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Research Article

Analytical Formulation of Non-Lightlike Offset Curves and Their Bertrand Structure in Minkowski Space

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Abstract

This paper investigates the geometric structure of offset curves derived from non-lightlike curves within the framework of Minkowski three-dimensional space. Offset curves, which consist of points located at a fixed distance along the normal direction from a base curve, are extended to Lorentzian geometry by analyzing their behavior under three different causal character cases: timelike, spacelike with timelike principal normal, and spacelike with timelike binormal. For each case, the curvature, torsion, and arc length of the offset curve are expressed in terms of the corresponding quantities of the original curve and two constants satisfying a specific linear condition that characterizes Bertrand curves. It is shown that the constructed offset curves preserve the principal normal direction of the base curve and satisfy the Bertrand condition, thus forming Bertrand pairs. The results provide a generalization of classical offset theory from Euclidean to Lorentzian geometry and offer new insights into the differential geometry of curves in pseudo-Riemannian spaces, with potential applications in relativistic kinematics and theoretical physics.

Keywords and 2020 Mathematics Subject Classification

Keywords: Bertrand curve — offset curve — Minkowski space — Lorentzian geometry — non-lightlike curve.
MSC: 53A35, 53A04

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1. Introduction

An offset curve, also referred to as a parallel curve, is mathematically defined as the locus of points that lie at a fixed perpendicular (normal) distance from a given base or generating curve. More specifically, at each point on the original curve, the corresponding point on the offset curve is obtained by moving along the normal vector direction by a prescribed constant magnitude. These curves are of significant importance in the context of computer-aided design (CAD) and computer-aided manufacturing (CAM) systems due to their wide range of practical applications. In particular, offset curves play a crucial role in numerically controlled (NC) machining processes. In such operations, the movement path of a cutting tool is often defined in terms of the offset curve, rather than the object's actual geometry. For instance, in two-axis NC milling using a circular cutter, the toolpath must be offset from the desired profile by a distance equal to the cutter's radius. This ensures that the center of the tool follows a trajectory that produces the desired material removal contour, as the physical cut is always formed at a normal offset from the tool's centerline. Thus, the offset curve effectively represents the machining envelope traced out by the cutting tool, enabling accurate shaping of mechanical components. Beyond machining, offset curves are also instrumental in path planning, tool compensation, surface modeling, and collision avoidance algorithms in modern design and manufacturing systems. Their computational generation involves sophisticated geometric and differential techniques, especially when the original curves exhibit high curvature, cusps, or self-intersections, making their study a topic of ongoing research in applied mathematics and geometric modeling [1–3].

Except for certain special cases such as straight lines and circles, the mathematical representation of offset curves is generally known to be highly nontrivial and intricate in structure [4]. This complexity arises from the differential-geometric behavior of the base curve, which can significantly influence the algebraic and analytical properties of its corresponding offsets. One notable challenge is that, in general, even when the original or generating curve is rational, its offset curves need not retain rationality [5]. This distinction becomes evident through classical examples: While the offset curves of a parabola remain rational, the offset curves associated with ellipses and hyperbolas do not possess such a property and are, in fact, non-rational in nature [6].

In another context, curves satisfying the linear relation

$$A\kappa + B\tau = 1$$

between their curvature κ and torsion τ are referred to as Bertrand curves for which there exist constants A and B. These curves are distinguished by a remarkable geometric property: they admit a Bertrand mate, that is, another curve sharing the same principal normal vector field at corresponding points. It is a well-established result in classical differential geometry that a space curve α in Euclidean 3- space admits an offset curve α^* that shares the same principal normal vector field if and only if α is a Bertrand curve [7].

Recent studies have shown that the concept of Bertrand curves can be extended to timelike ruled and developable surfaces in Minkowski space through the use of the E. Study mapping. In this way, it is possible to define pairs of timelike ruled surfaces that serve as Bertrand offsets of one another. Notably, such an offset relation holds if and only if a specific condition involving the dual invariants of the surfaces is satisfied. Furthermore, several new results have been established regarding the developability of these Bertrand offsets, enriching the geometric theory of ruled surfaces in the Lorentzian space [8]. In parallel, the notion of timelike V-Bertrand curves has been proposed, leading to the characterization of associated T, N, and B type Bertrand curves. These constructions further enable the definition of Bertrand curves and Bertrand-type surfaces, whose existence is supported through explicit examples. Together, these advancements broaden the theoretical landscape of offset geometry and open avenues for future research within Lorentzian differential geometry [9].

In this study, the offset geometry of non-lightlike curves is examined within the context of Minkowski 3-space. To establish the necessary theoretical background, an overview of offset curves in Euclidean 3-space is first provided. In this part, the fundamental definitions and essential geometric properties of offset curves are summarized, and the classical conditions under which these curves are constructed are reviewed.

Subsequently, Minkowski 3-space, denoted by $\mathbb{M}^3$, is introduced. Then, the basic geometric and kinematic properties of non-lightlike curves namely, spacelike and timelike curves are recalled. Lightlike (null) curves are not considered in this analysis due to the absence of a well-defined Frenet frame in such cases. Following this preliminary setup, the offset curves corresponding to non-lightlike curves are analyzed in detail. The construction of the offset curve is carried out for three distinct cases, each determined by the causal character of the Frenet frame vectors of the base curve. In each scenario, the mathematical expressions for the curvature, torsion, and arc length of the offset curve are derived. These quantities are expressed in terms of the curvature and torsion of the original curve, as well as constants A and B which arise from a linear relation involving these invariants.

In particular, attention is given to the condition $A\kappa + B\tau = 1$ where κ and τ denote the curvature and torsion of the curve, respectively. This condition is known to characterize Bertrand curves, which are distinguished by the existence of another curve sharing the same principal normal vector at corresponding points. It is demonstrated in this study that the offset curve constructed under the aforementioned conditions also satisfies the Bertrand condition, thereby constituting a Bertrand mate of the original curve.

