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ANALYSIS OF CONICS UNDER KANTOROVICH OPERATORS OF TWO VARIABLE

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Abstract. This study examines some shape-preserving properties of two-variable Kantorovich polynomials. We examine which types of conic equations transform into conic equations under two types of two-variable Kantorovich polynomials, single-index and double-index, and if so, which conic equations they transform into. While it is observed that conic equations transform into the same type of conic equations under the single-index two-variable Kantorovich polynomial, they are shown to transform into different types under the double-index two-variable Kantorovich polynomial, for example, a circle can transform into an ellipse or parabola under certain conditions. Furthermore, all the findings are supported by numerous graphical examples.

1. Introduction

One of the problems of approximation theory in mathematical analysis that has important results is finding a sequence of algebraic polynomials having geometric properties of a given function that converges to the given function. Such approximations are called "shape-preserving approximations" in the literature. The first studies on shape-preserving approximations with algebraic polynomials started with Chebyshev and Weierstrass in the mid-nineteenth century. Examples of important studies on form-preserving approximations of single-variable real-valued functions in terms of algebraic polynomials are the works of mathematicians Kocic and Milovanovic [12], Leviatan [13], [14] and Hu-Yu [9]. The first study on the preservation of the convexity property is the article [18]. This study states that there exists a sequence of convex algebraic polynomials that converges uniformly to a convex function defined on $[a, b]$. However, the first mathematician who could write these polynomials in explicit form was T. Popoviciu [20]. He used Bernstein polynomials for this purpose. Bernstein polynomials are defined as

$$B_m(h;x) = \sum_{k=0}^{m} p_{m,k}(x) h\left(\frac{k}{m}\right),$$

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for a bounded function h on the interval $I = [0,1]$, where $m \in \mathbb{N}$ and

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$$

[2]. With this result of Popoviciu, it was discovered that Bernstein polynomials have many form-preserving properties: Popoviciu [20] noticed that if s is a nonnegative integer and the function $f : I \to \mathbb{R}$ is s-convex, then for each $m \in \mathbb{N}$, the polynomials $B_m(f)$, which are the images of this function under the Bernstein operator, have the same property.

Lupas [16] showed that a function $h : I \to \mathbb{R}$ is a star function, that is, if $h(\lambda t) \leq \lambda h(t)$ holds for all $t \in I$ and $\lambda \in I$, then under certain conditions the Bernstein polynomials $B_m(h)$ $(m \in \mathbb{N})$ are also star functions. Lupas obtained this theorem for star functions f that are nonnegative, differentiable on the interval I and zero at 0, from the definition of Bruckner and Ostrow [3]. Tunç and Uzun [23] addressed the problem of whether the $\mathbb{B}$-concavity and $\mathbb{B}$-convexity of a function, as defined by Adilov and Yeşilce [1], are preserved by Bernstein Operators. It has been proven that Bernstein Operators do not preserve the $\mathbb{B}$-convexity property of functions, but they do preserve the $\mathbb{B}$-concavity property. Over time, many mathematicians have made great efforts to contribute to the preservation of different geometric properties [7].

Using the method of obtaining Bernstein polynomials of one-variable functions, Hildebrant and Schoenberg [8] defined the double-indexed and two-variable version of the Bernstein polynomials for two-variable real functions defined on $I^2 = [0,1] \times [0,1]$ as

$$\mathbb{B}_{m,n}(H;x,y) = \sum_{k=0}^{m}\sum_{j=0}^{n} p_{m,n,k,j}(x,y) H\Big(\frac{k}{m}, \frac{j}{n}\Big), \qquad (n,m \in \mathbb{N})$$

and studied the approximation properties, where

$$p_{m,n,k,j}(x,y) = p_{m,k}(x) p_{n,j}(y).$$

In the work of [11], it was shown that the sequence $\big(\mathbb{B}_{m,n}(H)\big)$ converges simultaneously to both itself and the partial derivatives of the function H. The (r,s)-convex functions defined by Popoviciu [19] are also preserved under the $\mathbb{B}_{m,n}$ operators [7]. In the work of [24], it was obtained that the form of B-convex two-variable real-valued functions is not preserved under these operators, but in some special cases the form of B-concave functions is preserved. Lorentz [15] and Dinghas [5] independently defined the single-index bivariate Bernstein polynomials on the simplex $\Delta_2 = \{(x,y) \in \mathbb{R}^2 : \ x+y \leq 1, \ x,y \geq 0\}$ for real-valued two-variable function H,

$$\mathbb{B}_m(H;x,y) = \sum_{k=0}^{m}\sum_{i=0}^{m-k} p_{m,k,i}(x,y) H\left(\frac{k}{m}, \frac{i}{m}\right), \quad (m \in \mathbb{N})$$

where

$$p_{m,k,i}(x,y) = \frac{m!}{k!i!(m-k-i)!} x^k y^i (1-x-y)^{m-k-i}$$

For a given function H that is continuous on the simplex, the sequence of bivariate polynomials $\big(\mathbb{B}_m(H)\big)$ converges uniformly to this function H [21], [15]. There are also some important studies on the shape-preserving properties of these bivariate polynomial sequences. In the study of [17], it was shown that the Schur-convexity

property is preserved, and in the study of [4], it was shown that the strong convexity property is preserved under these operators.

