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RESEARCH ARTICLE

Effect of Multiplicative Calculus on Special Ruled Surface Pairs

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ABSTRACT

This paper explores the application and advantages of multiplicative analysis in surface theory. Unlike additive methods, multiplicative analysis focuses on the interaction of variables through product-based relationships, offering a more accurate representation in contexts involving exponential growth, ratios, and scaling. One key advantage of multiplicative analysis is its ability to simplify complex problems by exploiting factorization and invariance properties, enabling more efficient problem-solving strategies. This study highlights both theoretical foundations and practical benefits of using multiplicative approaches in special ruled surface pairs for mathematical research. Hence, we define new special ruled surface pairs called mul-Bertrand, mul-involute-evolute and mul-Mannheim ruled surface pairs. Moreover, some illustrative examples are given to validate the results.

Keywords: *Multiplicative calculus, special partner surfaces, multiplicative Euclidean space.*

AMS Subject Classification (2020): *Primary: 53A04 ; Secondary: 11U10; 53A05.*

1. Introduction

Ruled surfaces, which are described as the set of points swept by a moving straight line in [9], have long been central objects of study in differential geometry due to their geometric simplicity and practical use in fields such as kinematics, computer-aided geometric design (CAGD) and architectural modeling. Classical surface pairs such as Mannheim, Bertrand, and involute-evolute ruled surfaces were extensively investigated in Euclidean and non-Euclidean settings, with foundational contributions in [17], [15], [18], [10], [16]. These pairs have provided key insights into the intrinsic and extrinsic geometries of developable and skew surfaces. In recent years, a growing body of research has focused on extending these classical notions to non-Newtonian frameworks, particularly multiplicative calculus, a branch of mathematical analysis first formalized by Stanley [19] and further developed in [1]. This approach replaces additive operations with multiplicative counterparts, enabling a more natural description of processes characterized by proportional change, scaling, and exponential growth. Multiplicative differential geometry, introduced rigorously by Georgiev [11] and expanded in collaboration with Zennir in [12], has laid the foundation for reinterpreting classical differential geometric concepts within this new analytical setting.

Unlike classical calculus, which is ill-suited for functions with multiplicative or exponential behavior, the multiplicative framework preserves positivity, offers consistent handling of relative rates of change, and naturally aligns with physical and economic systems that exhibit nonlinear growth patterns; see [7], [6], [2], [5]. Moreover, its application in geometry led to the emergence of new classes of curves and surfaces, such as multiplicative rectifying curves [3], multiplicative submanifolds [4], multiplicative tube surfaces [8], and even extensions into Lorentz–Minkowski spaces [13], [14].

Despite these developments, the analysis of special regular surface pairs within the multiplicative framework has not yet been studied. Inspired by classical formulations and fueled by the structure offered by multiplicative calculus, this paper aims to introduce and investigate new types of ruled surface pairs, such

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as multiplicative Mannheim, multiplicative Bertrand, and multiplicative involute-evolute surface couples, constructed under the principles of multiplicative differential geometry. Hence, the rest of this paper is structured as follows. A brief introduction to multiplicative differential geometry is given in Section 2. Section 3 outlines the construction of multiplicative ruled surface pairs. The results are then validated with some illustrated examples. Section 4 presents conclusions and potential advantages for further studies.

2. Preliminaries

In this section, some fundamental definitions and theorems about special ruled surface pairs and multiplicative space, respectively, are presented.

2.1. Differential geometry of ruled surfaces and some special ruled surface pairs

The set of points swept by a moving straight line is referred to as a ruled surface in differential geometry, see [9]. Hence, the parametric form of the ruled surface is expressed as follows:

$$\Phi(s,v)=\boldsymbol{r}(s)+v\boldsymbol{e}(s).$$

Any straight line $v\to\Phi(s_0,v)$ with fixed parameter $s=s_0$ is defined as a generator. Moreover, $\boldsymbol{r}(s)$ and $\boldsymbol{e}(s)$ are the base curve and the generator directions, respectively. If q is the arc length parameter of $\boldsymbol{e}(s)$, we have

$$q=\int_a^b ||\frac{d\boldsymbol{e}(s)}{ds}||ds.$$

The unit normal of Φ is computed by

$$\boldsymbol{N}(s,v)=\frac{\left(\frac{dr}{ds}+v\frac{d\boldsymbol{e}}{ds}\right)\times\boldsymbol{e}}{\left[\left(\frac{dr}{ds}+v\frac{d\boldsymbol{e}}{ds}\right)^2-\left(\boldsymbol{e}\frac{dr}{ds}\right)^2\right]^{1/2}}.$$

The unit normal of the ruled surface gradually gets closer to a limited direction as $\boldsymbol{t}$ goes down infinity. So, the direction is given as follows:

$$\boldsymbol{g}(s)|_{s=s_0}=\boldsymbol{N}(s,t)|_{s=s_0,t\to-\infty}=\left.\frac{\frac{d\boldsymbol{e}}{ds}\times\boldsymbol{e}}{\left\|\frac{d\boldsymbol{e}}{ds}\right\|}\right|_{s=s_0}.$$

The striction point, also known as the center point, is the point that is perpendicular to g. The central normal of the ruled surface, denoted by $\boldsymbol{t}$, is the direction of $\boldsymbol{N}$ at this location, as determined by

$$\boldsymbol{t}=\frac{\frac{d\boldsymbol{e}}{ds}}{\left\|\frac{d\boldsymbol{e}}{ds}\right\|}.$$