Through this work, classical results concerning offset and Bertrand curves in Euclidean geometry are extended to the setting of Minkowski 3-space. As a result, a broader understanding is provided regarding the geometric behavior of curves in Lorentzian manifolds, particularly in relation to offset constructions and their invariants. The findings are expected to contribute to both the theoretical framework of differential geometry in pseudo-Riemannian spaces and to potential applications in mathematical physics and kinematic modeling.

2. Preliminaries

Before proceeding to the main results concerning offset constructions of non-lightlike curves in Minkowski 3-space, it is essential to establish a rigorous mathematical foundation that includes both the classical and Lorentzian settings. This section is therefore devoted to a detailed exposition of the differential geometric preliminaries necessary for understanding the subsequent developments.

We begin by reviewing the theory of offset curves in the standard Euclidean 3-space, where the Frenet frame is well-defined for regular unit-speed curves, and offset constructions are formulated via displacements along the principal normal vector. The conditions under which such curves form Bertrand pairs are recalled, particularly emphasizing the linear relation between curvature and torsion that characterizes this special class of curves. Furthermore, the geometric and analytic consequences of this condition on the offset curve such as shared normal directions and inherited differential invariants are highlighted.

Subsequently, the geometry of Minkowski 3-space is introduced, where the indefinite metric structure leads to a causal classification of vectors and curves. In this Lorentzian context, the Frenet formalism must be adapted to account for timelike, spacelike, and null curves. Since null curves do not admit a well-posed Frenet frame, they are excluded from our analysis. Instead, our focus is placed on non-lightlike (timelike and spacelike) curves, for which the Frenet equations remain valid under appropriate modifications of the inner product and causal character.

Throughout this section, necessary background definitions, notation, and core formulae are presented to facilitate a self-contained and coherent understanding of the offset curve theory within both Euclidean and Lorentzian frameworks. This foundational material sets the stage for the analytical construction and geometric classification of non-lightlike offset curves explored in the subsequent sections of the paper.

Let $\alpha : I \to \mathbb{E}^3$ be unit speed curve with the Frenet frame $\{t,n,b\}$. The offset curve of α along the principal normal is defined by

$$\alpha^*(s) = \alpha(s) + \lambda(s)n(s). \tag{1}$$

If the curve α and its offset curve occur on an equal footing then the principal normals of α and α^* must be equal that is $n = n^*$. By differentiating the equation (1), we get

$$\alpha^{*\prime}(s) = (1-\lambda\kappa(s))t(s) + \lambda'(s)n(s) + \lambda(s)\tau(s)b(s).$$

Since $\alpha^*_s \perp n^*$, we obtain $\lambda_s = 0$. This means that the distance between the curve α and its offset curve α^* is a constant. Then the unit tangent vector field of the curve α^* can be obtained as

$$t^*(s) = \frac{(1-\lambda\kappa(s))t(s) + \lambda\tau(s)b(s)}{\sqrt{(1-\lambda\kappa(s))^2 + (\lambda\tau(s))^2}}\cdot \tag{2}$$

By differentiating the equation (2), we obtain

$$\begin{aligned} t^*_s(s) = &\left(\frac{1-\lambda\kappa(s)}{\sqrt{(1-\lambda\kappa(s))^2+(\lambda\tau(s))^2}}\right)_s t(s) + \left(\frac{(1-\lambda\kappa(s))\kappa(s)-\lambda\tau(s)^2}{\sqrt{(1-\lambda\kappa(s))^2+(\lambda\tau(s))^2}}\right) n(s) \\ &+ \left(\frac{\lambda\tau(s)}{\sqrt{(1-\lambda\kappa(s))^2+(\lambda\tau(s))^2}}\right)_s b(s). \end{aligned}$$

Since $t^*_s \parallel n^*$, then the coefficients of t add b in the above equation must be zero. By long and tedious computations, we can see that the relation

$$A\kappa(s) + B\tau(s) = 1$$

holds for some constants A and B. We know that the curves satisfying the above relations are called as Bertrand curves [7,10].

In Euclidean 3-space, an important geometric property of Bertrand curves is related to their offset constructions. A regular curve $\alpha : I \to \mathbb{E}^3$ is defined to be a Bertrand curve if there exists another curve α^* called its Bertrand mate, such that both curves share the same principal normal vector field at corresponding points. This geometric relationship is known to hold if and only if a specific linear condition between the curvature κ and torsion τ of the curve is satisfied. Namely, there must exist real constants A and B such that the relation

$$A\kappa(s) + B\tau(s) = 1.$$

When this condition is satisfied, the original curve α and its Bertrand mate α^* are related through explicit differential-geometric formulas. The offset curve α^* is obtained by translating the original curve in the direction of its principal normal vector $n(s)$ by a scalar multiple A, yielding

$$\alpha^*(s) = \alpha(s) + An(s).$$

The Frenet frame vectors of the offset curve can be expressed in terms of those of the original curve as follows: The tangent vector $t^*(s)$ is given by a normalized linear combination of the tangent and binormal vectors of the original curve, specifically

$$t^*(s) = \frac{Bt(s) + Ab(s)}{\sqrt{A^2+B^2}}$$

the normal vector $n^*(s)$ remains identical to $n(s)$, and the binormal vector $b^*(s)$ is given by

$$b^*(s) = \frac{Bb(s) - At(s)}{\sqrt{A^2 + B^2}}\cdot$$

Furthermore, the geometric invariants of the offset curve its curvature $\kappa^*(s)$, torsion $\tau^*(s)$, and differential arclength ds^* are explicitly computed as functions of the original curvature, torsion, and the constants A and B. These are expressed as