Inspired by the definition of the bivariate version of the Bernstein polynomial, Z. Ditzian [6] defined the single-index bivariate Kantorovich polynomials for integrable function $H \in L_1(\Delta_2)$ as

$$\mathbb{K}_m(H;x,y) = (m+1) \sum_{\substack{k,j=0 \\ k+j\leq m}} p_{m,k,j}(x,y) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} \int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}} H(s,t)dsdt$$

while defining the double-index bivariate Kantorovich polynomials for $H \in L_1(I^2)$ as

$$\mathbb{K}_{m,n}(H;x,y) = (m+1)(n+1) \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,n,k,j}(x,y) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} \int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} H(s,t)dsdt$$

where $p_{m,n,k,j}(x,y) = p_{m,k}(x)p_{n,j}(y)$. In this work, Ditzian not only defined these operators but also obtained the class of local saturation in cases where the conditions of the direct and saturation theorems do not coincide. However, the inverse results that need to be found remain open. Zhou [25] succeeded in giving direct and inverse results of multidimensional Kantorovich operators on simplexes as well as on cubes. A. Kajla and M. Goyal [10] gave the Voronovskaja-type asymptotic theorem for a generalization of these operators and investigated the degree of approximation of functions of the Lipschitz class.

Second-order two-variable algebraic equations geometrically specify some well-known shapes such as circles, ellipses, hyperbolas and parabolas. Such equations are called conic equations. Tunç and Alhazzori [22] investigated whether conic equations transform into a conic equation under both types of two-variable Bernstein polynomials given above, and if so, which conic equation they transform into. The subject of this study is whether the results obtained by Tunç and Alhazzori are valid for two-variable Kantorovich operators.

2. Conics under single-index bivariate Kantorovich polynomials

The moment formulas for the single-index two-variable version of the Kantorovich operators are given below.

Lemma 2.1. *The following equalities are hold for all $m \in \mathbb{N}$:*

(1) $\mathbb{K}_m\big(e_{0,0};x,y\big) = 1$
(2) $\mathbb{K}_m\big(e_{1,0};x,y\big) = \frac{2mx+1}{2(m+1)}$
(3) $\mathbb{K}_m\big(e_{0,1};x,y\big) = \frac{2my+1}{2(m+1)}$
(4) $\mathbb{K}_m\big(e_{1,1};x,y\big) = \frac{1}{4(m+1)^2}\,(4m(m-1)xy + 2m(x+y) + 1)$
(5) $\mathbb{K}_m\big(e_{2,0};x,y\big) = \frac{m^2x^2}{(m+1)^2} + \frac{mx(2-x)}{(m+1)^2} + \frac{1}{3(m+1)^2}$
(6) $\mathbb{K}_m\big(e_{0,2};x,y\big) = \frac{m^2y^2}{(m+1)^2} + \frac{my(2-y)}{(m+1)^2} + \frac{1}{3(m+1)^2}$

For $A,B,C,D,E,F \in \mathbb{R}$, the single-index Kantorovich operator $\mathbb{K}_m$ transforms the equation

$$G(x,y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{2.1}$$

to a conic equation as follows:

$$\mathbb{K}_m(G;x,y) = A'x^2 + B'xy + C'y^2 + D'x + E'y + F' = 0$$

where the coefficients can be computed using the equations in Lemma 2.1 as follows

Lemma 2.2.

$$\begin{aligned}
A' &= \frac{Am(m-1)}{(m+1)^2} \\
B' &= \frac{Bm(m-1)}{(m+1)^2} \\
C' &= \frac{Cm(m-1)}{(m+1)^2} \\
D' &= \frac{2mA}{(m+1)^2} + \frac{mB}{2(m+1)^2} + \frac{mD}{(m+1)} \\
E' &= \frac{2mC}{(m+1)^2} + \frac{mB}{2(m+1)^2} + \frac{mE}{(m+1)} \\
F' &= \frac{A+C}{3(m+1)^2} + \frac{B}{4(m+1)^2} + \frac{D+E}{2(m+1)} + F.
\end{aligned}$$

The equation (2.1) specifies a circle in the cartesian plane, if $A = C = 1, B = 0$ and $D^2 + E^2 - 4F > 0$. Let $A = C = 1, B = 0$,

$$\delta_0 = D^2 + E^2 - 4F \qquad \text{and} \qquad \delta_1 = D + E + 2F.$$

In the theorem given below, a necessary and sufficient condition is given for a conic equation specifying a circle under single-index two-variable Kantorovich operators.

Theorem 2.3. *Let the conic equation $G(x,y) = 0$ given by (2.1) with $A = C = 1$ denote a circle and let $m \in \mathbb{N}_2 = \{2,3,4,...\}$. Then, the equation $\mathbb{K}_m(G;x,y) = 0$ denotes a circle iff*

$$m\delta_0 + 2\delta_1 + \frac{8}{3}\left(1 - \left(\frac{m}{m+1}\right)^2\right) > 0.$$

Proof. Since $A = C = 1$ and $B = 0$, from the equations in Lemma 2.2

$$\begin{aligned}
A' = C' &= \frac{m(m-1)}{(m+1)^2} \\
B' &= 0 \\
D' &= \frac{2m}{(m+1)^2} + \frac{mD}{(m+1)} \\
E' &= \frac{2m}{(m+1)^2} + \frac{mE}{(m+1)} \\
F' &= \frac{2}{3(m+1)^2} + \frac{D+E}{2(m+1)} + F
\end{aligned}$$

are obtained. Thus, the equation given by (2.1) is viewed under the operator $\mathbb{K}_m$ as follows:

$$\begin{aligned}\tfrac{m(m-1)}{(m+1)^2}(x^2+y^2)+\Big(\tfrac{2m}{(m+1)^2}+\tfrac{mD}{m+1}\Big)x \quad &+\Big(\tfrac{2m}{(m+1)^2}+\tfrac{mE}{m+1}\Big)y\\ &+\Big(\tfrac{2}{3(m+1)^2}+\tfrac{D+E}{2(m+1)}+F\Big)=0\end{aligned}$$