Here, $\{\boldsymbol{e},\boldsymbol{t},\boldsymbol{g}\}$ defines the Frenet trihedron, where $\boldsymbol{e}$,$\boldsymbol{t}=\boldsymbol{e}_q=\frac{\boldsymbol{e}_s}{||\boldsymbol{e}_s||}$, $\boldsymbol{g}=\boldsymbol{e}\times\boldsymbol{e}_q=\frac{\boldsymbol{e}\times\boldsymbol{e}_s}{||\boldsymbol{e}_s||}$ refers to as the asymptotic normal, central normal and spherical indicatrix, respectively. Therefore, by calculating the derivative of these equations, we acquire

$$\begin{aligned}\boldsymbol{e}_q&=\frac{d\boldsymbol{e}}{dq}=\boldsymbol{t},\\ \boldsymbol{t}_q&=\frac{d\boldsymbol{t}}{dq}=\gamma\boldsymbol{g}-\boldsymbol{e},\\ \boldsymbol{g}_q&=\frac{d\boldsymbol{g}}{dq}=\gamma\boldsymbol{t},\end{aligned}\tag{2.1}$$

where γ is geodesic curvature. Furthermore, the striction curve of Φ is

$$\boldsymbol{c}(s)=\boldsymbol{r}(s)-\frac{\langle\boldsymbol{r}_s,\boldsymbol{e}_s\rangle}{\langle\boldsymbol{e}_s,\boldsymbol{e}_s\rangle}\cdot\boldsymbol{e}_s.$$

Definition 2.1. In $\mathbb{R}^3$, let Φ and Φ_1 be two ruled surfaces. At the striction points of their respective rulings, Φ_1 is considered a Bertrand offset surface of Φ if there is a one-to-one correspondence between their rulings, meaning that both surfaces possess a common central normal, see [18].

Let $\Phi(s,v) = \boldsymbol{r}(s) + v\boldsymbol{e}(s)$ be the parametric form of the base ruled surface. The base ruled surface Φ indicates that its Bertrand offset surface surface is represented by

$$\begin{aligned}\tilde{\Phi}(s,v) &= \tilde{c}(s) + v\tilde{\boldsymbol{e}}(s)\\ &= [\boldsymbol{c}(s) + \mathrm{R}\,\boldsymbol{t}(s)] + v[(\cos\mu)\boldsymbol{e}(s) + (\sin\mu)\boldsymbol{g}(s)],\end{aligned}$$

where R and μ are linear and the angular offset surfaces between the two ruled surfaces. The following matrix representation describes the relationship between geodesic Frenet trihedrons of Φ and its Bertrand offset surface surface $\tilde{\Phi}$:

$$\begin{pmatrix}\tilde{\boldsymbol{e}}\\ \tilde{\boldsymbol{t}}\\ \tilde{\boldsymbol{g}}\end{pmatrix} = \begin{pmatrix}\cos\mu & 0 & \sin\mu\\ 0 & 1 & 0\\ \cos\mu & 0 & -\sin\mu\end{pmatrix}\begin{pmatrix}\boldsymbol{e}\\ \boldsymbol{t}\\ \boldsymbol{g}\end{pmatrix},$$

where $\{\tilde{\boldsymbol{e}}, \tilde{\boldsymbol{t}}, \tilde{\boldsymbol{g}}\}$ are the Frenet vectors of $\tilde{\Phi}$ and $\{\boldsymbol{e}, \boldsymbol{t}, \boldsymbol{g}\}$ are the Frenet vectors of Φ, see [18].

Definition 2.2. The two ruled surfaces are said to be Mannheim offset surface surface of each other if two ruled surfaces have a relation between their rulings, which means that the asymptotic normal of Φ is the central normal of $\hat{\Phi}$ at the striction points of their respective rulings, see [17].

Let $\Phi(s,v) = \boldsymbol{r}(s) + v\hat{\boldsymbol{e}}(s)$ be the base ruled surface. Then, its Mannheim offset surface surface is expressed by

$$\begin{aligned}\hat{\Phi}(s,v) &= \hat{c}(s) + v\hat{\boldsymbol{e}}(s)\\ &= [\boldsymbol{c}(s) + \mathrm{R}\,\boldsymbol{g}(s)] + v[(\cos\psi)\boldsymbol{e}(s) + (\sin\psi)\boldsymbol{t}(s)].\end{aligned}$$

The following matrix representation describes the relationship between geodesic Frenet trihedrons between Φ and its Mannheim offset surface $\hat{\Phi}$:

$$\begin{pmatrix}\hat{\boldsymbol{e}}\\ \hat{\boldsymbol{t}}\\ \hat{\boldsymbol{g}}\end{pmatrix} = \begin{pmatrix}\cos\mu & \sin\mu & 0\\ 0 & 0 & 1\\ \sin\mu & -\cos\mu & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{e}\\ \boldsymbol{t}\\ \boldsymbol{g}\end{pmatrix},$$

where $\{\hat{\boldsymbol{e}}, \hat{\boldsymbol{t}}, \hat{\boldsymbol{g}}\}$ are the Frenet vectors of $\hat{\Phi}$ and $\{\boldsymbol{e}, \boldsymbol{t}, \boldsymbol{g}\}$ are the Frenet vectors of Φ, see [17].

Definition 2.3. If there is a correspondence between the rulings of the two ruled surfaces, they are called involute-evolute offset surfaces of each other. This indicates that the central normal of Φ is the spherical indicatrix vector of $\bar{\Phi}$ at the striction points of each ruling; see [15].

