Konuralp Journal of Mathematics, 13 (2) (2025) 241-249

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: *www.dergipark.gov.tr/konuralpjournalmath*

e-ISSN: 2147-625X

Left-Definite Theory for Fractal Sturm–Liouville Equations

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Abstract

In this paper, the left-definite theory of fractal Sturm–Liouville problems in the regular case is studied.

Keywords: *Fractal Sturm–Liouville problems, left-definiteness.*

2010 Mathematics Subject Classification: *34A08, 34B24, 34B37*

1. Introduction

Sturm–Liouville problems are one of the most researched topics in the theory of differential equations. The importance of these problems has increased when we try to solve partial differential equations with the Fourier method. They are still being intensively researched (see [2, 3, 4, 5, 6, 16, 22, 23]).

Problems on Hilbert space generated by the coefficients on the right side of a homogeneous differential equation are called right-definite problems. There is a very dense literature on such problems. Problems on Hilbert space generated by the coefficients on the left side of a homogeneous differential equation are called left-definite problems. The left-definite theory for Sturm–Liouville problems has been studied by many authors. Pleijel introduced this topic in [26, 27]. Later, in [8], a second-order linear differential equation is examined by Everitt. In [20], the author studied the left-definite theory for second-order differential operators with mixed boundary conditions. In [7], Bennewitz and Everitt are investigated the eigenvalue problems associated with second-order linear differential equations.

On the other hand, two significant extensions of the fundamentals of ordinary calculus are fractional calculus and fractal calculus. Fractal calculus is often confused with fractional calculus. These two calculi differ mostly in the following ways: The fractal derivative is local and shares numerous characteristics with the classical derivative, although the fractional derivative is not. Local fractional derivatives were first developed by Kolwankar and Gangal [17, 18]. Later, Parvate and Gangal define the F^{α}-calculus on fractal subsets of real numbers [24]. The relationship between the F^{α}-calculus and classical calculus is examined. Recently, a considerable number of researchers have engaged in work on this subject. (see [1, 9, 10, 11, 12, 24, 25, 13, 14, 28]). Çetinkaya and Golmankhaneh [9] investigated the basic properties of the fractal Sturm–Liouville problems. In the paper [1], Allahverdiev and Tuna obtained the existence and uniqueness of solutions to the fractal Sturm–Liouville problem.

In this paper, we study regular left-definite fractal Sturm–Liouville problems

$$l(u) := \frac{1}{r}\left[-(D_F^{\alpha})^2 u + v(\zeta)u\right] = \lambda u,\ \zeta \in [0,1], \tag{1.1}$$

$$AU(0) + BU(1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{1.2}$$

where

$$U = \begin{pmatrix} u \\ D_F^{\alpha} u \end{pmatrix},\ A = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix},$$

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and

$$B=\begin{pmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{pmatrix},\ rank\begin{pmatrix} \tau_{11} & \tau_{12} & \chi_{11} & \chi_{12} \\ \tau_{21} & \tau_{22} & \chi_{21} & \chi_{22} \end{pmatrix}=2.$$

v and $r>0$ are real-valued continuous functions and $v>\varepsilon r$ for some $\varepsilon>0$. To the authors' knowledge, there is no study in the literature on left-definite problems for the fractal Sturm–Liouville equations.

2. Preliminaries

The objective of this section is to present the fundamental concepts of fractal calculus (for further information, please refer to sources [24, 25, 19, 20]). The term "F" will be used throughout this study to denote a fractal subset of the real numbers.

Definition 2.1 ([24]). *Let*

$$P_J=\{a=\zeta_0<\zeta_1<\zeta_2<...<\zeta_n=b\},$$

where $J=[a,b]\subset\mathbb{R}$. *Then* $\sigma^{\alpha}[F,P_J]$ *is defined as*

$$\sigma^{\alpha}[F,P_J]=\sum_{i=0}^{n-1}\Gamma(\alpha+1)(\zeta_{i+1}-\zeta_i)^{\alpha}\,\theta(F,[\zeta_i,\zeta_{i+1}]),$$

where $\Gamma(.)$ *denotes the Gamma function,* $\alpha\in(0,1]$ *and*

$$\theta(F,J)=\begin{cases}1, & \text{if } F\cap J\neq\emptyset,\\ 0, & \text{otherwise.}\end{cases}$$

If $a=b$, $\sigma^{\alpha}[F,P_J]=0$.