Since $m>1$, if the common parentheses are used in this equation, the following equation is reached:

$$x^2+y^2+\frac{2+(m+1)D}{(m-1)}x+\frac{2+(m+1)E}{(m-1)}y+\frac{4+3(m+1)(D+E)+6(m+1)^2F}{6m(m-1)}=0.$$

For this last equation, define the following notations:

$$D''=\frac{2+(m+1)D}{(m-1)},\quad E''=\frac{2+(m+1)E}{(m-1)},$$
$$F''=\frac{4+3(m+1)(D+E)+6(m+1)^2F}{6m(m-1)}.$$

Considering these notations, we have

$$\begin{aligned}D''^2+E''^2-4F'' &= \tfrac{1}{(m-1)^2}\Big[m^2\delta_0+m(2\delta_0+2D+2E+4F)\\ &\quad+\Big(\delta_0+4D+4E+8F+\tfrac{16}{3}\Big)+\tfrac{1}{m}\Big(\tfrac{8}{3}+2D+2E+4F\Big)\Big]\\ &= \tfrac{1}{(m-1)^2}\Big[m^2\delta_0+2(\delta_0+\delta_1)m+\tfrac{16}{3}+4\delta_1+\delta_0+\tfrac{1}{m}\Big(\tfrac{8}{3}+2\delta_1\Big)\Big]\\ &= \tfrac{1}{m(m-1)^2}\Big[\delta_0 m(m+1)^2+2(m+1)^2\delta_1+\tfrac{8}{3}(m+1)^2-\tfrac{8m^2}{3}\Big]\\ &= \tfrac{(m+1)^2}{m(m-1)^2}\Big[m\delta_0+2\delta_1+\tfrac{8}{3}\Big(1-(\tfrac{m}{m+1})^2\Big)\Big].\end{aligned}$$

Therefore, the inequality $m\delta_0+2\delta_1+\frac{8}{3}\Big(1-(\frac{m}{m+1})^2\Big)>0$ hold iff the equation $\mathbb{K}_m(G;x,y)=0$ to define a circle for $m>1$. □

Remark. (1) *If $m=1$ in Theorem 2.3 then the equation $\mathbb{K}_m(f;x,y)=0$ defines a line. The equation of this line is* $(D+1)\,x+(E+1)\,y+\frac{1}{6}\Big(3\,(D+E)+12F+2\Big)=0$.

(2) *If the condition is satisfied in Theorem 2.3, then the equation $\mathbb{K}_m(G;x,y)=0$ denotes a circle with center*

$$\Big(-\frac{(m+1)D+2}{2m-2},-\frac{(m+1)E+2}{2m-2}\Big)$$

and radius

$$\frac{(m+1)}{2(m-1)}\sqrt{\frac{m\delta_0+2\delta_1+\frac{8}{3}\Big(1-(\frac{m}{m+1})^2\Big)}{m}}$$

for $m\neq 1$.

(3) *Since* $\delta_0 > 0$, *the condition* $m\delta_0 + 2\delta_1 + \frac{8}{3}\Big(1 - (\frac{m}{m+1})^2\Big) > 0$ *in Theorem 2.3 will be satisfied for every sufficiently large natural number* m.

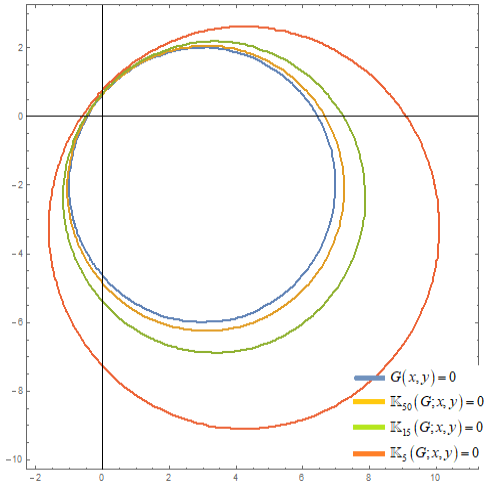

FIGURE 1. Images of the circle in example 2.1 under the operator $\mathbb{K}_m$ for $m = 5, 15, 50$.

Example 2.1. *If the circle defined by the equation* $G(x,y) = x^2+y^2-6x+4y-3 = 0$ *is considered, it is clear that*

$$\delta_0 = D^2 + E^2 - 4F = 64 \qquad \text{and} \qquad \delta_1 = D + E + 2F = -8.$$

Since

$$m\delta_0 + 2\delta_1 = 64m - 16 > 0$$

for each $m \in \mathbb{N}_2$ *by Theorem 2.3 the equation* $\mathbb{K}_m(G;x,y) = 0$ *defines a circle (see Figure 1).*

Theorem 2.4. *If the equation* $G(x,y) = 0$ *given by (2.1) denotes an ellipse, then for each* $m \in \mathbb{N}_2$ *the equation* $\mathbb{K}_m(G;x,y) = 0$ *also denotes an ellipse.*

Proof. Since the equation $G(x,y) = 0$ given by (2.1) denotes an ellipse, the inequality $B^2 - 4AC < 0$ holds. From Lemma 2.2, we have

$$\begin{aligned} B'^2 - 4A'C' &= \Big(\tfrac{m(m-1)}{(m+1)^2}B\Big)^2 - 4AC\Big(\tfrac{m(m-1)}{(m+1)^2}\Big)\Big(\tfrac{m(m-1)}{(m+1)^2}\Big) \\ &= \Big(\tfrac{m(m-1)}{(m+1)^2}\Big)^2\Big(B^2 - 4AC\Big) < 0 \end{aligned}$$

for every $m \in \mathbb{N}_2$. Thus the equation $\mathbb{K}_m(G;x,y) = 0$ defines an ellipse. □