Let $\Phi(s,v) = \boldsymbol{r}(s) + v\boldsymbol{e}(s)$ be the base ruled surface. According to the base surface Φ, its evolute offset surface is expressed by

$$\begin{aligned}\bar{\Phi}(s,v) &= \bar{c}(s) + v\bar{\boldsymbol{e}}(s)\\ &= [\boldsymbol{c}(s) + \mathrm{R}\,\boldsymbol{t}(s)] + v\boldsymbol{t}(s).\end{aligned}$$

The following matrix representation describes the relationship between geodesic Frenet trihedrons between Φ and its evolute offset surface $\bar{\Phi}$:

$$\begin{pmatrix}\bar{e}\\ \bar{t}\\ \bar{g}\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\ -\sin\mu & 0 & \cos\mu\\ \cos\mu & \sin\mu & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{e}\\ \boldsymbol{t}\\ \boldsymbol{g}\end{pmatrix},$$

where $\{\bar{e}, \bar{\boldsymbol{t}}, \bar{\boldsymbol{g}}\}$ are the Frenet vectors of $\bar{\Phi}$ and $\{\boldsymbol{e}, \boldsymbol{t}, \boldsymbol{g}\}$ are the Frenet vectors of Φ, see [15].

2.2. Multiplicative calculus

This section provides basic definitions and theorems concerning the multiplicative space formed using the exponential function of the generator $(\exp)$. The generating function α is selected as the $(\exp)$ function. Georgiev S. books are utilized to provide fundamental information; see [12]:

$$\begin{aligned}\alpha :\ \ &\mathbb{R} &&\to \mathbb{R}^+,\\ &a &&\to \alpha(m) = e^m\end{aligned}$$

and

$$\begin{aligned}\alpha^{-1}: \quad & \mathbb{R}^{+} && \to \mathbb{R},\\ & n && \to \alpha^{-1}(n)=\log n.\end{aligned}$$

It is possible to define a function from the real numbers to their positive side taking advantage of the generator function $(\exp)$. Consequently, the following is the definition of the set of real numbers in the multiplicative space:

$$\mathbb{R}_{*}=\{exp(m): m \in \mathbb{R}\}=\mathbb{R}^{+}.$$

Likewise, the following is a list of multiplicative numbers, both positive and negative.

$$\mathbb{R}_{*}^{+}=\{exp(m): m \in \mathbb{R}^{+}\}=(1, \infty)$$

and

$$\mathbb{R}_{*}^{-}=\{exp(m): m \in \mathbb{R}^{-}\}=(0,1).$$

The function $\exp$ is used in the following list to demonstrate the basic operations in multiplicative space. $\forall m, n \in \mathbb{R}_{*}, n \neq 1$,
$(i)\ m+_{*} n=e^{\log m+\log n}=mn$,
$(ii)\ m-_{*} n=e^{\log m-\log n}=\frac{m}{n}$,
$(iii)\ m ._{*} n=e^{\log m \log n}=m^{\log n}$,
$(iv)\ m /_{*} n=e^{\frac{\log m}{\log n}}=m^{\frac{1}{\log n}}$.
$(\mathbb{R}_{*},+_{*}, ._{*})$ form a multiplicative field. $m_{*} \in \mathbb{R}_{*}$, where $m_{*}=\exp(m)$, represents any multiplicative number that is a component of $\mathbb{R}_{*}$. Throughout the investigation, we will refer to as multiplicative numbers as $m \in \mathbb{R}_{*}$ rather than m_{*} in order to simplify notation. Additionally, $0_{*}=1$ and $1_{*}=e$ are the unit elements for multiplicative addition and multiplication, respectively.
Multiplication of absolute values defines multiplicative space. Absolute value is defined as additive since distance varies additively in Newtonian space. Given that distance is a multiplicative change, $|a|_{*}=a$ is its absolute value. (resp., $|m|_{*}=-_{*} m$) with $-_{*} m=1 / m$ for $k \in \mathbb{R}$. if $m \geq 0_{*}$ (resp., $m<0_{*}$,). In the multiplicative space we have

$$m^{k *}=m ._{*} m ._{*} \ldots ._{*} m=e^{(\log m)^{k}}$$

for $m \in \mathbb{R}_{*}$. Furthermore, we have

$$m^{\frac{1}{2} *}=e^{(\log m)^{\frac{1}{2}}}=* \sqrt{m}.$$

The vector definition in $p-$ dimensional multiplicative space $\mathbb{R}_{*}^{p}$ is denoted by

$$\mathbb{R}_{*}^{p}=\{(x_{1}, x_{2}, \ldots, x_{p}): x_{i} \in \mathbb{R}_{*}, i \in 1,2, \ldots, p\}.$$

