{N}$, we have the assertion.

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