Definition 2.2 ([24]). *The coarse-grained measure* $\gamma^{\alpha}_{\delta}(F,a,b)$ *is given by*

$$\gamma^{\alpha}_{\delta}(F,a,b)=\inf_{\{P_J:|P_J|\leq\delta\}}\sigma^{\alpha}[F,P_J],$$

where $\delta>0$, $a\leq 1$ *and*

$$|P_J|=\max_{0\leq i\leq n-1}(\zeta_{i+1}-\zeta_i)$$

for a partition P_J.

Definition 2.3 ([24]). *The measure function is given by*

$$\gamma^{\alpha}(F,a,b)=\lim_{\delta\to 0}\gamma^{\alpha}_{\delta}(F,a,b).$$

Definition 2.4 ([24]). *The fractal* γ*-dimension of* $F\cap J$ *is defined by*

$$\begin{aligned}\dim_{\gamma}(F\cap J)&=\inf\{\alpha:\gamma^{\alpha}(F,a,b)=0\}\\ &=\sup\{\alpha:\gamma^{\alpha}(F,a,b)=\infty\}.\end{aligned}$$

Definition 2.5 ([24]). *For a function* $\Psi:F\to\mathbb{R}$, *the* F*-limit of* Ψ *at a point* $\zeta\in F$ *is the number* A *such that for every* $\varepsilon>0$, *there exists a positive number* δ *satisfying*

$$y\in F \text{ and } |y-\zeta|<\delta\Longrightarrow|\Psi(y)-A|<\varepsilon.$$

If such an A *exists, it is denoted by*

$$A=F\text{-}\lim_{y\to\zeta}\Psi(y).$$

Definition 2.6 ([24]). *A function* Ψ *is the* F*-continuous at* $\zeta\in F$ *if*

$$F\text{-}\lim_{y\to\zeta}\Psi(y)=\Psi(\zeta)$$

holds.

Definition 2.7 ([24]). *The integral staircase function is defined by*

$$S^{\alpha}_{F}(\zeta)=\begin{cases}\gamma^{\alpha}(F,a_0,\zeta), & \text{if } \zeta\geq a_0,\\ -\gamma^{\alpha}(F,\zeta,a_0) & \text{otherwise,}\end{cases}$$

where $a_0\in\mathbb{R}$ *is fixed constant.*

Definition 2.8 ([24]). *For a function F on an α-perfect set. Then the F^{α}-derivative of Ψ is defined as*

$$D_F^{\alpha}\Psi(\zeta)=\begin{cases} F\text{-}\lim \frac{\Psi(y)-\Psi(\zeta)}{S_F^{\alpha}(y)-S_F^{\alpha}(\zeta)}, & \text{if } \zeta\in F,\\ 0, & \text{otherwise,}\end{cases}$$

provided that the F-lim exists.

Theorem 2.9 ([24, 20]). *Let $\Psi,\Phi:\mathbb{R}\to\mathbb{R}$ be F^{α}-differentiable functions. Then we have*

i) $D_F^{\alpha}(\Psi\Phi)(\zeta)=\Phi(\zeta)D_F^{\alpha}\Psi(\zeta)+\Psi(\zeta)D_F^{\alpha}\Phi(\zeta)$,

ii) $D_F^{\alpha}(a\Psi+b\Phi)(\zeta)=aD_F^{\alpha}\Psi(\zeta)+bD_F^{\alpha}\Phi(\zeta)$,

where $a,b\in\mathbb{R}$.