Additionally, the vector space $\mathbb{R}_{*}^{p}$ on $\mathbb{R}_{*}$ has the pair of operations fulfilling

$$u+_{*} v=(u_{1}+_{*} v_{1}, u_{2}+_{*} v_{2}, \ldots, u_{p}+_{*} v_{p})=(u_{1} v_{1}, u_{2} v_{2}, \ldots, u_{p} v_{p})$$

and

$$\begin{aligned} m ._{*} u & =(m ._{*} u_{1}, m ._{*} u_{2}, \ldots, m ._{*} u_{p}) \\ & =(u_{1}^{\log m}, u_{2}^{\log m}, \ldots, u_{p}^{\log m}) \\ & =e^{\log m \log u},\end{aligned}$$

where $u, v \in \mathbb{R}_{*}^{p}$. In the multiplicative vector space $\mathbb{R}_{*}^{p}$, assume that u and v are two multiplicative vectors. The multiplicative inner product is

$$\langle u, v\rangle_{*}=e^{\langle\log u, \log v\rangle}.$$

The multiplicative norm of u is

$$||u||_{*}=e^{\langle\log u, \log u\rangle^{\frac{1}{2}}}.$$

The multiplicative cross product has conventional algebraic and geometric properties. In the multiplicative vector space, let u and v represent two unit-direction multiplicative vectors. Let μ represent the multiplicative angle between unit vectors, as shown below:

$$\mu=\arccos_{*}(e^{\langle\log u, \log v\rangle}).$$

For $\mu \in \mathbb{R}_*$, the definitions multiplicative trigonometric functions are given as follows:

$$\begin{aligned} \sin_* \mu &= e^{\sin \log \mu}, \cos_* \mu = e^{\cos \log \mu}, \\ \tan_* \mu &= e^{\tan \log \mu}, \cot_* \mu = e^{\cot \log \mu}. \end{aligned}$$

The function f is assumed to be provided in the multiplicative space $\mathbb{R}_*$, where $s \in I \subset \mathbb{R}_*$. The multiplicative derivative of f is represented by

$$\begin{aligned} f^*(s) &= \lim_{h \to 0_*} (f(s +_* h) -_* f(s)) / * h \\ &= \lim_{h \to 1} \left(\frac{f(sh)}{f(s)}\right)^{\frac{1}{\log h}} \\ &= \lim_{h \to 1} e^{\frac{\log \frac{f(sh)}{f(s)}}{\log h}}. \end{aligned}$$

If the L'Hospital rule is applicable in this case, we get

$$f^*(s) = e^{\frac{sf'(s)}{f(s)}}.$$

Furthermore, the function f is referred to as a $*$ (multiplicative) differentiable function if it is continuous and differentiable in the multiplicative sense. Additionally, it complies with the chain rule and Leibniz's multiplicative derivative, see [12].
The inverse operator of the multiplicative derivative is used to define a multiplicative integral. The definition of the multiplicative indefinite integral of $f(s)$ is

$$\int_* f(s) ._* d_* s = e^{\int \frac{1}{s} \log f(s) ds}, \quad s \in \mathbb{R}_*.$$

Now, the definition of the multiplicative ruled surface to be used in the rest of the article is given as follows:

Definition 2.4. The multiplicative surface ϕ is said to be multiplicative ruled surface if its multiplicative local parameterization has the form

$$\phi(s, v) = r(s) +_* v ._* e(s), \quad ||e(s)||_* = 1_*, \tag{2.2}$$

where $r(s)$ and$e(s)$ denote the multiplicative base curve and multiplicative ruling, respectively in [11].

3. Main Results

In this section, taking the tools in multiplicative calculus into account, some ruled surface pairs are examined.

3.1. Multiplicative Bertrand ruled surface pairs

In this section, multiplicative Bertrand ruled surface pairs are defined, and then some important theorems are proved.

Definition 3.1. Assume that Φ and $\tilde{\Phi}$ are two multiplicative ruled surfaces. Then, $\tilde{\Phi}$ refers to as a multiplicative Bertrand offset surface (mul-Bertrand offset surface) of Φ if there is a one-to-one correspondence between their rulings, meaning that both surfaces have a provided central normal at the striction points of their respective rulings.

Theorem 3.1. *Let Φ and $\tilde{\Phi}$ be mul-Bertrand surface pairs. The distance and angle between these two multiplicative ruled surfaces are constant.*

Proof. From the definition of mul-Bertrand pairs, we have

$$\boldsymbol{t}(s) = \tilde{\boldsymbol{t}}(s),$$

where $s \in \mathbb{R}_*$. On the other hand, we have

$$\tilde{\boldsymbol{t}}(s) = \tilde{\boldsymbol{e}}^*(s) /_* ||\tilde{\boldsymbol{e}}^*(s)||_* \Rightarrow \tilde{\boldsymbol{e}}^*(s) = \lambda ._* \boldsymbol{t}(s),$$

where λ is a multiplicative scalar. Using the multiplicative chain rule and the multiplicative Frenet formulas, we write

$$\begin{aligned}&(\cos_* \mu ._* \boldsymbol{e}(s) +_* \sin_* \mu_* \boldsymbol{g}(s))^* = \lambda ._* \boldsymbol{t}(s)\\ &e^{-\mu \tan \mu} ._* \mu^* ._* \boldsymbol{e}(s) +_* \cos_* \mu ._* \boldsymbol{e}^*(s) +_* e^{\mu \cot \mu} ._* \mu^* ._* \boldsymbol{g}(s) +_* \sin_* \mu(s) ._* \boldsymbol{g}^*(s) = \lambda ._* \boldsymbol{t}(s),\end{aligned} \tag{3.1}$$

where

$$\boldsymbol{e}^*(s) = (d_* \boldsymbol{e} /_* d_* s) ._* (d_* s /_* d_* q) = \boldsymbol{t}(s) \tag{3.2}$$

and

$$\boldsymbol{g}^*(s) = (d_* \boldsymbol{g} /_* d_* s) ._* (d_* s /_* d_* q) = -_* \gamma ._* \boldsymbol{t}(s). \tag{3.3}$$

