Sakarya Üniversitesi Fen Bilimleri Dergisi
Sakarya University Journal of Science

e-ISSN : 2147-835X
Publisher : Sakarya University

Vol. 29, No. 4, 441-449, 2025
DOI: https://doi.org/10.16984/saufenbilder.1694462

Research Article

Dynamics and Expressions of Solutions of Nonlinear Difference Equations $x_{n+1} = \frac{x_{n-3}x_{n-6}}{\pm x_{n-2} \pm x_{n-2}x_{n-3}x_{n-6}}$

Burak Oğul[1*], Dağıstan Şimşek[2,3]

[1] Istanbul Aydın University, Faculty of Applied Science, Department of Management Information Systems, Istanbul, Türkiye, burakogul@aydin.edu.tr, ror.org/00qsyw664
[2] Konya Technical University, Faculty of Engineering and Natural Sciences, Department of Engineering Basic Sciences, Konya, Türkiye, dsimsek@ktun.edu.tr, ror.org/02s82rs08
[3] Kyrgyz Turkish Manas University, Faculty of Sciences, Department of Applied Mathematics and Informatics, Bishkek, Kyrgyzstan, ror.org/04frf8n21
[*]Corresponding Author

ARTICLE INFO

Keywords:
Difference equations
Stability
Recursive sequences

Article History:
Received: 07.05.2025
Revised: 23.07.2025
Accepted: 06.08.2025
Online Available: 27.08.2025

ABSTRACT

Nonlinear difference equations provide a framework for modeling natural phenomena in nonlinear sciences. In this paper, we investigate the periodicity, boundedness, oscillation, stability, and exact solutions of such equations. Employing the standard iteration method, we derive closed-form solutions and analyze the stability of equilibrium points using established theorems. Numerical simulations, implemented in Wolfram Mathematica, corroborate the theoretical findings. The proposed method can be readily extended to other rational recursive problems. This paper investigates the dynamical behavior of solutions to the rational difference equation

$$x_{n+1} = \frac{x_{n-3}x_{n-6}}{\pm x_{n-2} \pm x_{n-2}x_{n-3}x_{n-6}}$$

where the initial conditions are arbitrary nonzero real numbers. We analyze the stability properties, periodic solutions, and long-term behavior of this equation, employing both analytical and numerical approaches to characterize its dynamics.

1. Introduction

Rational difference equations have become an important area of study in recent years, mainly due to their applications in fields like mathematical biology, economics, and engineering. These equations are a useful tool for modeling discrete-time dynamical systems. The solutions can display complex behaviors, such as periodicity, bifurcations, and chaos. Analyzing key properties, such as stability, boundedness, and oscillation, is essential for understanding the system's dynamics and developing control strategies.

The literature on rational difference equations has evolved significantly, beginning with foundational studies on first and second-order equations and gradually extending to higher-order and nonlinear forms. The present work employs the definitions and theorems related to stability, periodicity, and global asymptotic stability as established in [1]. Early works by Kulenovic et al. [2] and DeVault et al. [3] established key stability results for linear fractional recursive sequences, while Amleh et al. [4] and Gibbons et al. [5] expanded these findings to more complex nonlinear structures. The theoretical framework was further solidified by Elaydi [6], whose comprehensive text provided essential tools for analyzing difference equations, including Poincaré-Perron theory and transformation methods.

Cite as: B. Oğul, D. Şimşek, "Dynamics and expressions of solutions of nonlinear difference equations x_(n+1)=(x_(n-3) x_(n-6))/(±x_(n-2)±x_(n-2) x_(n-3) x_(n-6))," Sakarya University Journal of Science, vol. 29, no. 4, pp. 441-449, 2025. https://doi.org/10.16984/saufenbilder.1694462

Recent research has focused on higher-order rational difference equations, with notable contributions from Elsayed [7], who investigated periodic solutions in sixth-order equations, and Agarwal and Elsayed [8], who derived closed-form solutions for fourth-order recursive sequences. The work of Stević et al. [9] introduced solvable product-type systems, offering new insights into exact solutions. Meanwhile, numerical and computational approaches, such as those by Aloqeili [10] and Karataş et al. [11], have complemented theoretical analyses by verifying stability and chaos in specific cases.

A parallel line of inquiry has explored the connections between differential and difference equations, as demonstrated by Karpenko et al. [12] and Bohner et al. [13], who examined the relationship between boundedness and oscillation in continuous and discrete systems. The latest advancements, including Almatrafi et al. [14] study on fractional difference equations, highlight the growing interest in extending classical results to more generalized frameworks.

High-order and nonlinear rational difference equations have garnered significant interest in recent years due to their ability to model complex and rich dynamical behaviors in discrete systems. A growing body of research has focused on exploring various qualitative aspects of such equations particularly those of very high order including boundedness, stability, periodicity, and chaotic dynamics. The studies in [15–23] contribute to this expanding literature by introducing novel forms of rational recursive equations and analyzing their global behavior, equilibrium structures, and long-term dynamics. These works collectively demonstrate that as the order and nonlinearity of a system increase, so does the diversity of its dynamical outcomes, making analytical and numerical investigations both challenging and essential.

The aim of this study is to examine several dynamical features, including equilibrium points, local and global behaviors, boundedness, and analytical solutions to nonlinear recursive equations.

$$x_{n+1} = \frac{x_{n-3}x_{n-6}}{\pm x_{n-2} \pm x_{n-2}x_{n-3}x_{n-6}}. \quad (1)$$

The initial values $x_{-6}, x_{-5}, x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary non-zero real numbers. This study also employs Wolfram Mathematica to depict several 2D figures, so validating the conclusions achieved.

2. Solution the Difference Equation $x_{n+1} = \frac{x_{n-3}x_{n-6}}{x_{n-2}+x_{n-2}x_{n-3}x_{n-6}}$

This section presents a particular form of the answers to the following difference equation, assuming the beginning conditions are arbitrary real integers,

$$x_{n+1} = \frac{x_{n-3}x_{n-6}}{x_{n-2} + x_{n-2}x_{n-3}x_{n-6}} \quad (2)$$

where, $x_{-6} = g$, $x_{-5} = f$, $x_{-4} = e$, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, $x_0 = a$.

Theorem 2.1. Suppose that $\{x_n\}_{n=-6}^{\infty}$ be a solution of Eq(2). Then,

$$x_{24n+1} = \frac{\prod_{i=0}^{2n-1}\left((1+3i)ad+1\right)\prod_{i=0}^{2n-1}\left((3+3i)cf+1\right)db^{2n}g^{2n+1}e^{2n}}{\prod_{i=0}^{2n}\left((1+3i)dg+1\right)\prod_{i=0}^{2n-1}\left((2+3i)be+1\right)a^{2n}c^{2n+1}f^{2n}},$$

$$x_{24n+2} = \frac{\prod_{i=0}^{2n-1}\left((1+3i)dg+1\right)\prod_{i=0}^{2n-1}\left((3+3i)be+1\right)a^{2n}c^{2n+1}f^{2n+1}}{\prod_{i=0}^{2n}\left((1+3i)cf+1\right)\prod_{i=0}^{2n-1}\left((2+3i)ad+1\right)g^{2n}b^{2n+1}e^{2n}},$$

$$x_{24n+3} = \frac{\prod_{i=0}^{2n-1}\left((3+3i)ad+1\right)\prod_{i=0}^{2n-1}\left((2+3i)cf+1\right)e^{2n+1}b^{2n+1}g^{2n}}{\prod_{i=0}^{2n}\left((1+3i)be+1\right)\prod_{i=0}^{2n-1}\left((3+3i)dg+1\right)a^{2n+1}c^{2n}f^{2n}},$$

$$x_{24n+4} = \frac{\prod_{i=0}^{2n}\left((1+3i)dg+1\right)\prod_{i=0}^{2n-1}\left((2+3i)be+1\right)a^{2n+1}c^{2n+1}f^{2n}}{\prod_{i=0}^{2n}\left((1+3i)ad+1\right)\prod_{i=0}^{2n-1}\left((3+3i)cf+1\right)g^{2n+1}b^{2n}e^{2n}},$$

$$x_{24n+5} = \frac{\prod_{i=0}^{2n}\left((1+3i)cf+1\right)\prod_{i=0}^{2n-1}\left((2+3i)ad+1\right)db^{2n+1}g^{2n+1}e^{2n}}{\prod_{i=0}^{2n}\left((2+3i)dg+1\right)\prod_{i=0}^{2n-1}\left((3+3i)be+1\right)a^{2n}c^{2n+1}f^{2n+1}},$$

$$x_{24n+6} = \frac{\prod_{i=0}^{2n}\left((1+3i)be+1\right)\prod_{i=0}^{2n-1}\left((3+3i)dg+1\right)a^{2n+1}c^{2n+1}f^{2n+1}}{\prod_{i=0}^{2n}\left((2+3i)cf+1\right)\prod_{i=0}^{2n-1}\left((3+3i)ad+1\right)g^{2n}b^{2n+1}e^{2n+1}},$$

$$x_{24n+7} = \frac{\prod_{i=0}^{2n}\left((1+3i)ad+1\right)\prod_{i=0}^{2n-1}\left((3+3i)cf+1\right)g^{2n+1}b^{2n+1}e^{2n+1}}{\prod_{i=0}^{2n}\left((2+3i)be+1\right)\prod_{i=0}^{2n}\left((1+3i)dg+1\right)f^{2n}c^{2n+1}a^{2n+1}},$$

$$x_{24n+8} = \frac{\prod_{i=0}^{2n}\left((2+3i)dg+1\right)\prod_{i=0}^{2n-1}\left((3+3i)be+1\right)a^{2n+1}c^{2n+1}f^{2n+1}}{\prod_{i=0}^{2n}\left((2+3i)ad+1\right)\prod_{i=0}^{2n}\left((1+3i)cf+1\right)e^{2n}b^{2n+1}g^{2n+1}},$$