$$\begin{aligned}
\kappa^*(s) &= \frac{B\kappa(s) - A\tau(s)}{(A^2 + B^2)\tau(s)},\\
\tau^*(s) &= \frac{1}{(A^2 + B^2)\tau(s)},\\
ds^* &= \sqrt{A^2 + B^2}\,\tau(s)ds,
\end{aligned}$$

respectively. Furthermore, for the constants $A^* = -A$ and $B^* = B$, the relation

$$A^*\kappa^*(s) + B^*\tau^*(s) = 1$$

holds, which shows that the offset curve α^* is itself a Bertrand curve. This recursive property highlights a deeper geometric structure in the theory of Bertrand curves, indicating that the offset operation under this condition preserves the Bertrand nature of the curve pair. Such results form a foundational aspect of classical differential geometry and serve as a bridge between geometric transformations and intrinsic curve properties [7].

In order to generalize the theory of curves from Euclidean geometry to a relativistic context, it is essential to consider the geometric structure of Minkowski 3-space, which serves as a fundamental model of three-dimensional spacetime equipped with a Lorentzian metric. This space, denoted by $\mathbb{M}^3$, possesses an indefinite product

$$\langle u, v\rangle = -u_1v_1 + u_2v_2 + u_3v_3$$

which classifies vectors and curves into timelike, spacelike, and lightlike types. Detailed discussions of this structure and its implications for constant mean curvature surfaces and quaternions can be found in the works of Inoguchi [11] and Özdemir and Ergin [12] and [13]. In particular, Özdemir and Ergin introduced rotation structures via timelike quaternions in Minkowski space [12], while the adaptation of parallel frames for non-lightlike curves was studied in [14]. For a comprehensive treatment of the Frenet frame construction for non-lightlike curves in this setting, see also the studies of Inoguchi [15] and Erdoğdu and Özdemir [16], which provide foundational results on Hasimoto-type structures in Minkowski 3-space.

To study the geometry of curves in Minkowski space, one must adapt the classical Frenet frame a moving orthonormal frame along the curve to the Lorentzian structure. This adaptation involves reinterpreting the curvature and torsion of a curve in light of its causal character. Specifically, the sign and magnitude of the inner products between the curve's tangent, normal, and binormal vectors influence how the curve bends and twists in space. Furthermore, new constants arise that describe the second and third causal characters of the curve, which reflect the nature of its normal and binormal directions. These modifications are essential for correctly formulating the differential geometry of curves in Minkowski space and serve as the foundation for further analysis in Lorentzian geometry. Let $\alpha(s)$ be a unit speed curve in $\mathbb{M}^3$, *i,e.*, $\langle\alpha', \alpha'\rangle = \varepsilon_1 = \pm 1$. The constant ε_1 is called the casual character of α. A unit speed curve α is said to be a Frenet curve if $\|\alpha''\| \neq 0$. Every Frenet curve admits a Frenet frame field (t, n, b) satisfying the equation:

$$\begin{bmatrix} t \\ n \\ b \end{bmatrix}_s = \begin{bmatrix} 0 & \varepsilon_2\kappa & 0 \\ -\varepsilon_1\kappa & 0 & -\varepsilon_3\tau \\ 0 & \varepsilon_2\tau & 0 \end{bmatrix}\begin{bmatrix} t \\ n \\ b \end{bmatrix}. \tag{3}$$

The functions $\kappa \geq 0$ and τ are called the curvature and torsion, respectively. The vector fields $t = \alpha'$, n and b are called, tangent, normal and binormal vector fields, respectively. Here, the symbols $\varepsilon_1, \varepsilon_2$ and ε_3 represent the causal characters of the Frenet frame vectors in Minkowski 3-space and they are essential in adapting the classical Frenet formalism to Lorentzian geometry. Specifically, ε_1 denotes the causal character of the tangent vector which takes the value -1 if the curve is timelike and $+1$ if it is spacelike. Similarly, $\varepsilon_2 = \langle n, n\rangle$ refers to the causal character of the principal normal vector, and $\varepsilon_3 = \langle b, b\rangle$ corresponds to that of the binormal vector. Due to the orthonormal structure of the Frenet frame in Lorentzian geometry, these constants satisfy the identity $\varepsilon_3 = -\varepsilon_1\varepsilon_2$. Introducing these signs is necessary to properly describe the geometric behavior of curves in Minkowski space and to preserve the consistency of the Serret–Frenet equations under the indefinite metric. This framework allows one to distinguish between timelike and spacelike evolutions of curves and ensures that the differential invariants are defined in accordance with the underlying causal structure [15–17].

3. Offset curve configurations of timelike and spacelike curves in Minkowski 3-space

Proposition 1. *A timelike curve $\alpha : I \to \mathbb{M}^3$ admits a non-ligthlike offset curve α^* which has the same principal normal as the curve α if and only if α is a Bertrand curve, that is the relation*

$$A\kappa + B\tau = 1$$

is satisfied for some constants A and B.

