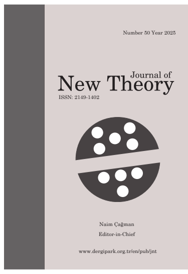
Journal of New Theory
dergipark.org.tr/tr/pub/jnt
Open Access



ISSN: 2149-1402 (51) (2025) 65-75 doi.org/10.53570/jnt.1696750

Sharp B-Maximal Function Estimates and Boundedness for Some Integral Operators to the Inequalities

Cansu Keskin [1], Havva Nur Turkak[2]

Article Info
Received: 10 May 2025
Accepted: 26 Jun 2025
Published: 30 Jun 2025
Research Article

Abstract— In this paper, we first establish the relation between B-maximal and sharp B-maximal functions generated by the generalized translation operator connected with the Laplace-Bessel differential operator. We then prove some sharp B-maximal function estimates and present an application using these sharp estimates to study singular integral operators. We finally obtain the boundedness of the Littlewood-Paley g-function related to the Laplace-Bessel differential operator on generalized B-Morrey spaces.

Keywords — *B-Morrey space, Laplace-Bessel differential operator, Littlewood-Paley g-function, sharp B-maximal function*

Mathematics Subject Classification (2020) 42B25, 42B35

1. Introduction

In 1938, Morrey [1] introduced the classical Morrey spaces, an extension of the classical Lebesgue spaces. In Morrey spaces, numerous researchers have studied the boundedness and compactness properties of maximal and singular integral operators. Due to the applications in the study of Morrey spaces, this space has aroused widespread interest and curiosity [2]. Thus far, many papers have focused on various Morrey spaces. They extended Morrey spaces to different settings. For example, Guliyev [3, 4], Sawano [5], and Nakai [6] introduced the generalized Morrey spaces. Moreover, they investigated the similar boundedness problems of maximal and singular integral operators in these spaces.

Additionally, weighted inequalities are crucial in Fourier analysis and have numerous applications in solving boundedness problems for certain integral operators. In particular, weight theory is critical in studying boundary value problems for the Laplace equation on Lipschitz regions. Muckenhoupt's characterization provides the foundation for defining the class A_p and developing weighted inequalities, ensuring that the Hardy–Littlewood maximal operator maps the weighted Lebesgue space $L^p(w) \equiv L^p(\mathbb{R}^n, w)$ onto itself.

The study of the Littlewood-Paley g-theory enjoys a natural motivation and great interest. Many works and topics have been studied. To study the dyadic decomposition of Fourier series, Littlewood and Paley [7–9] introduced the g-function of one dimension. The function g is basic in the Littlewood-Paley theory of Fourier series [10]. Littlewood and Paley proved that

[1]cansu.keskin@medeniyet.edu.tr (Corresponding Author); [2]havvanurturkak65@gmail.com
[1]Department of Mathematics, Faculty of Engineering and Natural Sciences, Istanbul Medeniyet University, İstanbul, Türkiye
[2]Ministry of Education, İstanbul, Türkiye

$$A_p\|f\|_p \leq \|g(f)\|_p \leq B_p\|f\|_p \tag{1.1}$$

where on the left side of the above inequality, it was assumed that $\int\limits_0^{2\pi} f(\theta)d\theta = 0$. Later, Stein [11] defined the following n-dimensional form of the Littlewood-Paley g-function and obtained the same norm inequality as (1.1),

$$g(f)(x) = \left(\int\limits_0^\infty t|\nabla u(x,t)|^2 dt\right)^{1/2}$$

where $u(x,t) = P_t * f(x)$ denotes the Poisson integral of f. Afterward, many mathematicians have studied Littlewood-Paley g-function of higher dimensions with more general kernels [12–16].

Over the past 30 years, considerable developments have been made to extend the classical Littlewood-Paley g-function to some different settings. Akbulut et al. [17] are interested in problems related to weighted inequalities for the g-Littlewood-Paley functions associated with the Laplace-Bessel differential operators. However, in [18], they establish some sharp maximal function estimates for certain Toeplitz-type operators (including the Littlewood-Paley operator) associated with some fractional integral operators with a general kernel. Moreover, Lerner [19] has established sharp weighted estimates for any convolution Calderón-Zygmund operator, for all $1 < p \leq 3/2$ and $3 \leq p < \infty$.