Theorem 2.10 ([24]). *Let Ψ is a bounded function on $F\cap J$. The F^{α}-integral of Ψ is defined by*

$$\int_a^b \Psi(\zeta)\,d_F^{\alpha}\zeta=\sup_{P_J}\sum_{i=0}^{n-1}\inf_{\zeta\in F\cap J}\Psi(\zeta)\left(S_F^{\alpha}(\zeta_{i+1})-S_F^{\alpha}(\zeta_i)\right)$$

$$=\inf_{P_J}\sum_{i=0}^{n-1}\sup_{\zeta\in F\cap J}\Psi(\zeta)\left(S_F^{\alpha}(\zeta_{i+1})-S_F^{\alpha}(\zeta_i)\right)$$

where $\zeta\in J$, and the infimum or supremum is taken over all partitions P_J.

Let

$$L_2^{\alpha}[0,1]=\left\{\Xi:\int_0^1|\Xi(\zeta)|^2\,d_F^{\alpha}\zeta<\infty\right\}.$$

Then the fractal Hilbert space $L_{2,r}^{\alpha}[0,1]$ has the following inner product

$$(\Xi,\Sigma)=\int_0^1 \Xi\overline{\Sigma}r\,d_F^{\alpha}\zeta.$$

3. The Hilbert space theory

Let

$$D_{\max}=\left\{u\in L_{2,r}^{\alpha}[0,1]:\begin{array}{c} D_F^{\alpha}u\in H,\ D_F^{2\alpha}u\in L_{2,r}^{\alpha}[0,1],\\ \\ \text{and } l(u)\in L_{2,r}^{\alpha}[0,1]\end{array}\right\}.$$

$$D_{\min}=\{u\in D_{\max}:u(0)=(D_F^{\alpha}u)(0)=u(1)=(D_F^{\alpha}u)(1)=0\}.$$

Then the *maximal* operator $\mathscr{L}_{\max}$ on $D_{\max}$ is given by

$$\mathscr{L}_{\max}u=l(u).$$

The minimal operator $\mathscr{L}_{\min}$ is obtained by restricting the operator $\mathscr{L}_{\max}$ to the set $D_{\min}$.
Eq. (1.1) gives Green's formula as

$$\int_0^1 l(u)\overline{z}r\,d_F^{\alpha}\zeta-\int_0^1 u\overline{l(z)}r\,d_F^{\alpha}\zeta=[u,z]_1-[u,z]_0, \tag{3.1}$$

where

$$[u,z]:=u(\overline{D_F^{\alpha}z})-(D_F^{\alpha}u)\overline{z}$$

$$=\left(z(\zeta)\,D_F^{\alpha}u(\zeta)\right)\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}u(\zeta)\\ D_F^{\alpha}u(\zeta)\end{pmatrix}\ (\zeta\in(0,1)).$$

For $u,z\in D_{\max}$, we see that

$$\int_0^1 l(u)\overline{z}r\,d_F^{\alpha}\zeta-\int_0^1 u\overline{l(z)}r\,d_F^{\alpha}\zeta$$

$$=\begin{pmatrix}Z^*(0) & Z^*(1)\end{pmatrix}\begin{pmatrix}-J&0\\0&J\end{pmatrix}\begin{pmatrix}U(0)\\U(1)\end{pmatrix}, \tag{3.2}$$

where

$$J=\begin{pmatrix}0&-1\\1&0\end{pmatrix}.$$

Theorem 3.1. *The domain $D_{\min}$ and $D_{\max}$ are dense in $L^{\alpha}_{2,r}[0,1]$, $\mathscr{L}_{\max}=\mathscr{L}^{*}_{\min}$, $\mathscr{L}^{*}_{\max}=\mathscr{L}_{\min}$.*