Remark. *By Theorem 2.4 if* $m = 1$ *then the equation* $\mathbb{K}_m(G;x,y) = 0$ *defines a line. The equation of this line is as follows:*

$$\Big(B + 4(A+D)\Big)x + \Big(B + 4(E+C)\Big)y + 2\Big(\frac{A+C}{3} + \frac{B}{4} + D + E + 4F\Big) = 0.$$

Example 2.2. *Considering the equation* $G(x,y) = 3x^2-4xy+2y^2-6x+4y-3 = 0$, *since*

$$B'^2 - 4A'C' = -8\Big(\frac{m(m-1)}{(m+1)^2}\Big)^2 < 0$$

for each $m \in \mathbb{N}_2$, *the equation* $\mathbb{K}_m(G;x,y) = 0$ *specifies an ellipse (see Figure 2). When* $m = 1$ *is taken, this equation specifies the line* $-2x + \frac{5}{2}y - \frac{10}{3} = 0$.

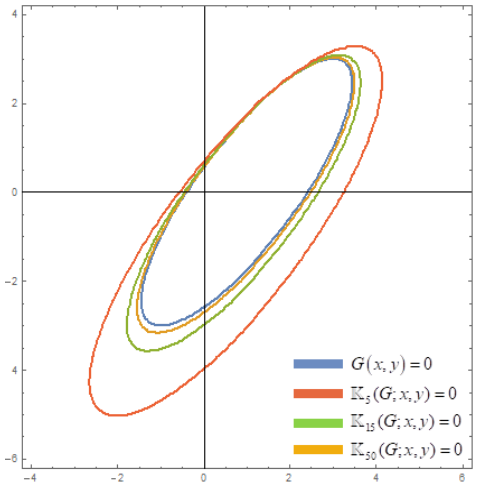

FIGURE 2. Images of the ellipse in Example 2.2 under the operator $\mathbb{K}_m$ for $m = 5, 15, 50$.

Since the proofs of the next two theorems are similar to Theorem 2.4, let us give these two theorems without proofs.

Theorem 2.5. *If the equation* $G(x,y) = 0$ *given by (2.1) specifies a hyperbola, then the equation* $\mathbb{K}_m(G;x,y) = 0$ *also specifies a hyperbola for each* $m \in \mathbb{N}_2$.

Theorem 2.6. *If the equation* $G(x,y) = 0$ *given by (2.1) specifies a parabola, then the equation* $\mathbb{K}_m(G;x,y) = 0$ *also specifies a parabola for each* $m \in \mathbb{N}_2$.

3. Conics under double-index bivariate Kantorovich polynomials

The following moment formulas are true for two-variable Kantorovich operators with double indexes [10].

Lemma 3.1. (1) $\mathbb{K}_{m,n}(e_{00};x,y) = 1$

(2) $\mathbb{K}_{m,n}(e_{10};x,y) = \frac{mx}{m+1} + \frac{1}{2(m+1)}$

(3) $\mathbb{K}_{m,n}(e_{01};x,y) = \frac{ny}{n+1} + \frac{1}{2(n+1)}$

(4) $\mathbb{K}_{m,n}(e_{11};x,y) = \frac{4mnxy\ +2mx\ +2ny\ +1}{4(m+1)(n+1)}$

(5) $\mathbb{K}_{m,n}(e_{20};x,y) = \frac{m^2x^2}{(m+1)^2} + \frac{mx(2-x)}{(m+1)^2} + \frac{1}{3(m+1)^2}$

(6) $\mathbb{K}_{m,n}(e_{02};x,y) = \frac{n^2y^2}{(n+1)^2} + \frac{ny(2-y)}{(n+1)^2} + \frac{1}{3(n+1)^2}$

The double-index Kantorovich operator $\mathbb{K}_{m,n}$ transforms the equation (2.1) to a conic equation as follows:

$$\mathbb{K}_{m,n}(G;x,y) = A'x^2 + B'xy + C'y^2 + D'x + E'y + F' = 0$$

where the coefficients can be computed using the equations in Lemma 3.1 as follows

Lemma 3.2.

$$\begin{aligned}
A' &= \frac{Am(m-1)}{(m+1)^2}; \qquad B' = \frac{Bmn}{(m+1)(n+1)}; \qquad C' = \frac{Cn(n-1)}{(n+1)^2};\\
D' &= \frac{2mA}{(m+1)^2} + \frac{mB}{2(m+1)(n+1)} + \frac{mD}{(m+1)};\\
E' &= \frac{nB}{2(m+1)(n+1)} + \frac{2nC}{(n+1)^2} + \frac{nE}{(n+1)};\\
F' &= \frac{A}{3(m+1)^2} + \frac{B}{4(m+1)(n+1)} + \frac{C}{3(n+1)^2} + \frac{E}{2(n+1)} + \frac{D}{2(m+1)} + F.
\end{aligned}$$

The theorem below provides a necessary and sufficient condition for a conic equation specifying a circle to also specify a circle under the two-variable Kantorovich operators with equal diagonals and even indices.