Highly inspired by [12–19], in this paper, we are interested in problems related to weighted inequalities for the Littlewood-Paley g-functions connected with the Laplace-Bessel differential operators Δ_ν. Moreover, we are motivated by the work of Akbulut et al. [20] in which there is a different setting of the Littlewood-Paley g-function has different settings. We obtained a similar Fefferman-Stein boundedness result for this operator on generalized B-Morrey spaces by utilizing B-sharp maximal functions related to the Laplace-Bessel operator.

The rest of the paper is organized as follows: Section 2 presents the basic notations needed throughout this paper. Section 3 concerns maximal functions related to the Laplace-Bessel differential operator and its properties. Section 4 defines the integral operator g. The last section indicates the boundedness of this integral operator on generalized B-Morrey spaces.

2. Preliminaries

This section presents the basic notations and concepts to be required. Throughout this paper, let Q denote a cube of $\mathbb{R}^n_+$, the upper half region of $\mathbb{R}^n$, n dimensional Euclidean space with sides parallel to the axes and $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$. Moreover, let $E(x,t) = \{y \in \mathbb{R}^n_+ \ ; \ |x-y| < t\}$ and $E(x,t)^c = \mathbb{R}^n_+ \backslash E(x,t)$. If E is a Lebesgue measurable set, then χ_E is the characteristic function of E, and the weighted Lebesgue measure of E denoted by $|E|_\nu$, where $|E|_\nu = \int\limits_E x_n^\nu dx$ such that $\nu > 0$. Besides, let $|E(0,r)|_\nu = w(n,\nu)r^{n+\nu}$, where

$$w(n,\nu) = \int\limits_{E(0,1)} x_n^\nu dx = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{2\Gamma\left(\frac{n+\nu-2}{2}\right)}$$

The weight w is a nonnegative locally integrable function on $\mathbb{R}^n_+$ that takes values in $(0, +\infty)$ almost everywhere.

Let the class $A_{p,\nu}$ consist of those weights w for which

$$\left(\frac{1}{|E|_\nu}\int\limits_E w(x)x_n^\nu\,dx\right)^{1/p}\left(\frac{1}{|E|_\nu}\int\limits_E w(x)^{-p'/p}\,x_n^\nu dx\right)^{1/p'}\le C$$

Here, p' is the dual of p such that $\frac{1}{p}+\frac{1}{p'}=1$, for $1<p<\infty$. The class $A_{1,\nu}$ is defined replacing the above inequality by

$$\frac{1}{|E|_\nu}\int\limits_E w(x)\,x_n^\nu dx\le C\operatorname*{ess\,inf}_{x\in E}w(x)$$

for every ball $E\subseteq\mathbb{R}_+^n$. Further, let $A_\infty=\bigcup\limits_{1\le p<\infty}A_{p,\nu}$. It is well known from [21] that if $w\in A_{p,\nu}$ with $1\le p<\infty$ (or $w\in A_\infty$), then w satisfies the doubling condition; that is, for all ball E, there exists an absolute constant $C>0$ such that $w(2E)\le C\,w(E)$. Furthermore, if $w\in A_\infty$, then for all ball B and all measurable subset E of a ball B, there exists a number $\delta>0$ independent of E and B such that $\frac{w(E)}{w(B)}\le C\left(\frac{|E|}{|B|}\right)^\delta$ [21]. Given a weight function w on $\mathbb{R}_+^n$, the weighted Lebesgue space $L_{p,w,\nu}(\mathbb{R}_+^n)$, for $1\le p<\infty$, is defined as the set of all functions f for which

$$\|f\|_{L_{p,w,\nu}}:=\left(\int\limits_{\mathbb{R}_+^n}|f(x)|^p w(x)\,x_n^\nu dx\right)^{1/p}<\infty$$

In addition, let $WL_{p,w,\nu}(\mathbb{R}^n)$, $1\le p<\infty$, denote the weighted weak Lebesgue space consisting of all measurable functions f such that

$$\|f\|_{WL_{p,w,\nu}}:=\sup_{\lambda>0}\lambda\Big[w\big(\{x\in\mathbb{R}^n:|f(x)|>\lambda\}\big)\Big]^{1/p}<\infty$$