Now, we formulate an operator$\mathscr{L}$ such that $\mathscr{L}_{\min}\subset\mathscr{L}\subset\mathscr{L}_{\max}$. It then follows that $\mathscr{L}^{*}$ satisfy the inequalities $\mathscr{L}_{\min}\subset\mathscr{L}^{*}\subset\mathscr{L}_{\max}$. The application of boundary conditions will give the result $\mathscr{L}=\mathscr{L}^{*}$, thereby establishing $\mathscr{L}$ as a self-adjoint operator.
Let C and D be matrices, with the following form:

$$C=\begin{pmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{pmatrix}$$

and

$$D=\begin{pmatrix}\delta_{11} & \delta_{12}\\ \delta_{21} & \delta_{22}\end{pmatrix}$$

such that the 4×4 matrix $\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ is nonsingular. Let $\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D}$ be 2×2 matrices satisfying

$$\begin{pmatrix}\widetilde{A} & \widetilde{B}\\ \widetilde{C} & \widetilde{D}\end{pmatrix}\begin{pmatrix}A & B\\ C & D\end{pmatrix}=\begin{pmatrix}-J & 0\\ 0 & J\end{pmatrix}.$$

From (3.2), we deduce that

$$\int_0^1 l(u)\bar{z}rd_F^{\alpha}\zeta-\int_0^1 u\overline{l(z)}rd_F^{\alpha}\zeta$$

$$=\left[\widetilde{A}Z(0)+\widetilde{B}Z(1)\right][AU(0)+BU(1)]$$

$$+\left[\widetilde{C}Z(0)+\widetilde{D}Z(1)\right][CU(0)+DU(1)].$$

To satisfy the original and conjugate boundary conditions, we get

$$AU(0)+BU(1)=0,\ CU(0)+DU(1)=\psi,$$

$$\widetilde{A}Z(0)+\widetilde{B}Z(1)=\phi,\ \widetilde{C}Z(0)+\widetilde{D}Z(1)=0,$$

where ψ and ϕ are arbitrary. Thus,

$$U(0)=J\widetilde{C}^{*}\psi,\ U(1)=-J\widetilde{D}^{*}\psi,$$

$$Z(0)=JA^{*}\phi,\ U(1)=-JB^{*}\phi.$$

Let us define the operator $\mathscr{L}$ by setting $\mathscr{L}u=l(u)$ for all $u\in D$, where

$$D=\left\{u\in D_{\max}:AU(0)+BU(1)=\begin{pmatrix}0\\0\end{pmatrix}\right\}.$$

Therefore, we have the following results.

Theorem 3.2. *$\mathscr{L}=\mathscr{L}^{*}$ if and only if $AJA^{*}=BJB^{*}$.*

Theorem 3.3. *Let $\mathscr{L}$ be a self-adjoint operator. Then its spectrum is constituted solely of eigenvalues with ∞ as their only limit. The associated eigenfunctions constitute a complete orthogonal set.*

4. Dirichlet formulas

Consider the following fractal expression

$$\langle l(u),z\rangle=\int_0^1\left[-(D_F^{\alpha})^2u+vu\right]\bar{z}d_F^{\alpha}\zeta.$$

Using fractal integration by parts, we conclude that

$$\langle l(u),z\rangle=-(D_F^{\alpha}u)\bar{z}\,|_0^1+\int_0^1[(D_F^{\alpha}u)(D_F^{\alpha}\bar{z})+vu\bar{z}]\,d_F^{\alpha}\zeta.$$

To produce a Sobolev space while preserving the self-adjointness of (1.1)-(1.2), we must adjust (1.2).
By (1.2), we see that

$$\begin{pmatrix}\tau_{11} & \chi_{11}\\ \tau_{21} & \chi_{21}\end{pmatrix}\begin{pmatrix}u(0)\\ u(1)\end{pmatrix}=\begin{pmatrix}-\tau_{12} & \chi_{12}\\ -\tau_{22} & \chi_{22}\end{pmatrix}\begin{pmatrix}D_F^{\alpha}u(0)\\ -D_F^{\alpha}u(1)\end{pmatrix}.$$