$$x_{24n+9} = \frac{\prod_{i=0}^{2n}\left((2+3i)cf+1\right)\prod_{i=0}^{2n-1}\left((3+3i)ad+1\right)db^{2n+1}c^{2n+1}f^{2n+1}}{\prod_{i=0}^{2n}\left((2+3i)ad+1\right)\prod_{i=0}^{2n}\left((1+3i)cf+1\right)e^{2n}b^{2n+1}g^{2n+1}},$$

$$x_{24n+10} = \frac{\prod_{i=0}^{2n}\left((1+3i)dg+1\right)\prod_{i=0}^{2n}\left((2+3i)be+1\right)a^{2n+1}c^{2n+2}f^{2n+1}}{\prod_{i=0}^{2n}\left((1+3i)ad+1\right)\prod_{i=0}^{2n}\left((3+3i)cf+1\right)b^{2n+1}e^{2n+1}g^{2n+1}},$$

$$x_{24n+11} = \frac{\prod_{i=0}^{2n}\left((1+3i)be+1\right)\prod_{i=0}^{2n}\left((2+3i)ad+1\right)b^{2n+2}e^{2n+1}g^{2n+1}}{\prod_{i=0}^{2n}\left((2+3i)dg+1\right)\prod_{i=0}^{2n}\left((3+3i)be+1\right)a^{2n+1}c^{2n+1}f^{2n+1}},$$

$$x_{24n+14} = \frac{\prod_{i=0}^{2n}\left((2+3i)dg+1\right)\prod_{i=0}^{2n}\left((3+3i)be+1\right)a^{2n+1}c^{2n+2}f^{2n+2}}{\prod_{i=0}^{2n+1}\left((1+3i)cf+1\right)\prod_{i=0}^{2n}\left((2+3i)ad+1\right)b^{2n+2}e^{2n+1}g^{2n+1}},$$

$$x_{24n+15} = \frac{\prod_{i=0}^{2n}\left((2+3i)cf+1\right)\prod_{i=0}^{2n}\left((3+3i)ad+1\right)b^{2n+2}e^{2n+2}g^{2n+1}}{\prod_{i=0}^{2n+1}\left((1+3i)be+1\right)\prod_{i=0}^{2n}\left((3+3i)dg+1\right)a^{2n+2}c^{2n+1}f^{2n+1}},$$

$$x_{24n+16} = \frac{\prod_{i=0}^{2n+1}\left((1+3i)dg+1\right)\prod_{i=0}^{2n}\left((2+3i)be+1\right)a^{2n+2}c^{2n+2}f^{2n+1}}{\prod_{i=0}^{2n+1}\left((1+3i)ad+1\right)\prod_{i=0}^{2n}\left((3+3i)cf+1\right)b^{2n+1}e^{2n+1}g^{2n+2}},$$

$$x_{24n+17} = \frac{\prod_{i=0}^{2n+1}\left((1+3i)cf+1\right)\prod_{i=0}^{2n}\left((2+3i)ad+1\right)b^{2n+2}e^{2n+1}g^{2n+2}d}{\prod_{i=0}^{2n+1}\left((2+3i)dg+1\right)\prod_{i=0}^{2n}\left((3+3i)be+1\right)a^{2n+1}c^{2n+2}f^{2n+2}},$$

$$x_{24n+18} = \frac{\prod_{i=0}^{2n+1}\left((1+3i)be+1\right)\prod_{i=0}^{2n}\left((3+3i)dg+1\right)a^{2n+2}c^{2n+2}f^{2n+2}}{\prod_{i=0}^{2n+1}\left((2+3i)cf+1\right)\prod_{i=0}^{2n}\left((3+3i)ad+1\right)b^{2n+2}e^{2n+2}g^{2n+1}},$$

$$x_{24n+19} = \frac{\prod_{i=0}^{2n+1}\left((1+3i)ad+1\right)\prod_{i=0}^{2n}\left((3+3i)cf+1\right)b^{2n+2}e^{2n+2}g^{2n+2}}{\prod_{i=0}^{2n+1}\left((2+3i)be+1\right)\prod_{i=0}^{2n+1}\left((1+3i)dg+1\right)a^{2n+2}c^{2n+2}f^{2n+1}},$$

$$x_{24n+20}=\frac{\prod_{i=0}^{2n+1}\left((2+3i)dg+1\right)\prod_{i=0}^{2n}\left((3+3i)be+1\right)a^{2n+2}c^{2n+2}f^{2n+2}}{\prod_{i=0}^{2n+1}\left((2+3i)ad+1\right)\prod_{i=0}^{2n+1}\left((1+3i)cf+1\right)b^{2n+2}e^{2n+1}g^{2n+2}},$$
$$x_{24n+21}=\frac{\prod_{i=0}^{2n+1}\left((2+3i)cf+1\right)\prod_{i=0}^{2n}\left((3+3i)ad+1\right)b^{2n+2}e^{2n+2}g^{2n+2}d}{\prod_{i=0}^{2n+1}\left((1+3i)be+1\right)\prod_{i=0}^{2n+1}\left((3+3i)dg+1\right)a^{2n+2}c^{2n+2}f^{2n+2}},$$
$$x_{24n+22}=\frac{\prod_{i=0}^{2n+1}\left((1+3i)dg+1\right)\prod_{i=0}^{2n+1}\left((2+3i)be+1\right)a^{2n+2}c^{2n+3}f^{2n+2}}{\prod_{i=0}^{2n+1}\left((1+3i)ad+1\right)\prod_{i=0}^{2n+1}\left((3+3i)cf+1\right)b^{2n+2}e^{2n+2}g^{2n+2}},$$
$$x_{24n+23}=\frac{\prod_{i=0}^{2n+1}\left((1+3i)cf+1\right)\prod_{i=0}^{2n+1}\left((2+3i)ad+1\right)b^{2n+3}e^{2n+2}g^{2n+2}}{\prod_{i=0}^{2n+1}\left((2+3i)dg+1\right)\prod_{i=0}^{2n+1}\left((3+3i)be+1\right)a^{2n+2}c^{2n+2}f^{2n+2}},$$
$$x_{24n+24}=\frac{\prod_{i=0}^{2n+1}\left((1+3i)be+1\right)\prod_{i=0}^{2n+1}\left((3+3i)dg+1\right)a^{2n+3}c^{2n+2}f^{2n+2}}{\prod_{i=0}^{2n+1}\left((2+3i)cf+1\right)\prod_{i=0}^{2n+1}\left((3+3i)ad+1\right)b^{2n+2}e^{2n+2}g^{2n+2}},$$

where, $x_{-6}=g$, $x_{-5}=f$, $x_{-4}=e$, $x_{-3}=d$, $x_{-2}=c$, $x_{-1}=b$, $x_0=a$.

Proof. For $n=0$ the result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$. That is,