Additionally, the generalized translate operator T^y is defined by

$$T^y f(x)=c_\nu\int\limits_0^\pi\dots\int\limits_0^\pi f\left(x'-y',(x_n,y_n)_\theta\right)\,d\mu(\theta)$$

where $c_\nu=\dfrac{\pi^{-\frac{1}{2}}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$, $(x_n,y_n)_\theta=\sqrt{x_n^2-2x_ny_n\cos\theta+y_n^2}$, and $d\mu\,(\theta)=\sin^{\nu-1}\theta\;\,d\theta$.

The operator T^y acts from $L_p(\mathbb{R}_+^n,d\mu)$ to $L_p(\mathbb{R}_+^n,d\mu)$ and satisfies the conditions $||T^yf||_p<||f||_p$, $T^y1=1$, and L_p-boundedness. We remark that the generalized translate operator T^y is closely related to the Laplace-Bessel differential operator Δ_ν defined by

$$\Delta_\nu=\sum_{i=1}^{n-1}\frac{\partial^2}{\partial x_i^2}+B\quad\text{ where }\quad B=\frac{\partial^2}{\partial x_n^2}+\frac{\nu}{x_n}\frac{\partial}{\partial x_n}\quad\text{ such that }\quad\nu>0$$

For $n=1$ and $n>1$, see [22–26]. The generalized translate operator T^y generates the corresponding B-convolution

$$(f\otimes g)(x)=\int\limits_{\mathbb{R}_+^n}f(y)T^yg(x)y_n^\nu dy$$

for which the following Young inequality holds:

$$\|f\otimes g\|_{L_{r,\nu}}\le\|f\|_{L_{p,\nu}}\|g\|_{L_{q,\nu}}$$

such that $1\le p,q\le r\le\infty$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$.

Lemma 2.1. [27] For all $x \in \mathbb{R}^n_+$, the following equality holds:

$$\int_{E_t} T^y g(x) y_n^\nu dy = \int_{E((x,0),t)} g\left(z', \sqrt{z_n^2 + \overline{z}_n^2}\right) d\mu(z', \overline{z_n})$$

where $E((x,0),t) = \{(z, \overline{z_n}) \in \mathbb{R}^n \times (0,\infty) : |(x - z, \overline{z_n})| < t\}$.

Lemma 2.2. [27] For all $x \in \mathbb{R}^n_+$, the following equality holds:

$$\int_{\mathbb{R}^n_+} T^y g(x) \varphi(y) M_\nu \chi_{E_r}(y) y_n^\nu dy = \int_{\mathbb{R}^n \times (0,\infty)} g\left(z', \sqrt{z_n^2 + \overline{z}_n^2}\right) \varphi(z', \overline{z_n}) M_\nu \chi_{E((x,0),r)}(z', \overline{z_n}) d\mu(z', \overline{z_n})$$

Lemmas 2.1 and 2.2 can be obtained via the following substitutions: $z' = x'$, $z_n = y_n \cos\theta$, and $\overline{z_n} = y_n \sin\alpha$, where $0 \le \theta < \pi$, $y \in \mathbb{R}^n_+$, and $(z, \overline{z_n}) \in \mathbb{R}^n \times (0,\infty)$. Let $\mathcal{S}'_+ = \mathcal{S}'_+\left(\mathbb{R}^n_+\right)$ denote the topological dual of $\mathcal{S}_+$, the collection of all tempered distributions on $\mathbb{R}^n_+$.

Definition 2.3. [27,28] Let ω be a positive measurable weight function. Then, $\mathcal{M}_{p,\omega,\nu}(\mathbb{R}^n_+)$ denotes the generalized B-Morrey spaces as the set of all locally integrable functions f with finite quasi-norm

$$||f||_{\mathcal{M}_{p,\omega,\nu}} = \sup_{x \in \mathbb{R}^n_+, r>0} \left(\frac{r^{-\frac{n+\nu}{p}}}{\omega(r)} \int_{E(0,r)} T^y[|f(x)|]^p \, y_n^\nu dy \right)^{\frac{1}{p}} < \infty$$

Note that

i. if $\omega(r) = r^{-\frac{n+\nu}{p}}$, then $\mathcal{M}_{p,\omega,\nu}(\mathbb{R}^n_+) \equiv L^p_\nu(\mathbb{R}^n_+)$.

ii. if $\omega(r) = r^{\frac{\lambda-n-\nu}{p}}$ and $0 \le \lambda < n+\nu$, then $\mathcal{M}_{p,\omega,\nu}(\mathbb{R}^n_+) = \mathcal{M}_{p,\lambda,\nu}(\mathbb{R}^n_+)$.