Proof. Let $\alpha : I \to \mathbb{M}^3$ be a timelike curve with the Frenet frame $\{t, n, b\}$ and α^* be the non-ligthlike offset curve of α with the Frenet Frame $\{t^*, n^*, b^*\}$. Then the offset curve α^* can be defined as

$$\alpha^*(s) = \alpha(s) + \lambda(s)n(s). \tag{4}$$

By using the Serret-Frenet equations for α given in equation (3) with the choice $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (-1, 1, 1)$, we get

$$\alpha^{*\prime}(s) = (1 + \lambda(s)\kappa(s))t(s) + \lambda'(s)n(s) - \lambda(s)\tau(s)b(s).$$

The main curve α and its offset curve α^* occur on an equal footing. So the principal normals must be coincide; $n^* = n$. Since $\alpha^{*\prime} \perp n^* = n$, then $\lambda'(s) = 0$. Thus λ is a constant function and we have

$$\begin{aligned} \langle \alpha^{*\prime}(s), \alpha^{*\prime}(s)\rangle = {} & \langle (1+\lambda\kappa(s)t(s) - \lambda\tau(s)b(s), (1+\lambda\kappa(s)t(s) - \lambda\tau(s)b(s)\rangle \\ = {} & (1+\lambda\kappa(s)^2 \langle t(s), t(s)\rangle + (\lambda\tau(s))^2 \langle b(s), b(s)\rangle \\ = {} & -(1+\lambda\kappa(s))^2 + (\lambda\tau(s))^2. \end{aligned} \tag{5}$$

Then, we find

$$t^*(s) = \frac{(1+\lambda\kappa(s)t(s) - \lambda\tau(s)b(s)}{\sqrt{-(1+\lambda\kappa(s))^2 + (\lambda\tau(s))^2}}. \tag{6}$$

Again by using the Serret- Frenet equations of α, we get

$$\begin{aligned} t^{*\prime}(s) = {} & \left(\frac{1+\lambda\kappa(s)}{\sqrt{-(1+\lambda\kappa(s))^2+(\lambda\tau(s))^2}}\right)_s t(s) + \left(\frac{1+\lambda\kappa(s)}{\sqrt{-(1+\lambda\kappa(s))^2+(\lambda\tau(s))^2}}\right) t'(s) \\ & + \left(\frac{-\lambda\tau(s)}{\sqrt{-(1+\lambda\kappa(s))^2+(\lambda\tau(s))^2}}\right)_s b(s) + \left(\frac{-\lambda\tau(s)}{\sqrt{-(1+\lambda\kappa(s))^2+(\lambda\tau(s))^2}}\right) b'(s) \\ = {} & \left(\frac{1+\lambda\kappa(s)}{\sqrt{-(1+\lambda\kappa(s))^2+(\lambda\tau(s))^2}}\right)_s t(s) + \left(\frac{(1+\lambda\kappa(s))\kappa(s) - \lambda\tau(s)^2}{\sqrt{-(1+\lambda\kappa(s))^2+(\lambda\tau(s))^2}}\right) n(s) \\ & + \left(\frac{-\lambda\tau(s)}{\sqrt{-(1+\lambda\kappa(s))^2+(\lambda\tau(s))^2}}\right)_s b(s). \end{aligned}$$

Since $t^{*\prime}(s) \parallel n^*$, the coefficients of $t(s)$ and $b(s)$ the above equation must be vanish. That is

$$\left(\frac{1+\lambda\kappa(s)}{\sqrt{-(1+\lambda\kappa(s))^2+(\lambda\tau(s))^2}}\right)_s = \left(\frac{-\lambda\tau(s)}{\sqrt{-(1+\lambda\kappa(s))^2+(\lambda\tau(s))^2}}\right)_s = 0.$$

The first equality implies

$$\frac{1+\lambda\kappa(s)}{\sqrt{-(1+\lambda\kappa(s))^2+(\lambda\tau(s))^2}} = C,$$

where C is constant of integration. If we expand the above equation, we get

$$1 = -\lambda\kappa(s) \pm \frac{C}{\sqrt{1+C^2}}\lambda\tau(s).$$

So, we obtain the relation

$$A\kappa(s) + B\tau(s) = 1$$

for the constants $A = -\lambda$ and $B = \pm\frac{C}{\sqrt{1+C^2}}\lambda$. Thus, α is a Bertrand curve. ■

Theorem 2. *Let $\alpha : I \to \mathbb{M}^3$ be a timelike curve with the Frenet frame $\{t, n, b\}$, α^* be the non-ligthlike offset curve of α with the Frenet Frame $\{t^*, n^*, b^*\}$ and*

$$A\kappa(s) + B\tau(s) = 1$$

for some constants A and B such that $A^2 - B^2 \neq 0$. Then, the curve α and its offset curve α^* are related by following relations;

$$\begin{aligned} \alpha^*(s) &= \alpha(s) - An(s) \\ t^*(s) &= \frac{Bt(s) + Ab(s)}{\sqrt{B^2 - A^2}}, \\ n^*(s) &= n(s), \\ b^*(s) &= \frac{-At(s) + Bb(s)}{\sqrt{B^2 - A^2}}. \end{aligned}$$

Moreover, the curvature, torsion and arclength parameter of the offset curve α^* are obtained as

$$\begin{aligned} \kappa^*(s) &= \frac{B\kappa(s) + A\tau(s)}{(B^2 - A^2)\tau(s)}, \\ \tau^*(s) &= \frac{B\kappa(s) + A\tau(s)}{(B^2 - A^2)\tau(s)}, \\ ds^* &= \sqrt{B^2 - A^2}\,\tau(s)ds, \end{aligned}$$

respectively. Furthermore, for the constants

$$A^* = \frac{A(B^2 - A^2)}{A^2 + B^2} \text{ and } B^* = \frac{B(B^2 - A^2)}{A^2 + B^2},$$

the relation

$$A^*\kappa^*(s) + B^*\tau^*(s) = 1$$

holds. So, the offset curve is also a Bertrand curve.