$$x_{24n-23}=\frac{\prod_{i=0}^{2n-3}\left((1+3i)ad+1\right)\prod_{i=0}^{2n-3}\left((3+3i)cf+1\right)db^{2n-2}g^{2n-1}e^{2n-2}}{\prod_{i=0}^{2n-2}\left((1+3i)dg+1\right)\prod_{i=0}^{2n-3}\left((2+3i)cf+1\right)a^{2n-2}c^{2n+1}f^{2n-2}},$$
$$x_{24n-22}=\frac{\prod_{i=0}^{2n-3}\left((1+3i)dg+1\right)\prod_{i=0}^{2n-3}\left((3+3i)be+1\right)a^{2n-2}c^{2n-1}f^{2n-1}}{\prod_{i=0}^{2n-2}\left((1+3i)cf+1\right)\prod_{i=0}^{2n-3}\left((2+3i)ad+1\right)g^{2n-2}b^{2n-1}e^{2n-2}},$$
$$x_{24n-21}=\frac{\prod_{i=0}^{2n-3}\left((3+3i)ad+1\right)\prod_{i=0}^{2n-3}\left((2+3i)cf+1\right)e^{2n-1}b^{2n-1}g^{2n-2}}{\prod_{i=0}^{2n-2}\left((1+3i)be+1\right)\prod_{i=0}^{2n-3}\left((3+3i)dg+1\right)a^{2n-1}c^{2n-2}f^{2n-2}},$$
$$x_{24n-20}=\frac{\prod_{i=0}^{2n-2}\left((1+3i)dg+1\right)\prod_{i=0}^{2n-3}\left((2+3i)be+1\right)a^{2n-1}c^{2n-1}f^{2n-2}}{\prod_{i=0}^{2n-2}\left((1+3i)ad+1\right)\prod_{i=0}^{2n-3}\left((3+3i)cf+1\right)g^{2n-1}b^{2n-2}e^{2n-2}},$$
$$x_{24n-19}=\frac{\prod_{i=0}^{2n-2}\left((1+3i)cf+1\right)\prod_{i=0}^{2n-3}\left((2+3i)ad+1\right)db^{2n-1}g^{2n-1}e^{2n-2}}{\prod_{i=0}^{2n-2}\left((2+3i)dg+1\right)\prod_{i=0}^{2n-3}\left((3+3i)be+1\right)a^{2n-2}c^{2n-1}f^{2n-1}},$$
$$x_{24n-18}=\frac{\prod_{i=0}^{2n-2}\left((1+3i)be+1\right)\prod_{i=0}^{2n-3}\left((3+3i)dg+1\right)a^{2n-1}c^{2n-1}f^{2n-1}}{\prod_{i=0}^{2n-2}\left((2+3i)cf+1\right)\prod_{i=0}^{2n-3}\left((3+3i)ad+1\right)g^{2n-2}b^{2n-1}e^{2n-1}},$$
$$x_{24n-17}=\frac{\prod_{i=0}^{2n-2}\left((1+3i)ad+1\right)\prod_{i=0}^{2n-3}\left((3+3i)cf+1\right)g^{2n-1}b^{2n-1}e^{2n-1}}{\prod_{i=0}^{2n-2}\left((2+3i)be+1\right)\prod_{i=0}^{2n-2}\left((1+3i)dg+1\right)f^{2n-2}c^{2n-1}a^{2n-1}},$$
$$x_{24n-16}=\frac{\prod_{i=0}^{2n-2}\left((2+3i)dg+1\right)\prod_{i=0}^{2n-3}\left((3+3i)be+1\right)a^{2n-1}c^{2n-1}f^{2n-1}}{\prod_{i=0}^{2n-2}\left((2+3i)ad+1\right)\prod_{i=0}^{2n-2}\left((1+3i)cf+1\right)e^{2n-2}b^{2n-1}g^{2n-1}},$$
$$x_{24n-15}=\frac{\prod_{i=0}^{2n-2}\left((2+3i)cf+1\right)\prod_{i=0}^{2n-3}\left((3+3i)ad+1\right)db^{2n-1}e^{2n-1}g^{2n-1}}{\prod_{i=0}^{2n-2}\left((2+3i)be+1\right)\prod_{i=0}^{2n-2}\left((1+3i)dg+1\right)a^{2n-1}c^{2n-1}f^{2n-1}}$$
$$x_{24n-14}=\frac{\prod_{i=0}^{2n-2}\left((1+3i)dg+1\right)\prod_{i=0}^{2n-2}\left((2+3i)be+1\right)a^{2n-1}c^{2n}f^{2n-1}}{\prod_{i=0}^{2n-2}\left((1+3i)ad+1\right)\prod_{i=0}^{2n-2}\left((3+3i)cf+1\right)b^{2n-1}e^{2n-1}g^{2n-1}},$$
$$x_{24n-13}=\frac{\prod_{i=0}^{2n-2}\left((1+3i)be+1\right)\prod_{i=0}^{2n-2}\left((2+3i)ad+1\right)b^{2n}e^{2n-1}g^{2n-1}}{\prod_{i=0}^{2n-2}\left((2+3i)dg+1\right)\prod_{i=0}^{2n-2}\left((3+3i)be+1\right)a^{2n-1}c^{2n-1}f^{2n-1}},$$
$$x_{24n-12}=\frac{\prod_{i=0}^{2n-2}\left((1+3i)be+1\right)\prod_{i=0}^{2n-2}\left((3+3i)dg+1\right)a^{2n}c^{2n-1}f^{2n-1}}{\prod_{i=0}^{2n-2}\left((2+3i)cf+1\right)\prod_{i=0}^{2n-2}\left((3+3i)ad+1\right)b^{2n-1}e^{2n-1}g^{2n-1}},$$
$$x_{24n-11}=\frac{\prod_{i=0}^{2n-2}\left((1+3i)ad+1\right)\prod_{i=0}^{2n-2}\left((3+3i)cf+1\right)b^{2n-1}e^{2n-1}g^{2n}d}{\prod_{i=0}^{2n-1}\left((1+3i)dg+1\right)\prod_{i=0}^{2n-2}\left((2+3i)be+1\right)a^{2n-1}c^{2n}f^{2n-1}}$$
$$x_{24n-10}=\frac{\prod_{i=0}^{2n-2}\left((2+3i)dg+1\right)\prod_{i=0}^{2n-2}\left((3+3i)be+1\right)a^{2n-1}c^{2n}f^{2n}}{\prod_{i=0}^{2n-1}\left((1+3i)cf+1\right)\prod_{i=0}^{2n-2}\left((2+3i)ad+1\right)b^{2n}e^{2n+1}g^{2n-1}},$$
$$x_{24n-9}=\frac{\prod_{i=0}^{2n-2}\left((2+3i)cf+1\right)\prod_{i=0}^{2n-2}\left((3+3i)ad+1\right)b^{2n}e^{2n}g^{2n-1}}{\prod_{i=0}^{2n-1}\left((1+3i)be+1\right)\prod_{i=0}^{2n-2}\left((3+3i)dg+1\right)a^{2n}c^{2n-1}f^{2n-1}},$$
$$x_{24n-8}=\frac{\prod_{i=0}^{2n-1}\left((1+3i)dg+1\right)\prod_{i=0}^{2n-2}\left((2+3i)be+1\right)a^{2n}c^{2n}f^{2n-1}}{\prod_{i=0}^{2n-1}\left((1+3i)ad+1\right)\prod_{i=0}^{2n-2}\left((3+3i)cf+1\right)b^{2n-1}e^{2n-1}g^{2n}},$$
$$x_{24n-7}=\frac{\prod_{i=0}^{2n-1}\left((1+3i)cf+1\right)\prod_{i=0}^{2n-2}\left((2+3i)ad+1\right)b^{2n}e^{2n-1}g^{2n}d}{\prod_{i=0}^{2n-1}\left((2+3i)dg+1\right)\prod_{i=0}^{2n-2}\left((3+3i)be+1\right)a^{2n-1}c^{2n}f^{2n}},$$
$$x_{24n-6}=\frac{\prod_{i=0}^{2n-1}\left((1+3i)be+1\right)\prod_{i=0}^{2n-2}\left((3+3i)dg+1\right)a^{2n}c^{2n+2}f^{2n}}{\prod_{i=0}^{2n-1}\left((2+3i)cf+1\right)\prod_{i=0}^{2n-2}\left((3+3i)ad+1\right)b^{2n}e^{2n}g^{2n-1}},$$
$$x_{24n-5}=\frac{\prod_{i=0}^{2n-1}\left((1+3i)ad+1\right)\prod_{i=0}^{2n-2}\left((3+3i)cf+1\right)b^{2n}e^{2n}g^{2n}}{\prod_{i=0}^{2n-1}\left((2+3i)be+1\right)\prod_{i=0}^{2n-1}\left((1+3i)dg+1\right)a^{2n}c^{2n}f^{2n-1}},$$
$$x_{24n-4}=\frac{\prod_{i=0}^{2n-1}\left((2+3i)dg+1\right)\prod_{i=0}^{2n-2}\left((3+3i)be+1\right)a^{2n}c^{2n}f^{2n}}{\prod_{i=0}^{2n-1}\left((2+3i)ad+1\right)\prod_{i=0}^{2n-1}\left((1+3i)cf+1\right)b^{2n}e^{2n-1}g^{2n}},$$
$$x_{24n-3}=\frac{\prod_{i=0}^{2n-1}\left((2+3i)cf+1\right)\prod_{i=0}^{2n-2}\left((3+3i)ad+1\right)b^{2n}e^{2n}g^{2n}d}{\prod_{i=0}^{2n-1}\left((1+3i)be+1\right)\prod_{i=0}^{2n-1}\left((3+3i)dg+1\right)a^{2n}c^{2n}f^{2n}},$$
$$x_{24n-2}=\frac{\prod_{i=0}^{2n-1}\left((1+3i)dg+1\right)\prod_{i=0}^{2n-1}\left((2+3i)be+1\right)a^{2n}c^{2n+1}f^{2n}}{\prod_{i=0}^{2n-1}\left((1+3i)ad+1\right)\prod_{i=0}^{2n-1}\left((3+3i)cf+1\right)b^{2n}e^{2n}g^{2n}},$$
$$x_{24n-1}=\frac{\prod_{i=0}^{2n-1}\left((1+3i)cf+1\right)\prod_{i=0}^{2n-1}\left((2+3i)ad+1\right)b^{2n+1}e^{2n}g^{2n}}{\prod_{i=0}^{2n-1}\left((2+3i)dg+1\right)\prod_{i=0}^{2n-1}\left((3+3i)be+1\right)a^{2n}c^{2n}f^{2n}},$$
$$x_{24n}=\frac{\prod_{i=0}^{2n-1}\left((1+3i)be+1\right)\prod_{i=0}^{2n-1}\left((3+3i)dg+1\right)a^{2n+1}c^{2n}f^{2n}}{\prod_{i=0}^{2n-1}\left((2+3i)cf+1\right)\prod_{i=0}^{2n-1}\left((3+3i)ad+1\right)b^{2n}e^{2n}g^{2n}}.$$

Now, it follows from (2) that

$$x_{24n+1}=\frac{x_{24n-3}x_{24n-6}}{x_{24n-2}+x_{24n-2}x_{24n-3}x_{24n-6}}$$

By substituting the previously obtained values of x_{24n-2}, x_{24n-3} and x_{24n-6} into the expression, the following equality is obtained.

$$x_{24n+1}=\frac{\prod_{i=0}^{2n-1}\left((1+3i)ad+1\right)\prod_{i=0}^{2n-1}\left((3+3i)cf+1\right)db^{2n}g^{2n+1}e^{2n}}{\prod_{i=0}^{2n}\left((1+3i)dg+1\right)\prod_{i=0}^{2n-1}\left((2+3i)be+1\right)a^{2n}c^{2n+1}f^{2n}}$$

In a similar manner,

$$x_{24n+2}=\frac{x_{24n-2}x_{24n-5}}{x_{24n-1}+x_{24n-1}x_{24n-2}x_{24n-5}}$$

By substituting the previously obtained values of x_{24n-1}, x_{24n-2} and x_{24n-5} into the expression, the following equality is obtained.

$$x_{24n+2}=\frac{\prod_{i=0}^{2n-1}\left((1+3i)dg+1\right)\prod_{i=0}^{2n-1}\left((3+3i)be+1\right)a^{2n}c^{2n+1}f^{2n+1}}{\prod_{i=0}^{2n}\left((1+3i)cf+1\right)\prod_{i=0}^{2n-1}\left((2+3i)ad+1\right)g^{2n}b^{2n+1}e^{2n}}.$$

Similarly, other relations can be obtained and thus, the proof has been completed.

2.1. Numerical investigation

This part is dedicated to validating the theoretical findings presented in Eq. (2).

Example 2.1.1. For Eq. (2) we consider following initial conditions. The graph of the solutions corresponding to the considered initial conditions is shown in Figure 1.

$x_{-6}=0.295$, $x_{-5}=0.305$, $x_{-4}=0.315$, $x_{-3}=0.325$, $x_{-2}=0.335$, $x_{-1}=0.345$, $x_0=0.355$.