In trying to solve for

$$\begin{pmatrix}D_F^{\alpha}u(0)\\ -D_F^{\alpha}u(1)\end{pmatrix},$$

we must consider the following

$$\begin{pmatrix} -\tau_{12} & \chi_{12} \\ -\tau_{22} & \chi_{22} \end{pmatrix}.$$

Three situations can arise.

i) Without loss of generality we may assume $\tau_{22}\chi_{12}-\tau_{12}\chi_{22}=1$. Then we find

$$\begin{pmatrix} D_F^\alpha u(0) \\ -D_F^\alpha u(1) \end{pmatrix} = \begin{pmatrix} \tau_{11}\chi_{22}-\tau_{21}\chi_{12} & \chi_{11}\chi_{22}-\chi_{12}\chi_{22} \\ \tau_{11}\tau_{22}-\tau_{12}\tau_{21} & \chi_{11}\tau_{22}-\chi_{21}\tau_{12} \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \end{pmatrix}.$$

Since

$$-(D_F^\alpha u)\bar{z}\,|_0^1 = \begin{pmatrix} \bar{z}(0) & \bar{z}(1) \end{pmatrix} \begin{pmatrix} D_F^\alpha u(0) \\ -D_F^\alpha u(1) \end{pmatrix},$$

we obtain

$$\langle l(u), z\rangle = \int_0^1 [(D_F^\alpha u)(D_F^\alpha \bar{z}) + vu\bar{z}]\, d_F^\alpha \zeta$$

$$+\begin{pmatrix} \bar{z}(0) & \bar{z}(1) \end{pmatrix} \begin{pmatrix} \tau_{11}\chi_{22}-\tau_{21}\chi_{12} & \chi_{11}\chi_{22}-\chi_{12}\chi_{22} \\ \tau_{11}\tau_{22}-\tau_{12}\tau_{21} & \chi_{11}\tau_{22}-\chi_{21}\tau_{12} \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \end{pmatrix}.$$

According to the self-adjointness criterion, the matrices mentioned above are symmetric. When defining a Sobolev inner product, the right side is used if they are positive.

ii) Let the matrix $\begin{pmatrix} -\tau_{12} & \chi_{12} \\ -\tau_{22} & \chi_{22} \end{pmatrix}$ is singular but not zero. Then its rows are linearly dependent, i.e., there exists a number ξ such that

$$\xi\tau_{12} = \tau_{22}, \text{ and } \xi\chi_{12} = \chi_{22}. \tag{4.1}$$

Assume without loss of generality that $\tau_{12}^2+\chi_{12}^2=1$. Then we find

$$\begin{pmatrix} \tau_{11} & \chi_{11} \\ \tau_{21}-\xi\tau_{11} & \chi_{21}-\xi\chi_{11} \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \end{pmatrix}$$

$$-\begin{pmatrix} -\tau_{12} & \chi_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D_F^\alpha u(0) \\ -D_F^\alpha u(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.2}$$

Let us define $u_0, D_F^\alpha u_0, u_1, D_F^\alpha u_1$ by

$$\begin{pmatrix} -\tau_{12} & \chi_{12} \\ \chi_{12} & \tau_{12} \end{pmatrix} \begin{pmatrix} u(0) \\ u(1) \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \tag{4.3}$$

$$\begin{pmatrix} -\tau_{12} & \chi_{12} \\ \chi_{12} & \tau_{12} \end{pmatrix} \begin{pmatrix} D_F^\alpha u(0) \\ -D_F^\alpha u(1) \end{pmatrix} = \begin{pmatrix} D_F^\alpha u_0 \\ D_F^\alpha u_1 \end{pmatrix}. \tag{4.4}$$

By virtue of (4.3) and (4.4), we see that

$$\begin{pmatrix} u(0) \\ u(1) \end{pmatrix} = \begin{pmatrix} -\tau_{12} & \chi_{12} \\ \chi_{12} & \tau_{12} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},$$

and

$$\begin{pmatrix} D_F^\alpha u(0) \\ -D_F^\alpha u(1) \end{pmatrix} = \begin{pmatrix} -\tau_{12} & \chi_{12} \\ \chi_{12} & \tau_{12} \end{pmatrix} \begin{pmatrix} D_F^\alpha u_0 \\ D_F^\alpha u_1 \end{pmatrix}.$$