Theorem 3.3. *Let the conic equation $G(x,y) = 0$ given by (2.1) with $A = C = 1$ denote a circle and let $m \in \mathbb{N}_2$. In this case, the necessary and sufficient condition for the equation $\mathbb{K}_{m,m}(G;x,y) = 0$ to define a circle is that*

$$m\delta_0 + 2\delta_1 + \frac{8}{3}\left(1 - \left(\frac{m}{m+1}\right)^2\right) > 0.$$

Proof. The proof is seen from the Theorem 2.3, since Lemma 2.1 and Lemma 3.1 are same for $m = n$ and $B = 0$. □

In the theorem given below, it is shown that a conic equation defining a circle transforms into an equation defining an ellipse under the double-indexed two-variable Kantorovich operators.

Theorem 3.4. *Let the equation $G(x,y) = 0$ given by (2.1) with $A = C = 1$ denote a circle and let $m, n \in \mathbb{N}_2$. Then, if $m \neq n$ then the equation $\mathbb{K}_{m,n}(G;x,y) = 0$ denotes an ellipse.*

Proof. Since $A = C = 1$ and $B = 0$, the following equations are obtained by Lemma 3.2:

$$\begin{aligned}
A' &= \frac{m(m-1)}{(m+1)^2}; \qquad B' = 0; \qquad C' = \frac{n(n-1)}{(n+1)^2};\\
D' &= \frac{2m}{(m+1)^2} + \frac{mD}{(m+1)}; \qquad E' = \frac{2n}{(n+1)^2} + \frac{nE}{(n+1)}\\
F' &= \frac{1}{3(m+1)^2} + \frac{1}{3(n+1)^2} + \frac{E}{2(n+1)} + \frac{D}{2(m+1)} + F.
\end{aligned}$$

Thus, we have

$$B'^2 - 4A'C' = -4A'C' = -4\Big(\frac{m(m-1)}{(m+1)^2}\Big)\Big(\frac{n(n-1)}{(n+1)^2}\Big) < 0$$

for all $n, m \in \mathbb{N}_2$. Since $m \neq n$, the equation $\mathbb{K}_{m,n}(G; x, y) = 0$ defines an ellipse. □

Theorem 3.5. *Let the equation $G(x, y) = 0$ given by (2.1) with $A = C = 1$ define a circle and let $m, n \in \mathbb{N}$. Then, if $m \neq n$ and $\min\{m, n\} = 1$, then the equation $\mathbb{K}_{m,n}(G; x, y) = 0$ defines a parabola.*

Proof. Let $m = 1$ and $n > 1$. In this case, if we consider that $A = C = 1$ and $B = 0$, then from Lemma 3.2

$$A' = B' = 0; \qquad C' = \frac{n(n-1)}{(n+1)^2}; \qquad D' = \frac{D+1}{2};$$
$$E' = \frac{2n}{(n+1)^2} + \frac{nE}{(n+1)}; \quad F' = \frac{1}{3(n+1)^2} + \frac{E}{2(n+1)} + \frac{1}{12}\big(12F + 3D + 1\big)$$

are obtained. Thus, $\mathbb{K}_{m,n}(G; x, y) = 0$ defines a parabola, since $B'^2 - 4A'C' = 0$. □

Example 3.1. *Considering the circle equation $G(x, y) = x^2 + y^2 + 3x - 8y - 2 = 0$, for each $m, n \in \mathbb{N}_2$ and since $m \neq n$, this equation defines an ellipse (see Figure 3).*

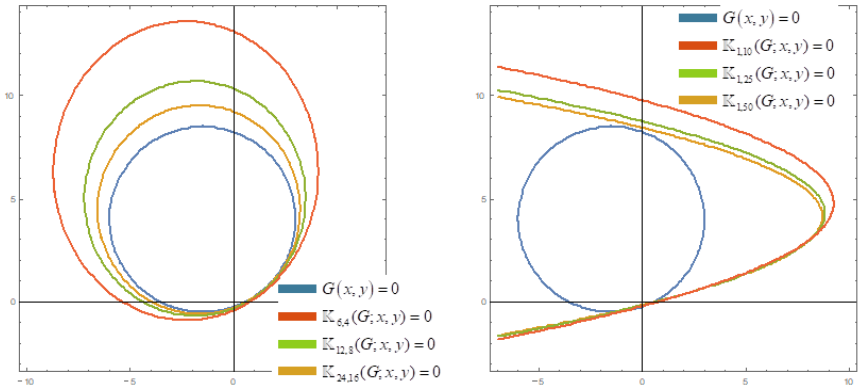

FIGURE 3. Images of the circle in Example 3.1 under the operator $\mathbb{K}_{m,n}$ for $(m, n) = (6, 4), (12, 8), (24, 16)$ (on left); $(m, n) = (1, 10), (1, 25), (1, 50)$ for images (on right)

Theorem 3.6. *Let the equation $G(x, y) = 0$ given by (2.1) denote an ellipse. Then*

(1) *For each $m, n \in \mathbb{N}_2$ satisfying the inequality $\frac{B^2}{4AC} < \frac{(m-1)(n-1)}{mn}$ the equation $\mathbb{K}_{m,n}(G; x, y) = 0$ denotes an ellipse.*
(2) *For each $m, n \in \mathbb{N}_2$ satisfying the inequality $\frac{B^2}{4AC} > \frac{(m-1)(n-1)}{mn}$ the equation $\mathbb{K}_{m,n}(G; x, y) = 0$ denotes an hyperbola.*
(3) *For each $m, n \in \mathbb{N}_2$ satisfying the inequality $\frac{B^2}{4AC} = \frac{(m-1)(n-1)}{mn}$ the equation $\mathbb{K}_{m,n}(G; x, y) = 0$ denotes a parabola.*