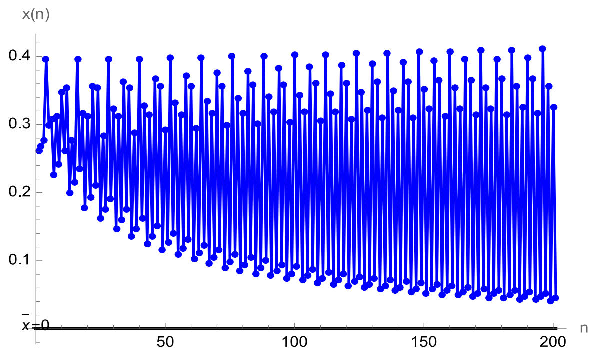

Figure 1. (Example 2.1.1)

3. Solution of Difference Equations $x_{n+1} = \frac{x_{n-3}x_{n-6}}{x_{n-2}-x_{n-2}x_{n-3}x_{n-6}}$

Here the specific form of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-3}x_{n-6}}{x_{n-2} - x_{n-2}x_{n-3}x_{n-6}}, \quad n \in N_{0}, \quad (3)$$

where the initial conditions are arbitrary nonzero positive real numbers, will be derived.

Theorem 3.1. Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of Eq. (3). Then for $n \in N_{0}$,

$$x_{24n+1} = -\frac{\prod_{i=0}^{2n-1}((1+3i)ad-1)\prod_{i=0}^{2n-1}((3+3i)cf-1)db^{2n}g^{2n+1}e^{2n}}{\prod_{i=0}^{2n}((1+3i)dg-1)\prod_{i=0}^{2n-1}((2+3i)be-1)a^{2n}c^{2n+1}f^{2n}},$$

$$x_{24n+2} = -\frac{\prod_{i=0}^{2n-1}((1+3i)dg-1)\prod_{i=0}^{2n-1}((3+3i)be-1)a^{2n}c^{2n+1}f^{2n+1}}{\prod_{i=0}^{2n}((1+3i)cf-1)\prod_{i=0}^{2n-1}((2+3i)ad-1)g^{2n}b^{2n+1}e^{2n}},$$

$$x_{24n+3} = -\frac{\prod_{i=0}^{2n-1}((3+3i)ad-1)\prod_{i=0}^{2n-1}((2+3i)cf-1)e^{2n+1}b^{2n+1}g^{2n}}{\prod_{i=0}^{2n}((1+3i)be-1)\prod_{i=0}^{2n-1}((3+3i)dg-1)a^{2n+1}c^{2n}f^{2n}},$$

$$x_{24n+4} = -\frac{\prod_{i=0}^{2n}((1+3i)dg-1)\prod_{i=0}^{2n-1}((2+3i)be-1)a^{2n+1}c^{2n+1}f^{2n}}{\prod_{i=0}^{2n}((1+3i)ad-1)\prod_{i=0}^{2n-1}((3+3i)cf-1)g^{2n+1}b^{2n}e^{2n}},$$

$$x_{24n+5} = -\frac{\prod_{i=0}^{2n}((1+3i)cf-1)\prod_{i=0}^{2n-1}((2+3i)ad-1)db^{2n+1}g^{2n+1}e^{2n}}{\prod_{i=0}^{2n}((2+3i)dg-1)\prod_{i=0}^{2n-1}((3+3i)be-1)a^{2n}c^{2n+1}f^{2n+1}},$$

$$x_{24n+6} = -\frac{\prod_{i=0}^{2n}((1+3i)be-1)\prod_{i=0}^{2n-1}((3+3i)dg-1)a^{2n+1}c^{2n+1}f^{2n+1}}{\prod_{i=0}^{2n}((2+3i)cf-1)\prod_{i=0}^{2n-1}((3+3i)ad-1)g^{2n}b^{2n+1}e^{2n+1}},$$

$$x_{24n+7} = -\frac{\prod_{i=0}^{2n}((1+3i)ad-1)\prod_{i=0}^{2n-1}((3+3i)cf-1)g^{2n+1}b^{2n+1}e^{2n+1}}{\prod_{i=0}^{2n}((2+3i)be-1)\prod_{i=0}^{2n}((1+3i)dg-1)f^{2n}c^{2n+1}a^{2n+1}},$$

$$x_{24n+8} = -\frac{\prod_{i=0}^{2n}((2+3i)dg-1)\prod_{i=0}^{2n-1}((3+3i)be-1)a^{2n+1}c^{2n+1}f^{2n+1}}{\prod_{i=0}^{2n}((2+3i)ad-1)\prod_{i=0}^{2n}((1+3i)cf-1)e^{2n}b^{2n+1}g^{2n+1}},$$

$$x_{24n+10} = -\frac{\prod_{i=0}^{2n}((1+3i)dg-1)\prod_{i=0}^{2n}((2+3i)be-1)a^{2n+1}c^{2n+2}f^{2n+1}}{\prod_{i=0}^{2n}((1+3i)ad-1)\prod_{i=0}^{2n}((3+3i)cf-1)b^{2n+1}e^{2n+1}g^{2n+1}},$$

$$x_{24n+11} = -\frac{\prod_{i=0}^{2n}((1+3i)be-1)\prod_{i=0}^{2n}((2+3i)ad-1)b^{2n+2}e^{2n+1}g^{2n+1}}{\prod_{i=0}^{2n}((2+3i)dg-1)\prod_{i=0}^{2n}((3+3i)be-1)a^{2n+1}c^{2n+1}f^{2n+1}},$$

$$x_{24n+12} = -\frac{\prod_{i=0}^{2n}((1+3i)be-1)\prod_{i=0}^{2n}((3+3i)dg-1)a^{2n+2}c^{2n+1}f^{2n+1}}{\prod_{i=0}^{2n}((2+3i)cf-1)\prod_{i=0}^{2n}((3+3i)ad-1)b^{2n+1}e^{2n+1}g^{2n+1}},$$

$$x_{24n+13} = -\frac{\prod_{i=0}^{2n}((1+3i)ad-1)\prod_{i=0}^{2n}((3+3i)cf-1)b^{2n+1}e^{2n+1}g^{2n+2}d}{\prod_{i=0}^{2n}((2+3i)cf-1)\prod_{i=0}^{2n}((3+3i)ad-1)b^{2n+1}e^{2n+1}g^{2n+1}},$$

$$x_{24n+14} = -\frac{\prod_{i=0}^{2n}((2+3i)dg-1)\prod_{i=0}^{2n}((3+3i)be-1)a^{2n+1}c^{2n+2}f^{2n+2}}{\prod_{i=0}^{2n+1}((1+3i)cf-1)\prod_{i=0}^{2n}((2+3i)ad-1)b^{2n+2}e^{2n+1}g^{2n+1}},$$

$$x_{24n+15} = -\frac{\prod_{i=0}^{2n}((2+3i)cf-1)\prod_{i=0}^{2n}((3+3i)ad-1)b^{2n+2}e^{2n+2}g^{2n+1}}{\prod_{i=0}^{2n+1}((1+3i)be-1)\prod_{i=0}^{2n}((3+3i)dg-1)a^{2n+2}c^{2n+1}f^{2n+1}},$$

$$x_{24n+16} = -\frac{\prod_{i=0}^{2n+1}((1+3i)dg-1)\prod_{i=0}^{2n}((2+3i)be-1)a^{2n+2}c^{2n+2}f^{2n+1}}{\prod_{i=0}^{2n+1}((1+3i)ad-1)\prod_{i=0}^{2n}((3+3i)cf-1)b^{2n+1}e^{2n+1}g^{2n+2}},$$

$$x_{24n+17} = -\frac{\prod_{i=0}^{2n+1}((1+3i)cf-1)\prod_{i=0}^{2n}((2+3i)ad-1)b^{2n+2}e^{2n+1}g^{2n+2}d}{\prod_{i=0}^{2n+1}((2+3i)dg-1)\prod_{i=0}^{2n}((3+3i)be-1)a^{2n+1}c^{2n+2}f^{2n+2}},$$

$$x_{24n+18} = -\frac{\prod_{i=0}^{2n+1}((1+3i)be-1)\prod_{i=0}^{2n}((3+3i)dg-1)a^{2n+2}c^{2n+2}f^{2n+2}}{\prod_{i=0}^{2n+1}((2+3i)cf-1)\prod_{i=0}^{2n}((3+3i)ad-1)b^{2n+2}e^{2n+2}g^{2n+1}},$$

$$x_{24n+19} = -\frac{\prod_{i=0}^{2n+1}((1+3i)ad-1)\prod_{i=0}^{2n}((3+3i)cf-1)b^{2n+2}e^{2n+2}g^{2n+2}}{\prod_{i=0}^{2n+1}((2+3i)be-1)\prod_{i=0}^{2n+1}((1+3i)dg-1)a^{2n+2}c^{2n+2}f^{2n+1}},$$

$$x_{24n+20} = -\frac{\prod_{i=0}^{2n+1}((2+3i)dg-1)\prod_{i=0}^{2n}((3+3i)be-1)a^{2n+2}c^{2n+2}f^{2n+2}}{\prod_{i=0}^{2n+1}((2+3i)ad-1)\prod_{i=0}^{2n+1}((1+3i)cf-1)b^{2n+2}e^{2n+1}g^{2n+2}},$$

$$x_{24n+21} = -\frac{\prod_{i=0}^{2n+1}((2+3i)cf-1)\prod_{i=0}^{2n}((3+3i)ad-1)b^{2n+2}e^{2n+2}g^{2n+2}d}{\prod_{i=0}^{2n+1}((1+3i)be-1)\prod_{i=0}^{2n+1}((3+3i)dg-1)a^{2n+2}c^{2n+2}f^{2n+2}},$$

$$x_{24n+22} = -\frac{\prod_{i=0}^{2n+1}((1+3i)dg-1)\prod_{i=0}^{2n+1}((2+3i)be-1)a^{2n+2}c^{2n+3}f^{2n+2}}{\prod_{i=0}^{2n+1}((1+3i)ad-1)\prod_{i=0}^{2n+1}((3+3i)cf-1)b^{2n+2}e^{2n+2}g^{2n+2}},$$