Proof. By Proposition 1, we get $\lambda = -A$. If we substitute $\lambda = -A$ into equation (4) and equation (6), we find

$$\alpha^*(s) = \alpha(s) - An(s),$$

$$t^*(s) = \frac{Bt(s) + Ab(s)}{\sqrt{B^2 - A^2}}.$$

By assumption, $n^*(s) = n(s)$. Since $b^* = -t^* \times n^*$, then we get

$$\begin{aligned} b^*(s) &= -\frac{B}{\sqrt{B^2 - A^2}} t(s) \times n(s) - \frac{A}{\sqrt{B^2 - A^2}} b(s) \times n(s) \\ &= -\frac{B}{\sqrt{B^2 - A^2}}(-b(s)) - \frac{A}{\sqrt{B^2 - A^2}} t(s) \\ &= \frac{-At(s) + Bb(s)}{\sqrt{B^2 - A^2}}. \end{aligned}$$

Again, substituting $\lambda = -A$ into equation (5) gives

$$ds^* = \sqrt{B^2 - A^2}\,\tau(s)ds.$$

Then, we have

$$
\begin{aligned}
\frac{d}{ds^*}t(s) &= \frac{dt^*(s)}{ds}\frac{ds}{ds^*} \\
&= \frac{1}{\sqrt{B^2-A^2}}(Bt'(s)+Ab'(s))\frac{1}{\sqrt{B^2-A^2}\tau(s)} \\
&= \frac{1}{(B^2-A^2)\tau(s)}[B\kappa(s)n(s)+A\tau(s)n(s)] \\
&= \frac{B\kappa(s)+A\tau(s)}{(B^2-A^2)\tau(s)}n(s), \\
\frac{d}{ds^*}b(s) &= \frac{db^*(s)}{ds}\frac{ds}{ds^*} \\
&= \frac{1}{\sqrt{B^2-A^2}}(-At'(s)+Bb'(s))\frac{1}{\sqrt{B^2-A^2}\tau(s)} \\
&= \frac{1}{(B^2-A^2)\tau(s)}[-A\kappa(s)n(s)+B\tau(s)n(s)] \\
&= \frac{-A\kappa(s)+B\tau(s)}{(B^2-A^2)\tau(s)}n(s).
\end{aligned}
$$

Also we obtain

$$
\begin{aligned}
\kappa^*(s) &= \left\langle \frac{dt^*(s)}{ds^*}, n^*(s) \right\rangle \\
&= \left\langle \frac{B\kappa(s)+A\tau(s)}{(B^2-A^2)\tau(s)}n(s), n(s) \right\rangle \\
&= \frac{B\kappa(s)+A\tau(s)}{(B^2-A^2)\tau(s)}, \\
\tau^*(s) &= \left\langle \frac{db^*(s)}{ds^*}, n^*(s) \right\rangle \\
&= \left\langle \frac{-A\kappa(s)+B\tau(s)}{(B^2-A^2)\tau(s)}n(s), n(s) \right\rangle \\
&= \frac{-A\kappa(s)+B\tau(s)}{(B^2-A^2)\tau(s)}.
\end{aligned}
$$

By choosing $A^* = \frac{A(B^2-A^2)}{A^2+B^2}$ and $B^* = \frac{B(B^2-A^2)}{A^2+B^2}$, we can easily see that

$$A^*\kappa^*(s)+B^*\tau^*(s) = 1.$$

■

Proposition 3. *A spacelike curve $\alpha : I \to \mathbb{M}^3$ with timelike principal normal admits a non-ligthlike offset curve α^* which has the same principal normal as the curve α if and only if α is a Bertrand curve, that is the relation*

$$A\kappa(s)+B\tau(s) = 1$$

is satisfied for some constants A and B.

Proof. Let $\alpha : I \to \mathbb{M}^3$ be a spacelike curve with timelike principal normal and Frenet frame $\{t,n,b\}$ and α^* be the non-ligthlike offset curve of α with the Frenet Frame $\{t^*,n^*,b^*\}$. Then the offset curve α^* can be defined as

$$\alpha^*(s) = \alpha(s)+\lambda(s)n(s) \tag{7}$$

By using the Serret-Frenet equations for α given in equation (3) with the choice $(\varepsilon_1,\varepsilon_2,\varepsilon_3) = (1,-1,1)$, we get

$$\alpha^{*\prime}(s) = (1-\lambda(s)\kappa(s))t(s)+\lambda'(s)n(s)-\lambda(s)\tau(s)b(s).$$

The main curve α and its offset curve α^* occur on an equal footing. So the principal normals must be coincide; $n^*(s) = n(s)$. Since $\alpha^{*\prime}(s) \perp n^*(s)$, then $\lambda'(s) = 0$. Thus λ is a constant function. And we have

$$\begin{aligned} \langle \alpha^{*\prime}(s), \alpha^{*\prime}(s) \rangle = & \ \langle (1-\lambda\kappa(s))t(s) - \lambda\tau(s)b(s), (1+\lambda\kappa(s)t(s) - \lambda\tau(s)b(s) \rangle \\ = & \ (1-\lambda\kappa(s))^2 \langle t(s), t(s) \rangle + (-\lambda\tau(s))^2 \langle b(s), b(s) \rangle \\ = & \ (1-\lambda\kappa(s))^2 + (\lambda\tau(s))^2. \end{aligned} \tag{8}$$

Then we find

$$t^*(s) = \frac{(1-\lambda\kappa(s))t(s) - \lambda\tau(s)b(s)}{\sqrt{(1-\lambda\kappa(s))^2 + (\lambda\tau(s))^2}}. \tag{9}$$