If we write Eq. (3.2) and Eq. (3.3) into Eq. (3.1) and exploit the multiplicative chain rule, we acquire

$$\mu^* ._* \left(e^{-\mu \tan \mu} ._* \boldsymbol{e}(s) +_* e^{\mu \cot \mu} ._* \boldsymbol{g}(s)\right) = 0_*.$$

Hence, we conclude that μ is constant. Since the directrix of the multiplicative ruled surface $\tilde{\Phi}$ is the multiplicative striction curve of the surface, we write

$$\tilde{c}(s) ._* \tilde{\boldsymbol{e}}(s) = 0_*,$$

where the multiplicative striction curve is the multiplicative curve is defined as the shortest distance with the help of a common perpendicular line between two multiplicative adjacent rulings in [11]. Since $\tilde{\boldsymbol{e}}^*(s) = \lambda ._* \tilde{\boldsymbol{t}}(s)$, the following equality is satisfied:

$$\lambda ._* \tilde{c}^*(s) ._* \tilde{\boldsymbol{t}}(s) = 0_*$$

and

$$\lambda ._* [c^*(s) +_* (R ._* \boldsymbol{t}(s))^* ._* \boldsymbol{t}(s)] = 0_*.$$

Therefore, we have

$$\lambda ._* [c^*(s) +_* (R^* ._* \boldsymbol{t}(s) +_* R ._* \boldsymbol{t}^*(s)) ._* \boldsymbol{t}(s)] = 0_*.$$

Applying the multiplicative chain rule, the equation yields $R^* = 0_*$. Thus, the proof is completed. □

Example 3.1. Assume that

$$\Phi(s, v) = (v ._* \sin_* s, s, v ._* \cos_* s)$$

is the base multiplicative ruled surface, where $s, v \in \mathbb{R}_*$. Thus, mul- Bertrand offset surface of Φ is

$$\tilde{\Phi}(s, v) = \left(\cos_* s +_* e /_* e^2 ._* v ._* \sin_* s, s +_* e^{\sqrt{\log 3}} /_* 2, -_* \sin_* s +_* e /_* e^2 ._* v ._* \cos_* s\right),$$

where $R = 1_*$ and $\mu = \pi /_* 3$.

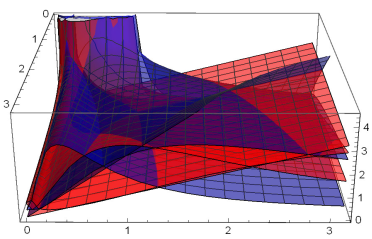

Figure 1. *The multiplicative base ruled surface Φ (Red) and its mul-Bertrand offset surface $\tilde{\Phi}$ (Blue), respectively*

3.2. Multiplicative Mannheim ruled surface pairs

In this section, multiplicative Mannheim ruled surface pairs are defined, and then some theorems are proved.

Definition 3.2. Assume that Φ and $\hat{\Phi}$ are two multiplicative ruled surfaces. Then, two ruled surfaces are said to be multiplicative Mannheim offset surfaces (mul-Mannheim offset surfaces) of one another if there is a correspondence between their rulings and the asymptotic normal of Φ is the central normal of $\hat{\Phi}$ at the striction points of their respective rulings.

Remark 3.1. Assume $\hat{\Phi}$ is the mul-Mannheim offset surface for Φ. Then R is constant if and only if Φ is multiplicative developable.

Theorem 3.2. *Let Φ and $\hat{\Phi}$ be mul-Mannheim ruled surface pairs. In order for the multiplicative ruled surface $\hat{\Phi}$ to be developable, the following equality is satisfied:*

$$\sin_* \theta +_* R \cdot_* \gamma \cdot_* (d_* s /_* d_* q) \cdot_* \cos_* \theta = 0_*, \tag{3.4}$$

where R is constant.

Proof. Assume that $\hat{\Phi}$ is developable. Then, we have

$$\hat{c}^* = \lambda \cdot_* \hat{\boldsymbol{e}}(s), \tag{3.5}$$

where λ is a multiplicative scalar. From Remark 3.1 and differentiating Eq. (3.5), we obtain

$$\begin{aligned}(d_*\hat{c}/_* d_* s) +_* R \cdot_* (d_* q/_* d_* s) \cdot_* (d_* \boldsymbol{g}/_* d_* q) +_* (d_* R/_* d_* s) \cdot_* \boldsymbol{g} &= \lambda \cdot_* [\cos_* \theta \cdot_* \boldsymbol{e} +_* \sin_* \theta \cdot_* \boldsymbol{t}] \cdot_* \boldsymbol{e} +_* R \cdot_* (d_* q/_* d_* s) \cdot_* (-_* \gamma \cdot_* \boldsymbol{t}) \\ &= \lambda \cdot_* \cos_* \theta \cdot_* \boldsymbol{e} +_* \lambda \cdot_* sin_* \theta \cdot_* \boldsymbol{t}\end{aligned}$$

Since the multiplicative inner product of $\boldsymbol{t}$ and $\boldsymbol{e}$ equals to 0_*, we obtain the following equation:

$$\sin_* \theta +_* R \cdot_* \gamma \cdot_* (d_* s /_* d_* q) \cdot_* \cos_* \theta = 0_*. \tag{3.6}$$

Conversely, suppose that Eq. (3.4) holds. The following equation can be written for the tangent of the multiplicative striction curve of $\hat{\Phi}$.