$$x_{24n+23} = -\frac{\prod_{i=0}^{2n+1}((1+3i)cf-1)\prod_{i=0}^{2n+1}((2+3i)ad-1)b^{2n+3}e^{2n+2}g^{2n+2}}{\prod_{i=0}^{2n+1}((2+3i)dg-1)\prod_{i=0}^{2n+1}((3+3i)be-1)a^{2n+2}c^{2n+2}f^{2n+2}},$$

$$x_{24n+24} = -\frac{\prod_{i=0}^{2n+1}((1+3i)be-1)\prod_{i=0}^{2n+1}((3+3i)dg-1)a^{2n+3}c^{2n+2}f^{2n+2}}{\prod_{i=0}^{2n+1}((2+3i)cf-1)\prod_{i=0}^{2n+1}((3+3i)ad-1)b^{2n+2}e^{2n+2}g^{2n+2}},$$

where, $x_{-6} = g$, $x_{-5} = f$, $x_{-4} = e$, $x_{-3} = d$, $x_{-2} = c$, $x_{-1} = b$, $x_0 = a$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$x_{24n-23} = -\frac{\prod_{i=0}^{2n-3}((1+3i)ad-1)\prod_{i=0}^{2n-3}((3+3i)cf-1)db^{2n-1}g^{2n-1}e^{2n-2}}{\prod_{i=0}^{2n-2}((1+3i)dg-1)\prod_{i=0}^{2n-3}((2+3i)be-1)a^{2n}c^{2n-1}f^{2n-2}},$$

$$x_{24n-22} = -\frac{\prod_{i=0}^{2n-3}((1+3i)dg-1)\prod_{i=0}^{2n-3}((3+3i)be-1)a^{2n-2}c^{2n-1}f^{2n-1}}{\prod_{i=0}^{2n-2}((1+3i)cf-1)\prod_{i=0}^{2n-3}((2+3i)ad-1)g^{2n-2}b^{2n-1}e^{2n-2}},$$

$$x_{24n-21} = -\frac{\prod_{i=0}^{2n-3}((3+3i)ad-1)\prod_{i=0}^{2n-3}((2+3i)cf-1)e^{2n-1}b^{2n-1}g^{2n-2}}{\prod_{i=0}^{2n-2}((1+3i)be-1)\prod_{i=0}^{2n-3}((3+3i)dg-1)a^{2n-1}c^{2n-2}f^{2n-2}},$$

$$x_{24n-20} = -\frac{\prod_{i=0}^{2n-2}((1+3i)dg-1)\prod_{i=0}^{2n-3}((2+3i)be-1)a^{2n-1}c^{2n+1}f^{2n-2}}{\prod_{i=0}^{2n-2}((1+3i)ad-1)\prod_{i=0}^{2n-3}((3+3i)cf-1)g^{2n-1}b^{2n-2}e^{2n-2}},$$

$$x_{24n-19} = -\frac{\prod_{i=0}^{2n-2}((1+3i)cf-1)\prod_{i=0}^{2n-3}((2+3i)ad-1)db^{2n-1}g^{2n-1}e^{2n-2}}{\prod_{i=0}^{2n-2}((2+3i)dg-1)\prod_{i=0}^{2n-3}((3+3i)be-1)a^{2n-2}c^{2n-1}f^{2n-1}},$$

$$x_{24n-18} = -\frac{\prod_{i=0}^{2n-2}((1+3i)be-1)\prod_{i=0}^{2n-3}((3+3i)dg-1)a^{2n-1}c^{2n-1}f^{2n-1}}{\prod_{i=0}^{2n-2}((2+3i)cf-1)\prod_{i=0}^{2n-3}((3+3i)ad-1)g^{2n-2}b^{2n-1}e^{2n-1}},$$

$$x_{24n-17} = -\frac{\prod_{i=0}^{2n-2}((1+3i)ad-1)\prod_{i=0}^{2n-3}((3+3i)cf-1)g^{2n-1}b^{2n-1}e^{2n-1}}{\prod_{i=0}^{2n-2}((2+3i)be-1)\prod_{i=0}^{2n-2}((1+3i)dg-1)f^{2n-2}c^{2n-1}a^{2n-1}},$$

$$x_{24n-16} = -\frac{\prod_{i=0}^{2n-2}((2+3i)dg-1)\prod_{i=0}^{2n-3}((3+3i)be-1)a^{2n-1}c^{2n-1}f^{2n-1}}{\prod_{i=0}^{2n-2}((2+3i)ad-1)\prod_{i=0}^{2n-2}((1+3i)cf-1)e^{2n-2}b^{2n-1}g^{2n-1}},$$

$$x_{24n-14} = -\frac{\prod_{i=0}^{2n-2}((1+3i)dg-1)\prod_{i=0}^{2n-2}((2+3i)be-1)a^{2n-1}c^{2n}f^{2n-1}}{\prod_{i=0}^{2n-2}((1+3i)ad-1)\prod_{i=0}^{2n-2}((3+3i)cf-1)b^{2n-1}e^{2n-1}g^{2n-1}},$$

$$x_{24n-13} = -\frac{\prod_{i=0}^{2n-2}((1+3i)be-1)\prod_{i=0}^{2n-2}((2+3i)ad-1)b^{2n}e^{2n-1}g^{2n-1}}{\prod_{i=0}^{2n-2}((2+3i)dg-1)\prod_{i=0}^{2n-2}((3+3i)be-1)a^{2n-1}c^{2n-1}f^{2n-1}},$$

$$x_{24n-12} = -\frac{\prod_{i=0}^{2n-2}((1+3i)be-1)\prod_{i=0}^{2n-2}((3+3i)dg-1)a^{2n}c^{2n-1}f^{2n-1}}{\prod_{i=0}^{2n-2}((2+3i)cf-1)\prod_{i=0}^{2n-2}((3+3i)ad-1)b^{2n-1}e^{2n-1}g^{2n-1}},$$

$$x_{24n-11} = -\frac{\prod_{i=0}^{2n-2}((1+3i)ad-1)\prod_{i=0}^{2n-2}((3+3i)cf-1)b^{2n-1}e^{2n-1}g^{2n}d}{\prod_{i=0}^{2n-2}((2+3i)cf-1)\prod_{i=0}^{2n-2}((3+3i)ad-1)b^{2n-1}e^{2n+1}g^{2n-1}},$$

$$x_{24n-10} = -\frac{\prod_{i=0}^{2n-2}((2+3i)dg-1)\prod_{i=0}^{2n-2}((3+3i)be-1)a^{2n-1}c^{2n}f^{2n}}{\prod_{i=0}^{2n-1}((1+3i)cf-1)\prod_{i=0}^{2n-2}((2+3i)ad-1)b^{2n}e^{2n-1}g^{2n-1}},$$

$$x_{24n-9} = -\frac{\prod_{i=0}^{2n-2}((2+3i)cf-1)\prod_{i=0}^{2n-2}((3+3i)ad-1)b^{2n}e^{2n}g^{2n}}{\prod_{i=0}^{2n-1}((1+3i)be-1)\prod_{i=0}^{2n-2}((3+3i)dg-1)a^{2n}c^{2n-1}f^{2n-1}},$$

$$x_{24n-8} = -\frac{\prod_{i=0}^{2n-1}((1+3i)dg-1)\prod_{i=0}^{2n-2}((2+3i)be-1)a^{2n}c^{2n}f^{2n-1}}{\prod_{i=0}^{2n-1}((1+3i)ad-1)\prod_{i=0}^{2n-2}((3+3i)cf-1)b^{2n-1}e^{2n-1}g^{2n}},$$

$$x_{24n-7} = -\frac{\prod_{i=0}^{2n-1}((1+3i)cf-1)\prod_{i=0}^{2n-2}((2+3i)ad-1)b^{2n}e^{2n-1}g^{2n}d}{\prod_{i=0}^{2n-1}((2+3i)dg-1)\prod_{i=0}^{2n-2}((3+3i)be-1)a^{2n-1}c^{2n}f^{2n}},$$

$$x_{24n-6} = -\frac{\prod_{i=0}^{2n-1}((1+3i)be-1)\prod_{i=0}^{2n-2}((3+3i)dg-1)a^{2n}c^{2n+2}f^{2n}}{\prod_{i=0}^{2n-1}((2+3i)cf-1)\prod_{i=0}^{2n-2}((3+3i)ad-1)b^{2n}e^{2n}g^{2n-1}},$$

$$x_{24n-5} = -\frac{\prod_{i=0}^{2n-1}((1+3i)ad-1)\prod_{i=0}^{2n-2}((3+3i)cf-1)b^{2n}e^{2n}g^{2n}}{\prod_{i=0}^{2n-1}((2+3i)be-1)\prod_{i=0}^{2n-1}((1+3i)dg-1)a^{2n}c^{2n}f^{2n-1}},$$

$$x_{24n-4} = -\frac{\prod_{i=0}^{2n-1}((2+3i)dg-1)\prod_{i=0}^{2n-2}((3+3i)be-1)a^{2n}c^{2n}f^{2n}}{\prod_{i=0}^{2n-1}((2+3i)ad-1)\prod_{i=0}^{2n-1}((1+3i)cf-1)b^{2n}e^{2n-1}g^{2n}},$$

$$x_{24n-3} = -\frac{\prod_{i=0}^{2n-1}((2+3i)cf-1)\prod_{i=0}^{2n-2}((3+3i)ad-1)b^{2n}e^{2n}g^{2n}d}{\prod_{i=0}^{2n-1}((1+3i)be-1)\prod_{i=0}^{2n-1}((3+3i)dg-1)a^{2n}c^{2n}f^{2n}},$$

$$x_{24n-2} = -\frac{\prod_{i=0}^{2n-1}((1+3i)dg-1)\prod_{i=0}^{2n-1}((2+3i)be-1)a^{2n}c^{2n+1}f^{2n}}{\prod_{i=0}^{2n-1}((1+3i)ad-1)\prod_{i=0}^{2n-1}((3+3i)cf-1)b^{2n}e^{2n}g^{2n}},$$

$$x_{24n-1} = -\frac{\prod_{i=0}^{2n-1}((1+3i)cf-1)\prod_{i=0}^{2n-1}((2+3i)ad-1)b^{2n+1}e^{2n}g^{2n}}{\prod_{i=0}^{2n-1}((2+3i)dg-1)\prod_{i=0}^{2n-1}((3+3i)be-1)a^{2n}c^{2n}f^{2n}},$$

$$x_{24n} = -\frac{\prod_{i=0}^{2n-1}((1+3i)be-1)\prod_{i=0}^{2n-1}((3+3i)dg-1)a^{2n+1}c^{2n}f^{2n}}{\prod_{i=0}^{2n-1}((2+3i)cf-1)\prod_{i=0}^{2n-1}((3+3i)ad-1)b^{2n}e^{2n}g^{2n}}$$