Proof. Since the conic equation $G(x,y) = 0$ given by (2.1) represents an ellipse, the inequality $B^2 - 4AC < 0$ holds, so that the inequality $AC > 0$ is true. Then the following equations are obtained by Lemma 3.2 for every $m, n \in \mathbb{N}_2$:

$$\begin{aligned} B'^2 - 4A'C' &= \left(\tfrac{m^2n^2}{(m+1)^2(n+1)^2}\right)B^2 - 4AC\left(\tfrac{m(m-1)}{(m+1)^2}\right)\left(\tfrac{n(n-1)}{(n+1)^2}\right) \\ &= \tfrac{m^2n^2}{(m+1)^2(n+1)^2}\left[B^2 - 4AC\tfrac{(m-1)(n-1)}{mn}\right] \end{aligned}$$

Since $B'^2 - 4A'C'$ and $B^2 - 4AC\frac{(m-1)(n-1)}{mn}$ have the same sign, we have the desired results in the theorem. □

Example 3.2. *If we consider the ellipse defined by the equation* $G(x,y) = 3x^2 + 6xy + 5y^2 - 6x + 4y - 3 = 0$, *we obtain*

$$B^2 - 4AC = (6)^2 - 4(3)(5) = -24 < 0 \qquad \text{and} \qquad \frac{B^2}{4AC} = \frac{(6)^2}{4(3)(5)} = \frac{3}{5}.$$

If $m = 30$, $n = 20$ *is taken into account, the equation* $\mathbb{K}_{m,n}(G;x,y) = 0$ *denotes an ellipse, since* $\frac{B^2}{4AC} = \frac{3}{5} < \frac{(m-1)(n-1)}{mn} = \frac{551}{600}$.

If $m = 3$, $n = 2$ *is taken into account, the equation* $\mathbb{K}_{m,n}(G;x,y) = 0$ *denotes a hyperbola, since* $\frac{B^2}{4AC} = \frac{3}{5} > \frac{(m-1)(n-1)}{mn} = \frac{1}{3}$.

Finally, if $m = 4$, $n = 5$ *is taken into account, the equation* $\mathbb{K}_{m,n}(G;x,y) = 0$ *denotes a parabola, since* $\frac{B^2}{4AC} = \frac{3}{5} = \frac{(m-1)(n-1)}{mn}$, *(see Figure 4).*

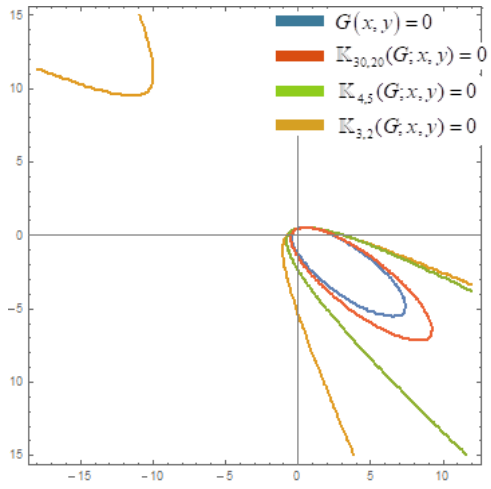

FIGURE 4. Images of the ellipse in example 3.2 under the operator $\mathbb{K}_{m,n}$ for $(m,n) = (3,2), (4,5), (30,20)$

Theorem 3.7. *Let the equation* $G(x,y) = 0$ *given by (2.1) denote an ellipse. Then, if* $m \neq n$ *and* $\min\{m,n\} = 1$ *then the equation* $\mathbb{K}_{m,n}(G;x,y) = 0$

(1) *denotes a parabola when* $B = 0$,
(2) *denotes a hyperbola when* $B \neq 0$.

Proof. Since it does not violate generality, we can take $m = 1$ and $n > 1$. In this case, the equations in Lemma 3.2 are as follows:

$$\begin{aligned} A' &= 0; \qquad B' = \frac{Bn}{2(n+1)}; \qquad C' = \frac{Cn(n-1)}{(n+1)^2}; \\ D' &= \frac{A+D}{2} + \frac{B}{4(n+1)}; \qquad E' = \frac{nB}{4(n+1)} + \frac{2nC}{(n+1)^2} + \frac{nE}{(n+1)}; \\ F' &= \frac{A+3D}{12} + \frac{B+4E}{8(n+1)} + \frac{C}{3(n+1)^2} + F. \end{aligned}$$

Since $B'^2 - 4A'C' = \left(\frac{n^2}{4(n+1)^2}\right)B^2$, the equation $\mathbb{K}_{m,n}(G;x,y) = 0$ defines a parabola for $B = 0$ while defines a hyperbola for $B \neq 0$. □

Example 3.3. (1) *If the ellipse specified by the equation* $G(x,y) = 2x^2 + 3y^2 - 3x + y - 1 = 0$ *is considered, since* $B = 0$ *the equation* $\mathbb{K}_{1,n}(G;x,y) = 0$ *specify a parabola for each* $n > 1$ *(see the figure on the left of Figure 5)*

(2) *If we consider the ellipse defined by the equation* $G(x,y) = 2x^2 + 3xy + 3y^2 - 3x + y - 1 = 0$, *since* $B \neq 0$,*the equation* $\mathbb{K}_{1,n}(G;x,y) = 0$ *specify a hyperbola for each* $n > 1$. *(see the figure on the right of Figure 5)*