Again by using the Serret- Frenet equations of α, we get

$$\begin{aligned} t^{*\prime}(s) = & \left(\frac{1-\lambda\kappa(s)}{\sqrt{(1-\lambda\kappa(s))^2 + (\lambda\tau(s))^2}} \right)_s t(s) + \left(\frac{1-\lambda\kappa(s)}{\sqrt{(1-\lambda\kappa(s))^2 + (\lambda\tau(s))^2}} \right) t'(s) \\ & + \left(\frac{-\lambda\tau(s)}{\sqrt{(1-\lambda\kappa(s))^2 + (\lambda\tau(s))^2}} \right)_s b(s) + \left(\frac{-\lambda\tau(s)}{\sqrt{(1-\lambda\kappa(s))^2 + (\lambda\tau(s))^2}} \right) b'(s) \\ = & \left(\frac{1-\lambda\kappa(s)}{\sqrt{(1-\lambda\kappa(s))^2 + (\lambda\tau(s))^2}} \right)_s t(s) + \left(\frac{-(1-\lambda\kappa(s))\kappa(s) + \lambda\tau(s)^2}{\sqrt{(1-\lambda\kappa(s))^2 + (\lambda\tau(s))^2}} \right) n(s) \\ & + \left(\frac{-\lambda\tau(s)}{\sqrt{(1-\lambda\kappa(s))^2 + (\lambda\tau(s))^2}} \right)_s b(s). \end{aligned}$$

Since $t^{*\prime}(s) \parallel n^*(s)$, the coefficients of $t(s)$ and $b(s)$ the above equation must be vanish. That is

$$\left(\frac{1-\lambda\kappa(s)}{\sqrt{(1-\lambda\kappa(s))^2 + (\lambda\tau(s))^2}} \right)_s = \left(\frac{-\lambda\tau(s)}{\sqrt{(1-\lambda\kappa(s))^2 + (\lambda\tau(s))^2}} \right)_s = 0.$$

The first equality implies

$$\frac{1-\lambda\kappa(s)}{\sqrt{(1-\lambda\kappa(s))^2 + (\lambda\tau(s))^2}} = C,$$

where C is constant of integration. If we expand the above equation, we get

$$1 = \lambda\kappa(s) \pm \frac{C}{\sqrt{1-C^2}} \lambda\tau(s).$$

So, we obtain the relation

$$A\kappa(s) + B\tau(s) = 1$$

for the constants $A = \lambda$ and $B = \pm\frac{C}{\sqrt{1-C^2}}\lambda$. Thus, α is a Bertrand curve. ■

Theorem 4. *Let $\alpha : I \to \mathbb{M}^3$ be a spacelike curve with timelike principal normal and the Frenet frame $\{t, n, b\}$, α^* be the non-ligthlike offset curve of α with the Frenet Frame $\{t^*, n^*, b^*\}$ and*

$$A\kappa(s) + B\tau(s) = 1$$

for some constants A and B. Then, the curve α and its offset curve α^* are related by following relations;

$$\begin{aligned} \alpha^*(s) &= \alpha(s) + An(s) \\ t^*(s) &= \frac{Bt(s) - Ab(s)}{\sqrt{A^2+B^2}}, \\ n^*(s) &= n(s), \\ b^*(s) &= \frac{At(s) + Bb(s)}{\sqrt{A^2+B^2}}. \end{aligned}$$

Moreover, the curvature, torsion and arc length parameter of the offset curve α^* *are obtained as*

$$\begin{aligned}
\kappa^*(s) &= \frac{B\kappa(s) - A\tau(s)}{(A^2+B^2)\tau(s)},\\
\tau^*(s) &= \frac{1}{(A^2+B^2)\tau(s)},\\
ds^* &= \sqrt{A^2+B^2}\tau ds,
\end{aligned}$$

respectively. Furthermore, for the constants

$$A^* = -A \text{ and } B^* = B,$$

the relation

$$A^*\kappa^*(s) + B^*\tau^*(s) = 1$$

is satisfied, that is the offset curve is also a Bertrand curve.

Proof. By Proposition 3, we get $\lambda = A$. If we substitute $\lambda = A$ into equations (7) and (9), we find

$$\alpha^*(s) = \alpha(s) + An(s),$$

$$t^*(s) = \frac{Bt(s) - Ab(s)}{\sqrt{A^2+B^2}}\cdot$$

By assumption, $n^*(s) = n(s)$. Since $b^*(s) = -t^*(s) \times n^*(s)$, then we get

$$\begin{aligned}
b^*(s) &= -\frac{B}{\sqrt{A^2+B^2}} t(s) \times n(s) + \frac{A}{\sqrt{A^2-B^2}} b(s) \times n(s)\\
&= -\frac{B}{\sqrt{A^2+B^2}}(-b(s)) + \frac{A}{\sqrt{A^2+B^2}} t(s)\\
&= \frac{At(s) + Bb(s)}{\sqrt{A^2+B^2}}\cdot
\end{aligned}$$

Again, substituting $\lambda = A$ into equation (8) gives

$$ds^* = \sqrt{A^2+B^2}\tau(s)ds.$$

Also we have

$$\begin{aligned}
\frac{d}{ds^*}t(s) &= \frac{dt^*(s)}{ds}\frac{ds}{ds^*}\\
&= \frac{1}{\sqrt{A^2+B^2}}(Bt'(s) - Ab'(s))\frac{1}{\sqrt{A^2+B^2}\tau(s)}\\
&= \frac{1}{(A^2+B^2)\tau(s)}[-B\kappa(s)n(s) + A\tau(s)n(s)]\\
&= \frac{-B\kappa(s) + A\tau(s)}{(A^2+B^2)\tau(s)} n(s),\\
\frac{d}{ds^*}b(s) &= \frac{db^*(s)}{ds}\frac{ds}{ds^*}\\
&= \frac{1}{\sqrt{A^2+B^2}}(At'(s) + Bb'(s))\frac{1}{\sqrt{A^2+B^2}\tau(s)}\\
&= \frac{1}{(A^2+B^2)\tau(s)}[-A\kappa(s)n(s) - B\tau(s)n(s)]\\
&= \frac{-A\kappa(s) - B\tau(s)}{(A^2+B^2)\tau(s)} n(s)\\
&= \frac{-1}{(A^2+B^2)\tau(s)} n(s).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\kappa^*(s) &= \left\langle \frac{dt^*(s)}{ds^*}, n^*(s) \right\rangle \\
&= \left\langle \frac{-B\kappa(s)+A\tau(s)}{(A^2+B^2)\tau(s)} n(s), n(s) \right\rangle \\
&= \frac{B\kappa(s)-A\tau(s)}{(A^2+B^2)\tau(s)}, \\
\tau^*(s) &= \left\langle \frac{db^*(s)}{ds^*}, n^*(s) \right\rangle \\
&= \left\langle \frac{-1}{(A^2+B^2)\tau(s)} n(s), n(s) \right\rangle \\
&= \frac{1}{(A^2+B^2)\tau(s)}
\end{aligned}$$