Now, it follows from (3) that

$$x_{24n+1} = \frac{x_{24n-3}x_{24n-6}}{x_{24n-2} + x_{24n-2}x_{24n-3}x_{24n-6}}$$

By substituting the previously obtained values of x_{24n-2}, x_{24n-3} and x_{24n-6} into the expression, the following equality is obtained.

$$x_{24n+1} = -\frac{\prod_{i=0}^{2n-1}((1+3i)ad-1)\prod_{i=0}^{2n-1}((3+3i)cf-1)db^{2n}g^{2n+1}e^{2n}}{\prod_{i=0}^{2n}((1+3i)dg-1)\prod_{i=0}^{2n-1}((2+3i)be-1)a^{2n}c^{2n+1}f^{2n}}$$

In a similar manner,

$$x_{24n+2} = \frac{x_{24n-2}x_{24n-5}}{x_{24n-1} + x_{24n-1}x_{24n-2}x_{24n-5}}$$

By substituting the previously obtained values of x_{24n-1}, x_{24n-2} and x_{24n-5} into the expression, the following equality is obtained.

$$x_{24n+2} = -\frac{\prod_{i=0}^{2n-1}\left((1+3i)dg-1\right)\prod_{i=0}^{2n-1}\left((3+3i)be-1\right)a^{2n}c^{2n+1}f^{2n+1}}{\prod_{i=0}^{2n}\left((1+3i)cf-1\right)\prod_{i=0}^{2n-1}\left((2+3i)ad-1\right)g^{2n}b^{2n+1}e^{2n}}.$$

Similarly, other relations can be obtained and thus, the proof has been completed.

3.1. Numerical investigation

This part is dedicated to validating the theoretical findings presented in Eq. (3).

Example 3.1.1. For Eq. (3) we consider following initial conditions. Figure 2 illustrates the solutions for the given initial conditions.

$x_{-6} = 2$, $x_{-5} = 3$, $x_{-4} = 4$, $x_{-3} = 5$, $x_{-2} = 6$, $x_{-1} = 7$, $x_0 = 8$.

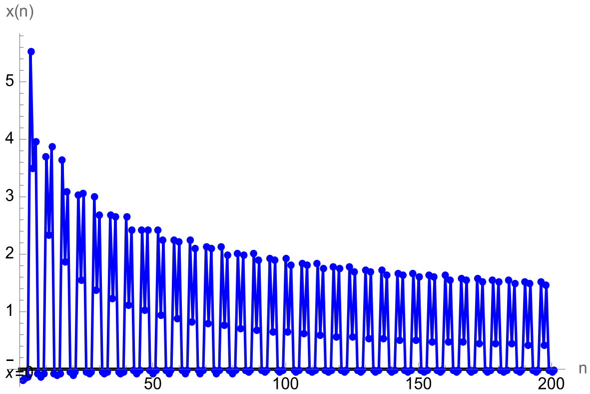

Figure 2. (Example 3.1.1)

4. Solution the Difference Equation $x_{n+1} = \frac{x_{n-3}x_{n-6}}{-x_{n-2}-x_{n-2}x_{n-3}x_{n-6}}$

In this section, we investigate the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-3}x_{n-6}}{-x_{n-2} - x_{n-2}x_{n-3}x_{n-6}}, \quad n \in N_0, (4)$$

where the initial conditions are arbitrary nonzero positive real numbers.

Theorem 4.1. Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of difference Eq.(4). Then for, $n = 0,1,2, \ldots$

$$x_{24n+1} = -\frac{(ad+1)^n(cf+1)^n db^{2n}g^{2n+1}e^{2n}}{(dg+1)^{n+1}(be+1)^n a^{2n}c^{2n+1}f^{2n}},$$

$$x_{24n+2} = -\frac{(dg+1)^n(be+1)^n a^{2n}c^{2n+1}f^{2n+1}}{(cf+1)^{n+1}(ad+1)^n g^{2n}b^{2n+1}e^{2n}},$$

$$x_{24n+3} = -\frac{(ad+1)^n(cf+1)^n e^{2n+1}b^{2n+1}g^{2n}}{(be+1)^{n+1}(dg+1)^n a^{2n+1}c^{2n}f^{2n}},$$

$$x_{24n+4} = \frac{(dg+1)^{n+1}(be+1)^n a^{2n+1}c^{2n+1}f^{2n}}{(ad+1)^{n+1}(cf+1)^n g^{2n+1}b^{2n}e^{2n}},$$

$$x_{24n+5} = -\frac{(ad+1)^n(cf+1)^{n+1} db^{2n+1}g^{2n+1}e^{2n}}{(dg+1)^n(be+1)^n a^{2n}c^{2n+1}f^{2n+1}},$$

$$x_{24n+6} = -\frac{(dg+1)^n(be+1)^{n+1} a^{2n+1}c^{2n+1}f^{2n+1}}{(cf+1)^n(ad+1)^n g^{2n}b^{2n+1}e^{2n+1}},$$

$$x_{24n+7} = \frac{(ad+1)^{n+1}(cf+1)^n e^{2n+1}b^{2n+1}g^{2n+1}}{(be+1)^n(dg+1)^{n+1} a^{2n+1}c^{2n+1}f^{2n}},$$

$$x_{24n+8} = -\frac{(dg+1)^n(be+1)^n a^{2n+1}c^{2n+1}f^{2n+1}}{(cf+1)^{n+1}(ad+1)^n g^{2n+1}b^{2n+1}e^{2n}},$$

$$x_{24n+9} = \frac{(ad+1)^n(cf+1)^n db^{2n+1}g^{2n+1}e^{2n+1}}{(dg+1)^{n+1}(be+1)^{n+1} a^{2n+1}e^{2n+1}f^{2n+1}},$$

$$x_{24n+10} = -\frac{(dg+1)^{n+1}(be+1)^n a^{2n+1}c^{2n+2}f^{2n+1}}{(cf+1)^{n+1}(ad+1)^{n+1} g^{2n+1}b^{2n+1}e^{2n+1}},$$

$$x_{24n+11} = \frac{(ad+1)^n(cf+1)^{n+1} e^{2n+1}b^{2n+2}g^{2n+1}}{(be+1)^n(dg+1)^n a^{2n+1}c^{2n+1}f^{2n+1}},$$

$$x_{24n+12} = -\frac{(dg+1)^{n+1}(be+1)^{n+1} a^{2n+2}c^{2n+1}f^{2n+1}}{(cf+1)^n(ad+1)^{n+1} e^{2n+1}b^{2n+1}g^{2n+1}},$$

$$x_{24n+13} = -\frac{(ad+1)^{n+1}(cf+1)^{n+1} db^{2n+1}g^{2n+2}e^{2n+1}}{(dg+1)^{n+1}(be+1)^n a^{2n+1}c^{2n+2}f^{2n+1}},$$

$$x_{24n+14} = \frac{(dg+1)^n(be-1)^{n+1} a^{2n+1}c^{2n+2}f^{2n+2}}{(ad+1)^n(cf+1)^{n+1} e^{2n+1}b^{2n+2}g^{2n+1}},$$

$$x_{24n+15} = -\frac{(ad+1)^{n+1}(cf+1)^n g^{2n+1}e^{2n+2}b^{2n+2}}{(dg+1)^{n+1}(be+1)^{n+1} a^{2n+2}c^{2n+1}f^{2n+1}},$$

$$x_{24n+16} = -\frac{(dg+1)^{n+1}(be+1)^n a^{2n+2}c^{2n+2}f^{2n+1}}{(cf+1)^{n+1}(ad+1)^{n+1} g^{2n+2}b^{2n+1}e^{2n+1}},$$

$$x_{24n+17} = -\frac{(ad+1)^n(cf+1)^{n+1} db^{2n+2}g^{2n+2}e^{2n+1}}{(dg+1)^{n+1}(be+1)^{n+1} a^{2n+1}c^{2n+2}f^{2n+2}},$$

$$x_{24n+18} = \frac{(dg+1)^{n+1}(be+1)^{n+1} a^{2n+2}c^{2n+2}f^{2n+2}}{(ad+1)^{n+1}(cf+1)^{n+1} g^{2n+1}b^{2n+2}e^{2n+2}},$$

$$x_{24n+19} = \frac{(ad+1)^{n+1}(cf+1)^{n+1} g^{2n+2}b^{2n+2}e^{2n+2}}{(dg+1)^{n+1}(be+1)^{n+1} a^{2n+2}c^{2n+2}f^{2n+1}},$$

$$x_{24n+20} = \frac{(dg+1)^{n+1}(be+1)^{n+1} a^{2n+2}c^{2n+2}f^{2n+2}}{(ad+1)^{n+1}(cf+1)^{n+1} g^{2n+2}b^{2n+2}e^{2n+1}},$$

$$x_{24n+21} = \frac{(ad+1)^{n+1}(cf+1)^{n+1} db^{2n+2}g^{2n+2}e^{2n+2}}{(dg+1)^{n+1}(be+1)^{n+1} a^{2n+2}c^{2n+2}f^{2n+2}},$$

$$x_{24n+22} = \frac{(dg+1)^{n+1}(be+1)^{n+1} a^{2n+2}c^{2n+3}f^{2n+2}}{(ad+1)^{n+1}(cf+1)^{n+1} g^{2n+2}b^{2n+2}e^{2n+2}},$$

$$x_{24n+23} = \frac{(ad+1)^{n+1}(cf+1)^{n+1} e^{2n+2}b^{2n+3}g^{2n+2}}{(dg+1)^{n+1}(be+1)^{n+1} a^{2n+2}c^{2n+2}f^{2n+2}},$$

$$x_{24n+24} = \frac{(dg+1)^{n+1}(be+1)^{n+1} a^{2n+3}c^{2n+2}f^{2n+2}}{(ad+1)^{n+1}(cf+1)^{n+1} g^{2n+2}b^{2n+2}e^{2n+2}}.$$