By (4.2), we deduce that

$$\begin{pmatrix} \tau_{11} & \chi_{11} \\ \tau_{21}-\xi\tau_{11} & \chi_{21}-\xi\chi_{11} \end{pmatrix} \begin{pmatrix} -\tau_{12} & \chi_{12} \\ \chi_{12} & \tau_{12} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$$

$$-\begin{pmatrix} -\tau_{12} & \chi_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -\tau_{12} & \chi_{12} \\ \chi_{12} & \tau_{12} \end{pmatrix} \begin{pmatrix} D_F^\alpha u_0 \\ D_F^\alpha u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} -\tau_{11}\tau_{12}+\chi_{11}\chi_{12} & \tau_{11}\chi_{12}+\chi_{11}\tau_{12} \\ -\tau_{12}[\tau_{21}-\xi\tau_{11}]+\chi_{12}[\chi_{21}-\xi\chi_{11}] & \chi_{12}[\chi_{21}-\xi\chi_{11}]+\tau_{12}[\chi_{21}-\xi\chi_{11}] \end{pmatrix}$$

$$\times\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D_F^\alpha u_0 \\ D_F^\alpha u_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives rise to two restrictions. Let

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$
$$= \begin{pmatrix} -\tau_{11}\tau_{12}+\chi_{11}\chi_{12} & \tau_{11}\chi_{12}+\chi_{11}\tau_{12} \\ -\tau_{12}\left[\tau_{21}-\xi\tau_{11}\right]+\chi_{12}\left[\chi_{21}-\xi\chi_{11}\right] & \chi_{12}\left[\chi_{21}-\xi\chi_{11}\right]+\tau_{12}\left[\chi_{21}-\xi\chi_{11}\right] \end{pmatrix}.$$

Thus, we find

$$Pu_0+Qu_1=D_F^{\alpha}u_0,\ Ru_0+Su_1=0.$$

Now the boundary terms

$$\begin{pmatrix} \bar{z}(0) & \bar{z}(1) \end{pmatrix}\begin{pmatrix} D_F^{\alpha}u(0) \\ -D_F^{\alpha}u(1) \end{pmatrix}$$
$$= \begin{pmatrix} \bar{z}(0) & \bar{z}(1) \end{pmatrix}\begin{pmatrix} -\tau_{12} & \chi_{12} \\ \chi_{12} & \tau_{12} \end{pmatrix}\begin{pmatrix} -\alpha_{12} & \chi_{12} \\ \chi_{12} & \tau_{12} \end{pmatrix}\begin{pmatrix} D_F^{\alpha}u_0 \\ D_F^{\alpha}u_1 \end{pmatrix}$$
$$= \begin{pmatrix} \bar{z}(0) & \bar{z}(1) \end{pmatrix}\begin{pmatrix} D_F^{\alpha}u_0 \\ D_F^{\alpha}u_1 \end{pmatrix}$$
$$= \bar{z}_0 D_F^{\alpha}u_0+\bar{z}_1 D_F^{\alpha}u_1.$$

Hence we restrict ourselves to a subspace where $z_1=0$ (and $u_1=0$) due to substitution can only be made for $D_F^{\alpha}u_0$.
Using the self-adjointness criterion

$$\alpha_{11}\tau_{12}-\tau_{12}\tau_{21}=\chi_{11}\chi_{22}-\chi_{12}\chi_{21}$$

and (4.1), we conclude that

$$\tau_{12}\left(\xi\tau_{11}-\tau_{21}\right)=\chi_{12}\left(\xi\chi_{11}-\chi_{21}\right).$$