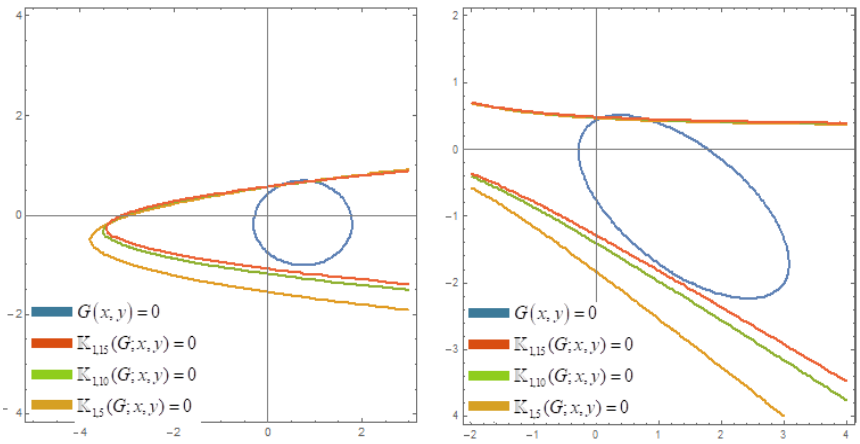

FIGURE 5. Images of the ellipses in Example 3.3 under the operator $\mathbb{K}_{m,n}$ for $(m,n) = (1,5), (1,10), (1,15)$

Theorem 3.8. *Let the conic equation* $G(x,y) = 0$ *given by (2.1) denote a hyperbola. Then, the equation* $\mathbb{K}_{m,n}(G;x,y) = 0$ *denotes a hyperbola for each* $m,n \in \mathbb{N}_2$.

Proof. Since the conic equation $G(x,y) = 0$ given by (2.1) denotes a hyperbola, the inequality $B^2 - 4AC > 0$ holds. Then the following equations are obtained by Lemma 3.2 for every $m,n \in \mathbb{N}_2$:

$$B'^2 - 4A'C' = \left(\frac{m^2n^2}{(m+1)^2(n+1)^2}\right)B^2 - 4AC\left(\frac{m(m-1)}{(m+1)^2}\right)\left(\frac{n(n-1)}{(n+1)^2}\right)$$

If $AC \leq 0$ then $B'^2 - 4A'C' > 0$ and therefore the equation $\mathbb{K}_{m,n}(G;x,y) = 0$ defines a hyperbola for each $m,n \in \mathbb{N}_2$. If $AC > 0$ then

$$B'^2 - 4A'C' = \frac{m^2n^2}{(m+1)^2(n+1)^2}\left[B^2 - 4AC\frac{(m-1)(n-1)}{mn}\right] > B^2 - 4AC > 0,$$

hence the equation $\mathbb{K}_{m,n}(G;x,y) = 0$ again defines a hyperbola for all $m,n \in \mathbb{N}_2$. □

Example 3.4. *If the hyperbola specified by the equation* $G(x,y) = x^2 + 6xy + y^2 - 5x + 3y - 2 = 0$ *is considered, then the equation* $\mathbb{K}_{m,n}(G;x,y) = 0$ *specifies a hyperbola for each* $m,n \in \mathbb{N}_2$ *(see Figure 6)*

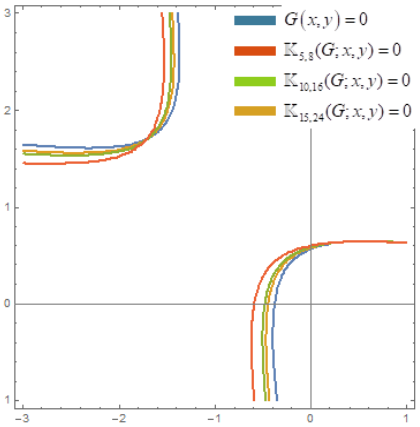

FIGURE 6. Images of the hyperbolas in Example 3.4 under the operator $\mathbb{K}_{m,n}$ for $(m,n) = (5,8),(10,16),(15,24)$

Theorem 3.9. *Let the conic equation* $G(x,y) = 0$ *given by (2.1) denote a parabola. Then,*

(1) *the equation* $\mathbb{K}_{m,n}(G;x,y) = 0$ *denotes a hyperbola for all* $m,n \in \mathbb{N}_2$, *if* $AC \neq 0$.
(2) *the equation* $\mathbb{K}_{m,n}(G;x,y) = 0$ *denotes a parabola for all* $m,n \in \mathbb{N}_2$, *if* $AC = 0$.

Proof. Since the conic equation $G(x,y) = 0$ given by (2.1) represents a parabola, the equality $B^2 - 4AC = 0$ is satisfied. Therefore, $AC \geq 0$ and A and C cannot be zero at the same time. Then the following equations are obtained by Lemma 3.2 for every $m,n \in \mathbb{N}_2$:

$$B'^2 - 4A'C' = \Big(\frac{m^2n^2}{(m+1)^2(n+1)^2}\Big)B^2 - 4AC\Big(\frac{m(m-1)}{(m+1)^2}\Big)\Big(\frac{n(n-1)}{(n+1)^2}\Big)$$

If $AC > 0$, then

$$B'^2 - 4A'C' = \frac{m^2n^2}{(m+1)^2(n+1)^2}\Big[B^2 - 4AC\Big(\frac{(m-1)(n-1)}{mn}\Big)\Big] > B^2 - 4AC = 0$$

so the equation $\mathbb{K}_{m,n}(G;x,y) = 0$ defines a hyperbola for all $m,n \in \mathbb{N}_2$.