Proof. Suppose that, $n > 0$ and that our assumption holds for $n-1$.

$$x_{24n-23} = -\frac{(ad+1)^{n-1}(cf+1)^n db^{2n-2}g^{2n-1}e^{2n-2}}{(dg+1)^n(be+1)^{n-1}a^{2n-2}c^{2n-1}f^{2n-2}},$$
$$x_{24n-22} = -\frac{(dg+1)^{n-1}(be+1)^{n-1}a^{2n-2}c^{2n-1}f^{2n-1}}{(cf+1)^n(ad+1)^{n-1}g^{2n-2}b^{2n-1}e^{2n-2}},$$
$$x_{24n-21} = -\frac{(ad+1)^{n-1}(cf+1)^{n-1}e^{2n-1}b^{2n-1}g^{2n-2}}{(be+1)^n(dg+1)^{n-1}a^{2n-1}c^{2n-2}f^{2n-2}},$$
$$x_{24n-20} = \frac{(dg+1)^n(be+1)^{n-1}a^{2n-1}c^{2n-1}f^{2n-2}}{(ad+1)^n(cf+1)^{n-1}g^{2n-1}b^{2n-2}e^{2n-2}},$$
$$x_{24n-19} = -\frac{(ad+1)^{n-1}(cf+1)^n db^{2n-1}g^{2n-1}e^{2n-2}}{(dg+1)^{n-1}(be+1)^{n-1}a^{2n-2}c^{2n-1}f^{2n-1}},$$
$$x_{24n-18} = -\frac{(dg+1)^{n-1}(be+1)^n a^{2n-1}c^{2n-1}f^{2n-1}}{(cf+1)^{n-1}(ad+1)^{n-1}g^{2n-2}b^{2n-1}e^{2n-1}},$$
$$x_{24n-17} = \frac{(ad+1)^n(cf+1)^{n-1}e^{2n-1}b^{2n-1}g^{2n-1}}{(be+1)^{n-1}(dg+1)^{n-1}a^{2n-1}c^{2n-1}f^{2n-2}},$$
$$x_{24n-16} = -\frac{(dg+1)^{n-1}(be+1)^{n-1}a^{2n-1}c^{2n-1}f^{2n-1}}{(cf+1)^n(ad+1)^{n-1}g^{2n-1}b^{2n-1}e^{2n-2}},$$
$$x_{24n-15} = \frac{(ad+1)^{n-1}(cf+1)^{n-1}db^{2n-1}g^{2n-1}e^{2n-1}}{(dg+1)^n(be+1)^n a^{2n-1}e^{2n-1}f^{2n-1}},$$
$$x_{24n-14} = -\frac{(dg+1)^n(be+1)^{n-1}a^{2n-1}c^{2n}f^{2n-1}}{(cf+1)^n(ad+1)^n g^{2n-1}b^{2n-1}e^{2n-1}},$$
$$x_{24n-13} = \frac{(ad+1)^{n-1}(cf+1)^n e^{2n-1}b^{2n}g^{2n-1}}{(be+1)^{n-1}(dg+1)^{n-1}a^{2n-1}c^{2n-1}f^{2n-1}},$$
$$x_{24n-12} = -\frac{(dg+1)^n(be+1)^n a^{2n}c^{2n-1}f^{2n-1}}{(cf+1)^{n-1}(ad+1)^n e^{2n-1}b^{2n-1}g^{2n-1}},$$
$$x_{24n-11} = -\frac{(ad+1)^n(cf+1)^n db^{2n-1}g^{2n}e^{2n-1}}{(dg+1)^n(be+1)^{n-1}a^{2n-1}c^{2n}f^{2n-1}},$$
$$x_{24n-10} = \frac{(dg+1)^{n-1}(be-1)^n a^{2n-1}c^{2n}f^{2n}}{(ad+1)^{n-1}(cf+1)^n e^{2n-1}b^{2n}g^{2n-1}},$$
$$x_{24n-9} = -\frac{(ad+1)^n(cf+1)^{n-1}g^{2n-1}e^{2n}b^{2n}}{(dg+1)^n(be+1)^n a^{2n}c^{2n-1}f^{2n-1}},$$
$$x_{24n-8} = -\frac{(dg+1)^n(be+1)^{n-1}a^{2n}c^{2n}f^{2n-1}}{(cf+1)^n(ad+1)^n g^{2n}b^{2n-1}e^{2n-1}},$$
$$x_{24n-7} = -\frac{(ad+1)^{n-1}(cf+1)^n db^{2n}g^{2n}e^{2n-1}}{(dg+1)^n(be+1)^n a^{2n-1}c^{2n}f^{2n}},$$
$$x_{24n-6} = \frac{(dg+1)^n(be+1)^n a^{2n}c^{2n}f^{2n}}{(ad+1)^n(cf+1)^n g^{2n-1}b^{2n}e^{2n}},$$
$$x_{24n-5} = \frac{(ad+1)^n(cf+1)^n g^{2n}b^{2n}e^{2n}}{(dg+1)^n(be+1)^n a^{2n}c^{2n}f^{2n-1}},$$
$$x_{24n-4} = \frac{(dg+1)^n(be+1)^n a^{2n}c^{2n}f^{2n}}{(ad+1)^n(cf+1)^n g^{2n}b^{2n}e^{2n-1}},$$
$$x_{24n-3} = \frac{(ad+1)^n(cf+1)^n db^{2n}g^{2n}e^{2n}}{(dg+1)^n(be+1)^n a^{2n}c^{2n}f^{2n}},$$
$$x_{24n-2} = \frac{(dg+1)^n(be+1)^n a^{2n}c^{2n+1}f^{2n}}{(ad+1)^n(cf+1)^n g^{2n}b^{2n}e^{2n}},$$
$$x_{24n-1} = \frac{(ad+1)^n(cf+1)^n e^{2n}b^{2n+1}g^{2n}}{(dg+1)^n(be+1)^n a^{2n}c^{2n}f^{2n}},$$
$$x_{24n} = \frac{(dg+1)^n(be+1)^n a^{2n+1}c^{2n}f^{2n}}{(ad+1)^n(cf+1)^n g^{2n}b^{2n}e^{2n}}.$$

Now, it follows from Eq. (4) that

$$x_{24n+1} = \frac{x_{24n-3}x_{24n-6}}{-x_{24n-2} - x_{24n-2}x_{24n-3}x_{24n-6}}.$$

By substituting the previously obtained values of x_{24n-2}, x_{24n-3} and x_{24n-6} into the expression, the following equality is obtained.

$$x_{24n+1} = -\frac{(ad+1)^n(cf+1)^n db^{2n}g^{2n+1}e^{2n}}{(dg+1)^{n+1}(be+1)^n a^{2n}c^{2n+1}f^{2n}}$$

In a similar manner,

$$x_{24n+2} = \frac{x_{24n-2}x_{24n-5}}{-x_{24n-1} - x_{24n-1}x_{24n-2}x_{24n-5}}.$$

By substituting the previously obtained values of x_{24n-1}, x_{24n-2} and x_{24n-5} into the expression, the following equality is obtained.

$$x_{24n+2} = -\frac{(dg+1)^n(be+1)^n a^{2n}c^{2n+1}f^{2n+1}}{(cf+1)^{n+1}(ad+1)^n g^{2n}b^{2n+1}e^{2n}}$$

Similarly, other relations can be obtained and thus, the proof has been completed.

Theorem 4.2. Eq. (4) has on equilibrium point which 0, and this equilibrium point is not locally asymptotically stable.

Proof. For the equilibriums of Eq. (4), we have

$$\bar{x} = \frac{\bar{x}^2}{-\bar{x}(1+\bar{x}^2)},$$

then

$$\bar{x}^2 + \bar{x}^4 = 0, \quad \bar{x}^2(1+\bar{x}^2) = 0.$$

Thus the equilibrium point of Eq. (4) is $\bar{x} = 0$. Let $f: (0,\infty)^3 \to (0,\infty)$ be the function defined by

$$f(l,o,t) = \frac{ot}{-l - lot}.$$

Therefore it follows that,

$$f_l(l,o,t) = \frac{-ot}{l^2(1+ot)}, \; f_o(l,o,t) = \frac{t}{(1+ot)^2},$$
$$f_t(l,o,t) = \frac{o}{(1+ot)^2}.$$

We see that,

$$f_l(\bar{x},\bar{x},\bar{x}) = -1, \; f_o(\bar{x},\bar{x},\bar{x}) = 1$$
$$f_t(\bar{x},\bar{x},\bar{x}) = 1$$

4.1. Numerical investigations

This part is dedicated to validating the theoretical findings presented in Eq. (4).

Example 4.1.1. For Eq. (4) we consider following initial conditions. The solutions for the specified initial conditions are depicted in Figure 3.