$$\begin{aligned} d_* \hat{c} /_* d_* s &= d_* (c +_* R \cdot_* \boldsymbol{g}) /_* d_* s \\ &= \boldsymbol{e} -_* R \cdot_* \gamma \cdot_* (d_* q /_* d_* s) \cdot_* \boldsymbol{t} \\ &= 1 /_* \cos_* \theta \cdot_* [(\cos_* \theta) \cdot_* \boldsymbol{e} +_* (\sin_* \theta) \cdot_* \boldsymbol{t}] \\ &= 1 /_* \cos_* \theta \cdot_* \hat{\boldsymbol{e}}. \end{aligned}$$

So, we conclude that $\hat{\Phi}$ is a multiplicative developable ruled surface. □

Theorem 3.3. *Assume that Φ is a multiplicative developable ruled surface. In order for the multiplicative ruled surfaces Φ and $\hat{\Phi}$ to be a mul-Mannheim ruled surface offset surfaces, the following equality is provided:*

$$d_* \gamma /_* d_* s = (1/_* R) \cdot_* (e +_* R^{2*} \cdot_* \gamma^{2*} \cdot_* (d_* q /_* d_* s)^{2*}) -_* (1/_* (d_* q/_* d_* s)) \cdot_* \gamma \cdot_* (d_*^2 q /_* d_* s^2). \tag{3.7}$$

Proof. Assume that Φ and $\hat{\Phi}$ are mul-Mannheim ruled surface pairs. Thus, we have

$$R \cdot_* \gamma \cdot_* (d_* q /_* d_* s) = -_* \tan_* \theta. \tag{3.8}$$

From the multiplicative chain rule, we write

$$(d_* \hat{\boldsymbol{e}} /_* d_* s) = -_* \sin_* \theta ((d_* \theta /_* d_* s) +_* (d_* q /_* d_* s)) \cdot_* \boldsymbol{e} +_* \cos_* ((d_* \theta /_* d_* s) +_* (d_* q /_* d_* s)) \cdot_* \boldsymbol{t} +_* \gamma \cdot_* (d_* q /_* d_* s) \sin_* \theta \cdot_* \boldsymbol{g}.$$

By definion of $\hat{\boldsymbol{t}}$, we have

$$(d_* \theta /_* d_* s) = -_* (d_* q /_* d_* s). \tag{3.9}$$

Taking the multiplicative derivative with respect to the parameter s and using the multiplicative chain rule, we obtain

$$(d_* \gamma /_* d_* s) = (1/_* R) \cdot_* \left(1_* +_* R^{2_*} \cdot_* \gamma^{2_*} \cdot_* (d_* q /_* d_* s)^{2_*}\right) -_* (1/_* (d_* q /_* d_* s)) \cdot_* \gamma \cdot_* d_*^2 q /_* (d_*^2 s).$$

Conversely, assume that Eq. (3.7) is satisfied. For a non-zero constant scalar R, the multiplicative ruled surface is expressed as follows:

$$\hat{\Phi} = \hat{c}(s) +_* t ._* \hat{\boldsymbol{e}}(s), \tag{3.10}$$

where $\hat{c}(s) = c(s) +_* R ._* \boldsymbol{g}(s)$.
Now, we prove that $\hat{\Phi}$ is a mul-Mannheim offset surface of Φ. Since $\hat{\Phi}$ is multiplicative developable ruled surface, we have

$$(d_*\hat{c}/_*d_*s) = (d_*s_{1_*}/_*d_*s)._*\hat{\boldsymbol{e}}. \tag{3.11}$$

where s and s_1 are the arc-length parameters of the multiplicative striction curves c and $\hat{c}$, respectively. From $\hat{c}(s) = c(s) +_* R ._* \boldsymbol{g}(s)$ and Eq.(3.11), we get

$$(d_*\hat{c}/_*d_*s)._*\hat{\boldsymbol{e}} = \boldsymbol{e} - R._*\gamma._*(d_*q/_*d_*s)._*\boldsymbol{t} \tag{3.12}$$

If we take the multiplicative derivative with respect to the arc length parameter s in the above equation, we calculate

$$\begin{aligned}
(d_*^2s_1/_*d_*s^2)._*\hat{\boldsymbol{e}} +_* (d_*s_1/_*d_*s)._*(d_*\hat{\boldsymbol{e}}/_*d_*s) \quad &= \quad R._*\gamma._*(d_*^2q/_*d_*^2s)._*\boldsymbol{e} \\
&+_* \quad ((d_*q/_*d_*s) -_* R._*(d_*\gamma/_*d_*s)._*(d_*q/_*d_*s) -_* R._*\gamma._*(d_*^2q/_*d_*^2s))._*\boldsymbol{t} \\
&-_* \quad R._*\gamma^{2*}._*(d_*^2q/_*d_*^2s)._*\boldsymbol{g}.
\end{aligned}$$

From the definition of $\hat{\boldsymbol{t}}$, we write

$$(d_*^2s_1/_*d_*s^2)._*\hat{\boldsymbol{e}} +_* (d_*s_1/_*d_*s)._*\lambda._*\hat{\boldsymbol{t}} = R._*\gamma._*(d_*^2q/_*d_*s^2)._*\boldsymbol{e} -_* R^{2*}._*\gamma^{2*}._*(d_*^3q/_*d_*s^3)._*\boldsymbol{t} -_* R._*\gamma^{2*}._*(d_*^2q/_*d_*s^2)._*\boldsymbol{g}, \tag{3.13}$$

where λ is a scalar. Taking the cross product of Eq. (3.12) with the above equation, we have

$$(d_*s_1/_*d_*s)^{2_*}._*\lambda._*\hat{\boldsymbol{g}} = R^{2_*}._*\gamma^{3_*}._*(d_*q/_*d_*s)^{3_*}._*\boldsymbol{e} +_* R._*\gamma^{2_*}._*(d_*q/_*d_*s)^{2_*}._*\boldsymbol{t}. \tag{3.14}$$