If $AC = 0$ then $B = 0$ and therefore $B'^2 - 4A'C' = 0$. Thus, the equation $\mathbb{K}_{m,n}(G; x, y) = 0$ defines a parabola. □

Remark. *Let the equation $G(x, y) = 0$ given by (2.1) define a parabola and let $AC > 0$. Then, for all $m, n \in \mathbb{N}$ such that $m \neq n$ and $\min\{m, n\} = 1$, the equation $\mathbb{K}_{m,n}(G; x, y) = 0$ also defines a hyperbola. Because under these conditions $B'^2 - 4A'C' \geq B^2/16 > 0$.*

Example 3.5.
(1) *If the parabola defined by the equation $G(x, y) = 4x^2 - 8xy + 4y^2 - x + 3y - 3 = 0$ is considered, since $AC = 16 \neq 0$, then the equation $\mathbb{K}_{m,n}(G; x, y) = 0$ specifies a hyperbola for each $m, n \in \mathbb{N}_2$ (see Figure 7).*
(2) *If the parabola specified by the equation $G(x, y) = 4y^2 - x + 3y - 3 = 0$ is considered, since $AC = 0$, then the equation $\mathbb{K}_{m,n}(G; x, y) = 0$ specifies a parabola for each $m, n \in \mathbb{N}_2$ (see Figure 8).*

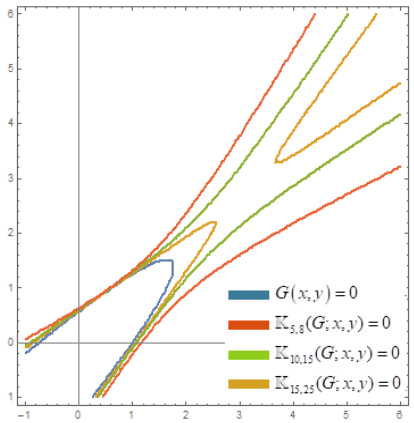

FIGURE 7. Example 3.5, images of parabola in (1) under $\mathbb{K}_{m,n}$ operator for $(m, n) = (5, 8), (10, 15), (15, 25)$

4. Conclusion

The equation $\mathbb{K}_{m,n}(G; x, y) = 0$ is a quadratic equation in two variables, where $G(x, y) = 0$ is the equation of a conic and $\mathbb{K}_{m,n}$ is the even-indexed two-variable Kantorovich operator. If the equation $G(x, y) = 0$ describes a circle, then for sufficiently large m the equations $\mathbb{K}_{m,m}(G; x, y) = 0$ also describe circles. However, when the indices are different, that is, for $m \neq n$ the equations $\mathbb{K}_{m,n}(G; x, y) = 0$ describe ellipses or parabolas. More clearly, in the infinite matrix of equations $\left(\mathbb{K}_{m,n}(G; x, y) = 0\right)_{m,n=1}^{\infty}$ all equations in the first column and first row represent a parabola, equations in the diagonal represent a circle, and other coordinates represent an ellipse.

If the conic equation $G(x, y) = 0$ represents an ellipse, then the conic equations $\mathbb{K}_{m,n}(G; x, y) = 0$ may in some cases represent an ellipse, parabola or hyperbola. However, if the infinite matrix of conic equations $\left(\mathbb{K}_{m,n}(f; x, y) = 0\right)_{m,n=1}^{\infty}$ is

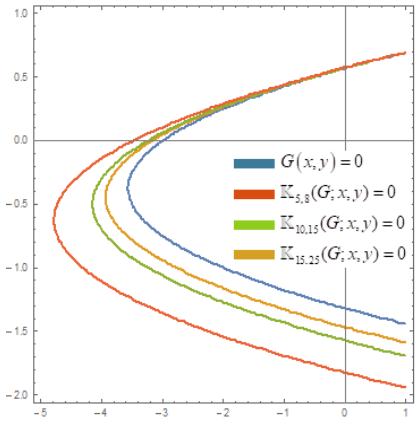

FIGURE 8. Example 3.5, images of parabola in (2) for $(m,n) = (5,8),(10,15),(15,25)$ $\mathbb{K}_{m,n}$ operator under images

considered, all equations in the first column and first row can represent a parabola or hyperbola, depending on the coefficient B in the equation $G(x,y) = 0$.

If the conic equation $G(x,y) = 0$ represents a hyperbola, then the conic equations $\mathbb{K}_{m,n}(G;x,y) = 0$ also represent a hyperbola. However, in the matrix $\left(\mathbb{K}_{m,n}(G;x,y) = 0\right)_{m,n=1}^{\infty}$, all equations in the first column and first row can define a parabola when the coefficient B in the equation $G(x,y) = 0$ is equal to zero.

If the conic equation $G(x,y) = 0$ defines a parabola, then the conic equations $\mathbb{K}_{m,n}(G;x,y) = 0$ define either a hyperbola or a parabola, depending on whether the product of the coefficients A and C in the equation $G(x,y) = 0$ is zero or not. However, in the matrix $\left(\mathbb{K}_{m,n}(G;x,y) = 0\right)_{m,n=1}^{\infty}$, all equations in the first row and first column specify a hyperbola.

The equation $\mathbb{K}_m(G;x,y) = 0$ is a second-order two-variable equation, where $G(x,y) = 0$ is a general conic equation and $\mathbb{K}_m$ is a single-index version of the two-variable Kantorovich operators. If $G(x,y) = 0$ denotes a circle, then for all sufficiently large m, the equation $\mathbb{K}_m(G;x,y) = 0$ denotes a circle. If the conic equation $G(x,y) = 0$ denotes an ellipse, a hyperbola, and a parabola, respectively, then the equations $\mathbb{K}_m(G;x,y) = 0$ denote an ellipse, a hyperbola, and a parabola, respectively. From these results, it is clear that the $\mathbb{K}_m$ Kantorovich operators preserve the geometry of the conic equations.

In the light of the above discussions, many different two-variable operators in the literature can be considered and examined whether the conic equations preserve their type.

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