Taking $A^* = -A$, $B^* = B$ and using the relation $A\kappa(s)+B\tau(s)=1$, it is easily seen that

$$A^*\kappa^*(s)+B^*\tau^*(s)=1.$$

■

Proposition 5. *A spacelike curve $\alpha : I \to \mathbb{M}^3$ with timelike binormal admits a non-ligthlike offset curve α^* which has the same principal normal as the curve α if and only if α is a Bertrand curve, that is the relation*

$$A\kappa(s)+B\tau(s)=1$$

is satisfied for some constants A and B.

Proof. Let $\alpha : I \to \mathbb{M}^3$ be a spacelike curve with timelike binormal and Frenet frame $\{t,n,b\}$ and α^* be the non-ligthlike offset curve of α with the Frenet Frame $\{t^*,n^*,b^*\}$. Then the offset curve α^* can be defined as

$$\alpha^*(s) = \alpha(s)+\lambda(s)n(s) \tag{10}$$

By using the Serret-Frenet equations for α given in equation (3) with the choice $(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(1,1,-1)$, we get

$$\alpha^{*\prime}(s) = (1-\lambda(s)\kappa(s))t(s)+\lambda'(s)n(s)+\lambda(s)\tau(s)b(s).$$

The main curve α and its offset curve α^* occur on an equal footing. So the principal normals must be coincide; $n^*(s)=n(s)$. Since $\alpha^{*\prime}(s) \perp n^*(s)$, then $\lambda'(s)=0$. Thus λ is a constant function. And we have

$$\begin{aligned}
\langle \alpha^{*\prime}(s), \alpha^{*\prime}(s)\rangle &= \langle (1-\lambda\kappa(s))t(s)+\lambda\tau(s)b(s), (1+\lambda\kappa(s)t(s)+\lambda\tau(s)b(s)\rangle \\
&= (1-\lambda\kappa(s))^2\langle t(s),t(s)\rangle + (-\lambda\tau(s))^2\langle b(s),b(s)\rangle \\
&= (1-\lambda\kappa(s))^2-(\lambda\tau(s))^2.
\end{aligned} \tag{11}$$

Then we find

$$t^*(s) = \frac{(1-\lambda\kappa(s))t(s)+\lambda\tau(s)b(s)}{\sqrt{(1-\lambda\kappa(s))^2-(\lambda\tau(s))^2}}. \tag{12}$$

Again by using the Serret- Frenet equations of α, we get

$$\begin{aligned}
t^{*\prime}(s) &= \left(\frac{1-\lambda\kappa(s)}{\sqrt{(1-\lambda\kappa(s))^2-(\lambda\tau(s))^2}}\right)_s t(s) + \left(\frac{1-\lambda\kappa(s)}{\sqrt{(1-\lambda\kappa(s))^2-(\lambda\tau(s))^2}}\right) t'(s) \\
&+ \left(\frac{\lambda\tau(s)}{\sqrt{(1-\lambda\kappa(s))^2-(\lambda\tau(s))^2}}\right)_s b(s) + \left(\frac{\lambda\tau(s)}{\sqrt{(1-\lambda\kappa(s))^2-(\lambda\tau(s))^2}}\right) b'(s) \\
&= \left(\frac{1-\lambda\kappa(s)}{\sqrt{(1-\lambda\kappa(s))^2-(\lambda\tau(s))^2}}\right)_s t(s) + \left(\frac{(1-\lambda\kappa(s))\kappa(s)+\lambda\tau(s)^2}{\sqrt{(1-\lambda\kappa(s))^2-(\lambda\tau(s))^2}}\right) n(s) \\
&+ \left(\frac{\lambda\tau(s)}{\sqrt{(1-\lambda\kappa(s))^2-(\lambda\tau(s))^2}}\right)_s b(s)
\end{aligned}$$

Since $t^{*\prime}(s) \parallel n^{*}(s)$, the coefficients of $t(s)$ and $b(s)$ the above equation must be vanish. That is

$$\left(\frac{1-\lambda\kappa(s)}{\sqrt{(1-\lambda\kappa(s))^2-(\lambda\tau(s))^2}}\right)_s = \left(\frac{\lambda\tau(s)}{\sqrt{(1-\lambda\kappa(s))^2-(\lambda\tau(s))^2}}\right)_s = 0.$$

The first equality implies

$$\frac{1-\lambda\kappa(s)}{\sqrt{(1-\lambda\kappa(s))^2-(\lambda\tau(s))^2}} = C$$

where C is constant of integration. If we expand the above equation, we get

$$1 = \lambda\kappa(s) \pm \frac{C}{\sqrt{C^2-1}}\lambda\tau(s).$$

So, we obtain the relation

$$A\kappa(s) + B\tau(s) = 1$$

for the constants $A = \lambda$ and $B = \pm\frac{C}{\sqrt{C^2-1}}\lambda$. Thus, α is a Bertrand curve. ■