Then, we acquire

$$(d_*s_1/_*d_*s)^{3_*}._*\lambda._*\hat{\boldsymbol{t}} = -_*(R._*\gamma^{2_*}._*(d_*q/_*d_*s)^{2_*} +_* R^{3_*}._*\gamma^{4_*}._*(d_*q/_*d_*s)^{4_*})\boldsymbol{g}. \tag{3.15}$$

Therefore, the multiplicative developable ruled surface $\hat{\Phi}$ is a Mannheim offset surface of the multiplicative ruled surface Φ. □

Example 3.2. Let the multiplicative ruled surface

$$\Phi(s,v) = \left(\cos_* s -_* e^{\sqrt{\log 2}}/_*2._*v._*\sin_* s, \sin_* s +_* e^{\sqrt{\log 2}}/_*2._*v._*\cos_* s, e^{\sqrt{\log 2}}/_*2._*v\right) \tag{3.16}$$

be the base surface, where $s, v \in \mathbb{R}_*$. Then, mul- Mannheim ruled surface offset surface of Φ is given as follows:

$$\begin{aligned}
\hat{\Phi}(s,v) \quad = \quad & (\cos_* s -_* e^{\sqrt{\log 2}}/_*2._*\sin_* s -_* (1_* +_* e^{\sqrt{\log 2}}/_*2)._*v._*\cos_* s._*\sin_* s, \\
& \sin_* s -_* e^{\sqrt{\log 2}}/_*2._*\cos_* s +_* e^{\sqrt{\log 2}}/_*2._*v._*\cos_*^{2*} s -_* \sin_*^{2*} s, \\
& e^{\sqrt{\log 2}}/_*2._*s +_* e^{\sqrt{\log 2}}/_*2._*v._*\cos_* s),
\end{aligned}$$

where $R = 1_*$.

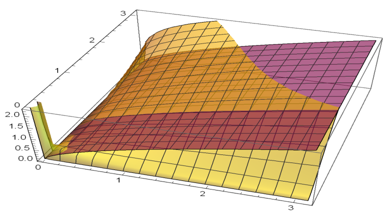

Figure 2. *The multiplicative base ruled surface Φ (Yellow) and its mul-Mannheim offset surface surface $\hat{\Phi}$ (Pink), respectively*

3.3. Multiplicative Involute-evolute ruled surface pairs

In this section, multiplicative involute-evolute ruled surface pairs are defined, and then some theorems are proved.

Definition 3.3. Assume that Φ and $\bar{\Phi}$ are two multiplicative ruled surfaces. $\bar{\Phi}$ refers to as a multiplicative evolute offset surface surface (mul-evolute offset surface surface) of Φ (or Φ refers to as a multiplicative involute offset surface surface (mul-involute offset surface surface) of $\bar{\Phi}$) if there is a one-to-one correspondence between their rulings such that the central normal of Φ and the spherical indicatrix vector of $\bar{\Phi}$ are linearly dependent at the striction points of their corresponding rulings.

Theorem 3.4. *Assume that $\bar{\Phi}$ is the mul-evolute offset surface surface of Φ. θ is also constant iff γ is constant. In contrast if $\theta \neq 0$.*

Proof. By definition of $\bar{\boldsymbol{t}}$, we have

$$\bar{\boldsymbol{t}} = t_q^* /_* ||t_q||_*. \tag{3.17}$$

Using the multiplicative Frenet formulas, we compute as follows:

$$\bar{\boldsymbol{t}} = \left(\gamma /_* e^{\sqrt{1_* +_* \gamma^{2*}}}\right) \cdot_* \boldsymbol{g} -_* \left(1_* /_* e^{\sqrt{1_* +_* \gamma^{2*}}}\right) \cdot_* \boldsymbol{e}. \tag{3.18}$$

Hence, we write

$$\sin_* \theta = \left(1_* /_* e^{\sqrt{1_* +_* \gamma^{2*}}}\right) \quad \text{and} \quad \cos_* \theta = \left(\gamma /_* e^{\sqrt{1_* +_* \gamma^{2*}}}\right). \tag{3.19}$$

Consequently, if γ is constant, then θ is constant, and if θ is constant for $\theta \neq 0$, then γ is constant. □

Let $\bar{\Phi}$ be the mul-evolute offset surface surface of Φ. The distribution parameter of the multiplicative ruled surface $\bar{\Phi}$ is defined as follows:

$$P_{\bar{\boldsymbol{e}}} = \det\left(\bar{c}^*(s), \bar{\boldsymbol{e}}(s), \bar{\boldsymbol{e}}^*(s)\right) /_* ||\bar{\boldsymbol{e}}(s)||_*^{2_*}. \tag{3.20}$$

Then, we calculate

$$P_{\bar{\boldsymbol{e}}} = \left(1_* /_* (1_* +_* \gamma^{2*})\right) \cdot_* \left[P_{\boldsymbol{e}} +_* \gamma \cdot_* < e, c^*(s) >_*\right]. \tag{3.21}$$

Theorem 3.5. *Assume that $\bar{\Phi}$ is a mul-evolute offset surface surface of a developable multiplicative ruled surface Φ. $\bar{\Phi}$ is multiplicative developable if the multiplicative spherical indicatrix curve $e(s)$ of Φ is a multiplicative geodesic curve.*

Assume that $\bar{\Phi}$ is a mul-evolute offset surface surface of Φ. If Φ is a closed surface, then $\Phi(s + P, v)$ exists for some positive integer P.