Hence, we obtain

$$\eta\tau_{12}=\xi\chi_{11}-\chi_{21},\ \eta\chi_{12}=\xi\tau_{11}-\tau_{21},$$

where η is parameter.
We find that $R=0$ when these are used to simplify R. The $R-S$ constraint vanishes. Then we find

$$\langle l(u),z\rangle=\int_0^1\left[\left(D_F^{\alpha}u\right)\left(D_F^{\alpha}\bar{z}\right)+vu\bar{z}\right]d_F^{\alpha}\zeta$$
$$+\left(-\tau_{12}\bar{z}(0)+\chi_{12}\bar{z}(1)\right)\left(-\tau_{11}\tau_{12}+\chi_{11}\chi_{12}\right)\left(-\tau_{12}u(0)+\chi_{12}u(1)\right).$$

iii) Let $\begin{pmatrix} -\tau_{12} & \chi_{12} \\ -\tau_{22} & \chi_{22} \end{pmatrix}$ is the zero matrix. Then we have

$$\begin{pmatrix} \tau_{11} & \chi_{11} \\ \tau_{21} & \chi_{21} \end{pmatrix}\begin{pmatrix} u(0) \\ u(1) \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the coefficient matrix has rank 2, $u(0)=0$ and $u(1)=0$. Hence, we get

$$\langle l(u),z\rangle=\int_0^1\left[\left(D_F^{\alpha}u\right)\left(D_F^{\alpha}\bar{z}\right)+vu\bar{z}\right]d_F^{\alpha}\zeta.$$

5. The Sobolev space theory

Let us consider the following operator

$$l(u)=\frac{1}{r}\left[-\left(D_F^{\alpha}\right)^2u+vu\right],$$

whose domain is constrained by boundary conditions

$$AU(0)+BU(1)=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We will define three inner products according to the above three cases.

$$\langle u,z\rangle_{H^1}=\int_0^1\left[\left(D_F^{\alpha}u\right)\left(D_F^{\alpha}\bar{z}\right)+vu\bar{z}\right]d_F^{\alpha}\zeta$$

$$+\begin{pmatrix} \bar{z}(0) & \bar{z}(1) \end{pmatrix}\begin{pmatrix} \tau_{11}\chi_{22}-\tau_{21}\chi_{12} & \chi_{11}\chi_{22}-\chi_{12}\chi_{22} \\ \tau_{11}\tau_{22}-\tau_{12}\tau_{21} & \chi_{11}\tau_{22}-\chi_{21}\tau_{12} \end{pmatrix}\begin{pmatrix} u(0) \\ u(1) \end{pmatrix}, \tag{5.1}$$

where we assume that the following matrix

$$\begin{pmatrix} \tau_{11}\chi_{22}-\tau_{21}\chi_{12} & \chi_{11}\chi_{22}-\chi_{12}\chi_{22} \\ \tau_{11}\tau_{22}-\tau_{12}\tau_{21} & \chi_{11}\tau_{22}-\chi_{21}\tau_{12} \end{pmatrix}$$

is positive.

$$\langle u,z\rangle_{H^1}=\int_0^1 [(D_F^\alpha u)(D_F^\alpha \bar{z})+vu\bar{z}]\,d_F^\alpha\zeta$$

$$+(-\tau_{12}\bar{z}(0)+\chi_{12}\bar{z}(1))(-\tau_{11}\tau_{12}+\chi_{11}\chi_{12})(-\tau_{12}u(0)+\chi_{12}u(1)), \tag{5.2}$$

where

$$-\tau_{11}\tau_{12}+\chi_{11}\chi_{12}\geq 0$$

and

$$\chi_{12}u(0)+\tau_{12}u(1)=0.$$

$$\langle u,z\rangle_{H^1}=\int_0^1 [(D_F^\alpha u)(D_F^\alpha \bar{z})+vu\bar{z}]\,d_F^\alpha\zeta, \tag{5.3}$$

where $u(0)=u(1)=0$.
Then the Sobolev space H^1 is generated by the inner product (5.1), (5.2) or (5.3).
Let

$$D_s=\left\{ u\in H^1: \begin{array}{c} D_F^\alpha u\in H,\ D_F^{2\alpha}u\in L_{2,r}^\alpha[0,1], \\ AU(0)+BU(1)=\begin{pmatrix}0\\0\end{pmatrix}, \\ \text{and } l(u)\in H^1 \end{array}\right\},$$

where $AJA^*=BJB^*$, and $v>\varepsilon r$ for some $\varepsilon>0$.
Let us define the operator L by setting $Lu=l(u)$ for all $u\in D_s$.