$x_{-6} = 0.22, \; x_{-5} = 0.23, \; x_{-4} = 0.24, \; x_{-3} = 0.25, \; x_{-2} = 0.26, x_{-1} = 0.27, x_0 = 0.28.$

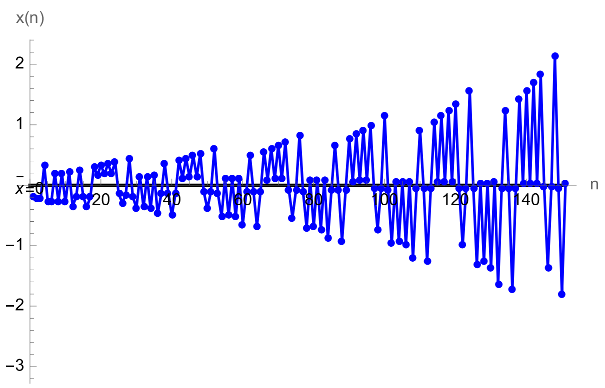

Figure 3. (Example 4.1.1)

5. Solution the Difference Equation $x_{n+1} = \frac{x_{n-3}x_{n-6}}{-x_{n-2}+x_{n-2}x_{n-3}x_{n-6}}$

In this section, we investigate the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-3}x_{n-6}}{-x_{n-2} + x_{n-2}x_{n-3}x_{n-6}}, \quad n \in N_0, (5)$$

where the initial conditions are arbitrary nonzero positive real numbers.

Theorem 5.1 Let $\{x_n\}_{n=-6}^{\infty}$ be a solution of difference Eq. (5).

$$x_{24n+1} = \frac{(ad-1)^n(cf-1)^n db^{2n}g^{2n+1}e^{2n}}{(dg-1)^{n+1}(be-1)^n a^{2n}c^{2n+1}f^{2n}},$$
$$x_{24n+2} = \frac{(dg-1)^n(be-1)^n a^{2n}c^{2n+1}f^{2n+1}}{(cf-1)^{n+1}(ad-1)^n g^{2n}b^{2n+1}e^{2n}},$$
$$x_{24n+3} = \frac{(ad-1)^n(cf-1)^n e^{2n+1}b^{2n+1}g^{2n}}{(be-1)^{n+1}(dg-1)^n a^{2n+1}c^{2n}f^{2n}},$$
$$x_{24n+4} = \frac{(dg-1)^{n+1}(be-1)^n a^{2n+1}c^{2n+1}f^{2n}}{(ad-1)^{n+1}(cf-1)^n g^{2n+1}b^{2n}e^{2n}},$$
$$x_{24n+5} = \frac{(ad-1)^n(cf-1)^{n+1} db^{2n+1}g^{2n+1}e^{2n}}{(dg-1)^n(be-1)^n a^{2n}c^{2n+1}f^{2n+1}},$$
$$x_{24n+6} = \frac{(dg-1)^n(be-1)^{n+1} a^{2n+1}c^{2n+1}f^{2n+1}}{(cf-1)^n(ad-1)^n g^{2n}b^{2n+1}e^{2n+1}},$$
$$x_{24n+7} = \frac{(ad-1)^{n+1}(cf-1)^n e^{2n+1}b^{2n+1}g^{2n+1}}{(be-1)^n(dg-1)^{n+1} a^{2n+1}c^{2n+1}f^{2n}},$$
$$x_{24n+8} = \frac{(dg-1)^n(be-1)^n a^{2n+1}c^{2n+1}f^{2n+1}}{(cf-1)^{n+1}(ad-1)^n g^{2n+1}b^{2n+1}e^{2n}},$$
$$x_{24n+9} = \frac{(ad-1)^n(cf-1)^n db^{2n+1}g^{2n+1}e^{2n+1}}{(dg-1)^{n+1}(be-1)^{n+1} a^{2n+1}c^{2n+1}f^{2n+1}},$$
$$x_{24n+10} = \frac{(dg-1)^{n+1}(be-1)^n a^{2n+1}c^{2n+2}f^{2n+1}}{(cf-1)^{n+1}(ad-1)^{n+1} g^{2n+1}b^{2n+1}e^{2n+1}},$$
$$x_{24n+11} = \frac{(ad-1)^n(cf-1)^{n+1} e^{2n+1}b^{2n+2}g^{2n+1}}{(be-1)^n(dg-1)^n a^{2n+1}c^{2n+1}f^{2n+1}},$$
$$x_{24n+12} = \frac{(dg-1)^{n+1}(be-1)^{n+1} a^{2n+2}c^{2n+1}f^{2n+1}}{(cf-1)^n(ad-1)^{n+1} e^{2n+1}b^{2n+1}g^{2n+1}},$$
$$x_{24n+13} = \frac{(ad-1)^{n+1}(cf-1)^{n+1} db^{2n+1}g^{2n+2}e^{2n+1}}{(dg-1)^{n+1}(be-1)^n a^{2n+1}c^{2n+2}f^{2n+1}},$$
$$x_{24n+14} = \frac{(dg-1)^n(be-1)^{n+1} a^{2n+1}c^{2n+2}f^{2n+2}}{(ad-1)^n(cf-1)^{n+1} e^{2n+1}b^{2n+2}g^{2n+1}},$$
$$x_{24n+15} = \frac{(ad-1)^{n+1}(cf-1)^n g^{2n+1}e^{2n+2}b^{2n+2}}{(dg-1)^{n+1}(be-1)^{n+1} a^{2n+2}c^{2n+1}f^{2n+1}},$$
$$x_{24n+16} = \frac{(dg-1)^{n+1}(be-1)^n a^{2n+2}c^{2n+2}f^{2n+1}}{(cf-1)^{n+1}(ad-1)^{n+1} g^{2n+2}b^{2n+1}e^{2n+1}},$$
$$x_{24n+17} = \frac{(ad-1)^n(cf-1)^{n+1} db^{2n+2}g^{2n+2}e^{2n+1}}{(dg-1)^{n+1}(be-1)^{n+1} a^{2n+1}c^{2n+2}f^{2n+2}},$$
$$x_{24n+18} = \frac{(dg-1)^{n+1}(be-1)^{n+1} a^{2n+2}c^{2n+2}f^{2n+2}}{(ad-1)^{n+1}(cf-1)^{n+1} g^{2n+1}b^{2n+2}e^{2n+2}},$$
$$x_{24n+19} = \frac{(ad-1)^{n+1}(cf-1)^{n+1} g^{2n+2}b^{2n+2}e^{2n+2}}{(dg-1)^{n+1}(be-1)^{n+1} a^{2n+2}c^{2n+2}f^{2n+1}},$$
$$x_{24n+20} = \frac{(dg-1)^{n+1}(be-1)^{n+1} a^{2n+2}c^{2n+2}f^{2n+2}}{(ad-1)^{n+1}(cf-1)^{n+1} g^{2n+2}b^{2n+2}e^{2n+1}},$$
$$x_{24n+21} = \frac{(ad-1)^{n+1}(cf-1)^{n+1} db^{2n+2}g^{2n+2}e^{2n+2}}{(dg-1)^{n+1}(be-1)^{n+1} a^{2n+2}c^{2n+2}f^{2n+2}},$$
$$x_{24n+22} = \frac{(dg-1)^{n+1}(be-1)^{n+1} a^{2n+2}c^{2n+3}f^{2n+2}}{(ad-1)^{n+1}(cf-1)^{n+1} g^{2n+2}b^{2n+2}e^{2n+2}},$$
$$x_{24n+23} = \frac{(ad-1)^{n+1}(cf-1)^{n+1} e^{2n+2}b^{2n+3}g^{2n+2}}{(dg-1)^{n+1}(be-1)^{n+1} a^{2n+2}c^{2n+2}f^{2n+2}},$$
$$x_{24n+24} = \frac{(dg-1)^{n+1}(be-1)^{n+1} a^{2n+3}c^{2n+2}f^{2n+2}}{(ad-1)^{n+1}(cf-1)^{n+1} g^{2n+2}b^{2n+2}e^{2n+2}}.$$

where the initial conditions are arbitrary nonzero real numbers.

Proof. The proof resembles that of Theorem 4.1 and will hence be omitted.

Theorem 5.2. Eq. (5) has two equilibrium points which $\pm\sqrt{2}$, and these equilibrium points aren't locally asymptotically stable.
Proof. The proof resembles that of Theorem 4.2 and will hence be omitted.

5.1. Numerical investigations

This part is dedicated to validating the theoretical findings presented in Eq. (5).

Example 5.1.1. For Eq. (5) we consider following initial conditions. The solutions for the specified initial conditions are depicted in Figure 4.

$x_{-6} = 0.105$, $x_{-5} = 0.115$, $x_{-4} = 0.120$, $x_{-3} = 0.125$, $x_{-2} = 0.130$, $x_{-1} = 0.127$, $x_0 = 0.117$.

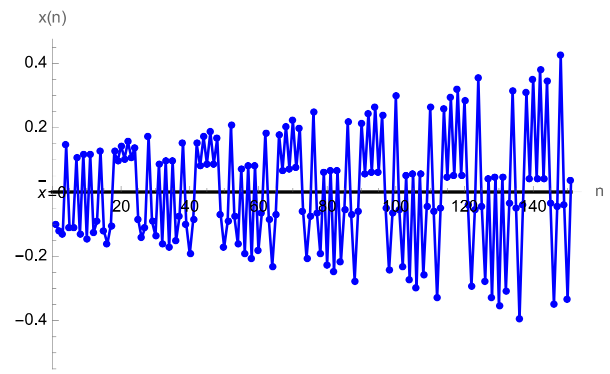

Figure 4. (Example 5.1.1)

4. Conclusion

This article thoroughly examines the qualitative properties of difference equations. It thoroughly analyzes local stability, periodicity, oscillation, and solutions. Conventional iterative approaches are utilized to obtain precise answers for the pertinent equations.

Article Information Form

Authors' Contribution

Conceptualization, F.A. and S.A.; methodology, F.A.; validation, F.A., and S.A.; investigation, F.A.; resources, F.A.; writing—original draft preparation, F.A.; writing—review and editing, F.A.; visualization, S.A. All authors have read and agreed to the published version of the manuscript.

The Declaration of Conflict of Interest/ Common Interest

No conflict of interest or common interest has been declared by authors.

Artificial Intelligence Statement

No artificial intelligence tools were used while writing this article.

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