Theorem 3.6. *Let $\bar{\Phi}$ be the mul-evolute offset surface surface of the multiplicative closed ruled surface Φ with period P. $\bar{\Phi}$ refers to as a multiplicative ruled surface if and only if $R = R(s)$ denotes a function with period P.*

Remark 3.2. Let $\bar{\Phi}$ be a mul-evolute offset surface surface of Φ. Assume that $\Phi_{\boldsymbol{t}}$ and $\bar{\Phi}_{\boldsymbol{t}}$ are the multiplicative ruled surfaces generated by the striction points and central normals of Φ and $\bar{\Phi}$, respectively. Then $\bar{\Phi}_{\boldsymbol{t}}$ is the mul-evolute offset surface surface of $\Phi_{\boldsymbol{t}}$.
Furthermore, the multiplicative ruled surfaces obtained by choosing asymptotic normals instead of central normals for $\Phi_{\boldsymbol{g}}$ and $\bar{\Phi}_{\boldsymbol{g}}$ cannot be the mul-evolute offset surface surface of $\Phi_{\boldsymbol{g}}$.

Example 3.3. Let

$$\begin{aligned}
\Phi(s, v) \quad = \quad & (\cos_* s \cdot_* e^{\sqrt{\log 2}} \cdot_* s -_* e^{\sqrt{\log 2}} /_* 2 \cdot_* v \cdot_* \sin_* s \cdot_* e^{\sqrt{\log 2}} /_* 2 \cdot_* s, \\
& \sin_* s \cdot_* e^{\sqrt{\log 2}} /_* 2 \cdot_* s +_* e^{\sqrt{\log 2}} /_* 2 \cdot_* v \cdot_* \cos_* s \cdot_* e^{\sqrt{\log 2}} /_* 2 \cdot_* s \\
& e^{\sqrt{\log 2}} /_* 2 \cdot_* s +_* e^{\sqrt{\log 2}} /_* 2 \cdot_* v)
\end{aligned}$$

be the multiplicative base ruled surface, where $s, v \in \mathbb{R}_*$. Then, mul- evolute offset surface surface of Φ is given as follows:

$$\bar{\Phi}(s, v) = \left(-_* \cos_* s \cdot_* e^{\sqrt{\log 2}} /_* 2 \cdot_* s, -_* \sin_* s \cdot_* e^{\sqrt{\log 2}} /_* 2 \cdot_* s, 0_*\right), \tag{3.22}$$

where $R = 1_*$.

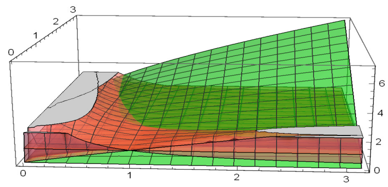

Figure 3. *The multiplicative base ruled surface Φ (Red) and its mul-evolute offset surface surface $\tilde{\Phi}$ (Green), respectively*

4. Conclusion

Multiplicative calculus, developed as an alternative to classical analysis, has recently attracted considerable attention among researchers. It has a wide range of applications in various fields such as science, engineering, and mathematics. Moreover, multiplicative calculus offers advantages over classical calculus in differential geometry by simplifying expressions involving growth rates and ratios. It provides a natural framework for analyzing geometric objects with exponential or scale-invariant properties, leading to more intuitive formulations in certain curvature and transformation problems.
The study of special pairs of ruled surfaces—such as Bertrand, Mannheim, and Involute-Evolute partner surfaces—plays a significant role in the differential geometry of surfaces due to their intrinsic and extrinsic geometric relations. These surface pairs are characterized by specific correspondences between their rulings and directrices, leading to structured relationships between their Gaussian and mean curvatures. In particular, the interplay between the curvatures of these ruled surface pairs often yields insightful geometric invariants and facilitates a deeper understanding of the underlying geometry. Moreover, the condition for a ruled surface to be developable, i.e., to have zero Gaussian curvature, bears special significance, as it implies the surface can be unfolded onto a plane without distortion. This property has important theoretical implications in geometry and topology, as well as practical applications in fields such as architectural design and manufacturing. Investigating these special ruled surface pairs under the lens of curvature theory and developability thus provides a fertile ground for both theoretical exploration and applied geometric modeling.
Taking these geometric advantages into consideration, this study seeks to address the following central question: Can special ruled surface pairs be redefined within the framework of multiplicative calculus? Therefore, in this context, the specific ruled surface pairs to be examined are referred to as the mul-Bertrand, mul-Mannheim, and mul-involute–evolute ruled surface pairs, respectively. Therefore, we can give the diagram below to understand the relationship better:

Mul-Bertrand offset ruled surface	$t \,//\, \tilde{t}$
Mul-Mannheim offset ruled surface	$g \,//\, \hat{e}$
Mul-involute-evolute offset ruled surface	$t \,//\, \bar{e}$

Furthermore, the study evaluates how these newly constructed special ruled surface pairs are influenced by the multiplicative differential structure, and their geometric behavior is analyzed in detail.

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