Theorem 6. *Let $\alpha : I \to \mathbb{M}^3$ be a spacelike curve with timelike binormal and the Frenet frame $\{t,n,b\}$, α^* be the non-ligthlike offset curve of α with the Frenet Frame $\{t^*,n^*,b^*\}$ and*

$$A\kappa(s) + B\tau(s) = 1$$

for some constants A and B. Then, the curve α and its offset curve α^* are related by following relations;

$$\begin{aligned} \alpha^*(s) &= \alpha(s) + An(s) \\ t^*(s) &= \frac{Bt(s)+Ab(s)}{\sqrt{B^2-A^2}}, \\ n^*(s) &= n(s), \\ b^*(s) &= \frac{-At(s)+Bb(s)}{\sqrt{B^2-A^2}}. \end{aligned}$$

Moreover, the curvature, torsion and arc length parameter of the offset curve α^* are obtained as

$$\begin{aligned} \kappa^*(s) &= \frac{B\kappa(s)+A\tau(s)}{(B^2-A^2)\tau(s)}, \\ \tau^*(s) &= \frac{-A\kappa(s)+B\tau(s)}{(B^2-A^2)\tau(s)}, \\ ds^* &= \sqrt{B^2-A^2}\,\tau(s)ds, \end{aligned}$$

respectively. Furthermore, for the constants

$$A^* = \frac{A(B^2-A^2)}{A^2+B^2} \text{ and } B^* = \frac{B(B^2-A^2)}{A^2+B^2},$$

the relation

$$A^*\kappa^*(s) + B^*\tau^*(s) = 1$$

holds, that is the offset curve is also a Bertrand curve.

Proof. By Proposition 5, we get $\lambda = A$. If we substitute $\lambda = A$ into equations (10) and (12), we find

$$\begin{aligned} \alpha^*(s) &= \alpha(s) + An(s), \\ t^*(s) &= \frac{Bt(s)+Ab(s)}{\sqrt{B^2-A^2}}. \end{aligned}$$

By assumption, $n^*(s) = n(s)$. Since $b^* = -t^* \times n^*$, then we get

$$\begin{aligned}
b^*(s) &= -\frac{B}{\sqrt{B^2-A^2}} t(s) \times n(s) - \frac{A}{\sqrt{B^2-A^2}} b(s) \times n(s) \\
&= -\frac{B}{\sqrt{B^2-A^2}} (-b(s)) - \frac{A}{\sqrt{B^2-A^2}} t(s) \\
&= \frac{-At(s) + Bb(s)}{\sqrt{B^2-A^2}} \cdot
\end{aligned}$$

Again, substituting $\lambda = A$ into equation (11) gives

$$ds^* = \sqrt{B^2-A^2}\,\tau(s) ds.$$

And we have

$$\begin{aligned}
\frac{d}{ds^*} t(s) &= \frac{dt^*(s)}{ds} \frac{ds}{ds^*} \\
&= \frac{1}{\sqrt{B^2-A^2}} (Bt'(s) + Ab'(s)) \frac{1}{\sqrt{B^2-A^2}\,\tau(s)} \\
&= \frac{1}{(B^2-A^2)\tau(s)} [B\kappa(s) n(s) + A\tau(s) n(s)] \\
&= \frac{B\kappa(s) + A\tau(s)}{(B^2-A^2)\tau(s)} n(s), \\
\frac{d}{ds^*} b(s) &= \frac{db^*(s)}{ds} \frac{ds}{ds^*} \\
&= \frac{1}{\sqrt{B^2-A^2}} (-At'(s) + Bb'(s)) \frac{1}{\sqrt{B^2-A^2}\,\tau(s)} \\
&= \frac{1}{(B^2-A^2)\tau(s)} [-A\kappa(s) n(s) + B\tau(s) n(s)] \\
&= \frac{-A\kappa(s) + B\tau(s)}{(B^2-A^2)\tau(s)} n(s).
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\kappa^*(s) &= \left\langle \frac{dt^*(s)}{ds^*}, n^*(s) \right\rangle \\
&= \left\langle \frac{B\kappa(s) + A\tau(s)}{(B^2-A^2)\tau(s)} n(s), n(s) \right\rangle \\
&= \frac{B\kappa(s) + A\tau(s)}{(B^2-A^2)\tau(s)}, \\
\tau^*(s) &= \left\langle \frac{db^*(s)}{ds^*}, n^*(s) \right\rangle \\
&= \left\langle \frac{-A\kappa(s) + B\tau(s)}{(B^2-A^2)\tau(s)} n(s), n(s) \right\rangle \\
&= \frac{-A\kappa(s) + B\tau(s)}{(B^2-A^2)\tau(s)} \cdot
\end{aligned}$$

By choosing

$$A^* = \frac{A(B^2-A^2)}{A^2+B^2} \text{ and } B^* = \frac{B(B^2-A^2)}{A^2+B^2},$$

we can easily see that

$$A^* \kappa^*(s) + B^* \tau^*(s) = 1.$$

■

4. Conclusion

This study has extended the classical concept of offset curves from Euclidean space to Minkowski 3-space, focusing on non-lightlike curves.

i. Three distinct causal configurations were analyzed in detail: Timelike curves, spacelike curves with timelike principal normals and spacelike curves with timelike binormals.

ii. For each case, explicit expressions for the curvature, torsion, and arc length of the offset curves were derived in terms of the original curve's differential invariants.

   It was shown that if the original curve satisfies a linear relation of the form $A\kappa + B\tau = 1$ then the offset curve shares the same principal normal vector and forms a Bertrand pair with the original curve.

iii. Furthermore, the offset curve constructed under this condition also satisfies a corresponding Bertrand relation, which confirms that it is itself a Bertrand curve.

iv. The study generalizes known results in Euclidean differential geometry to the Lorentzian space, contributing to the theory of curves in pseudo-Riemannian spaces.

v. The findings provide potential theoretical foundations for applications in relativistic kinematics, spacetime modeling, and geometric methods in mathematical physics.

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