Theorem 5.1. *The operator L is bounded below.*

Proof. In each case, we see that

$$\langle Lu,u\rangle=\langle u,u\rangle_{H^1}\geq \varepsilon\langle u,u\rangle.$$

Hence,

$$\langle (L-\varepsilon)u,u\rangle\geq 0.$$

□

Corollary 5.2. *The operator L^{-1} exists and is given as*

$$L^{-1}f(\zeta)=\int_0^1 G(\zeta,\xi)f(\xi)r(\xi)\,d_F^\alpha\xi.$$

L^{-1} is bounded by $1/\varepsilon$.

Proof. Let $Lu=f$. Then we conclude that

$$\langle f,u\rangle=\langle f,L^{-1}f\rangle\geq\varepsilon\langle L^{-1}f,L^{-1}f\rangle.$$

From Schwartz's inequality, we have

$$\left\|L^{-1}f\right\|\leq\frac{1}{\varepsilon}\|f\|.$$

□

Theorem 5.3. *The operator L is symmetric.*

Proof. By the Dirichlet formula, we find

$$\langle Lu, z\rangle = \langle u, z\rangle_{H^1},$$

where $u \in D_s$ and $z \in H^1$. Assume that $z \in D_s$. Then we get

$$\langle Lu, Lz\rangle = \langle u, Lz\rangle_{H^1}. \tag{5.4}$$

Hence,

$$\langle Lu, Lz\rangle = \langle Lu, z\rangle_{H^1}. \tag{5.5}$$

By (5.4) and (5.5), we deduce that

$$\langle Lu, z\rangle_{H^1} = \langle u, Lz\rangle_{H^1}.$$

□

Theorem 5.4. *The operator L^{-1} exists and is bounded.*

Proof. Let $L^{-1}u = f$. From the Dirichlet formula, we get

$$\langle f, L^{-1}f\rangle = \langle L^{-1}f, L^{-1}f\rangle_{H^1}.$$

By Schwartz's inequality, we deduce that

$$\langle L^{-1}f, L^{-1}f\rangle_{H^1} = \left\|L^{-1}f\right\|_{H^1}^2 \leq \|f\| \frac{1}{\varepsilon} \|f\| \leq \left(\frac{1}{\varepsilon}\right)^2 \|f\|_{H^1}^2,$$

i.e.,

$$\left\|L^{-1}\right\|_{H^1} \leq \frac{1}{\varepsilon}.$$

□

Theorem 5.5. *The operator L is self-adjoint in H^1.*

Proof. By Theorem 5.2, we see that L is self-adjoint in H^1 since the range of the operator L is H^1. □

Theorem 5.6. *The spectrum of L consists of the same eigenvalues as $\mathscr{L}$, with same eigenfunctions. These eigenfunctions form a complete orthogonal set in H^1.*

Proof. See [21, 20]. □

6. Conclusion

This study focused on the left-definite problem in the regular case of fractal Sturm–Liouville problems. The results obtained here provide a foundation that can be further extended to investigate higher-order fractal differential problems.

Article Information

Acknowledgements: The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Author's contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflict of Interest Disclosure: No potential conflict of interest was declared by the author.

Supporting/Supporting Organizations: No grants were received from any public, private or non-profit organizations for this research.

Ethical Approval and Participant Consent: It is declared that during the preparation process of this study, scientific and ethical principles were followed and all the studies benefited from are stated in the bibliography.

Plagiarism Statement: This article was scanned by the plagiarism program. No plagiarism detected.

Availability of data and materials: Not applicable

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