3. B-Maximal Functions

This section includes a modified version of the sharp maximal operator, as introduced by Fefferman and Stein [29]. A variant of the sharp maximal function, namely the sharp B-maximal function $M_\nu^\# f$, associated with the Laplace–Bessel differential operator, was introduced in [30] as follows:

$$M_\nu^\# f(x) = \sup_{x \in Q} \inf_c \frac{1}{|Q|_\nu} \int_Q \left| T^y f(x) - c \right| y_n^\nu \, dy \approx \sup_{x \in Q} \frac{1}{|Q|_\nu} \int_Q |T^y f(x) - f_Q| y_n^\nu \, dy$$

Here, $f_Q = \frac{1}{|Q|_\nu} \int_Q T^y f(x) y_n^\nu \, dy$ denotes the average of f over E. Moreover, for $\delta > 0$,

$$M_{\nu,\delta}^\# f(x) = M_\nu^\# \left(|f|^\delta\right)(x)^{\frac{1}{\delta}}$$

which is useful for the sharp B-maximal operator below. We denote the Hardy-Littlewood maximal function, i.e., B-maximal function, by $M_\nu f$, defined as follows [30]:

$$M_\nu f(x) = \sup_{r>0} \frac{1}{|Q|_\nu} \int_Q T^y |f(x)| y_n^\nu \, dy$$

It is well known that the B-maximal function provides control over the mean value of a function concerning any radially decreasing function in $L_{1,\nu}$. Moreover, boundedness estimates for M_ν can be established in the framework of generalized B-Morrey spaces.

Theorem 3.1. [20] Let $1 \le p < \infty$ and ω be positive measurable weight function on $\mathbb{R}^n_+$ satisfying the condition

$$\int_{t}^{\infty} \omega(x,\tau)\frac{d\tau}{\tau} \leq C\omega(x,t) \tag{3.1}$$

where the constant C is independent of x and t. Then, for $p > 1$, the maximal operator M_ν is bounded on $\mathcal{M}_{p,\omega,\nu}(\mathbb{R}^n_+)$ and for $p = 1$, it is bounded on $\mathcal{M}_{1,\omega,\nu}(\mathbb{R}^n_+)$.

The proof can be obtained using a similar way to the one employed in the proof of Theorem 4.1 in [20].

The following inequalities, inspired by the work of Fefferman and Stein [29], will be used in the remainder of this section.

Lemma 3.2. [29] Let $1 \leq p < \infty$ and ω be an A_∞ weight. Then, there exists a constant C such that the following inequality holds for every function f for which the left-hand side is finite:

$$\int_{\mathbb{R}^n_+} |M_\nu f(x)|^p \omega(x)\ x_n^\nu\, dx \leq C \int_{\mathbb{R}^n_+} \left|M_\nu^{\#} f(x)\right|^p \omega(x)\ x_n^\nu\, dx$$

4. Littlewood-Paley g-Function

This section is devoted to defining and investigating the Littlewood-Paley g-function associated with the Laplace-Bessel differential operator in the generalized B-Morrey space $\mathcal{M}_{p,w,\nu}(\mathbb{R}^n_+)$.

Definition 4.1. [31] Let $f \in S(\mathbb{R}^n_+)$, the space of infinitely differentiable functions on $\mathbb{R}^n_+$ that decrease rapidly at infinity together with all their derivatives. For $t > 0$, the Poisson-type integral $u_t(f)$ is defined by

$$u(f)(x,t) = u_t(f)(x) := \int_{\mathbb{R}^n_+} p_t(y)\, T^y f(x)\, y_n^\nu dy$$

where $x \in \mathbb{R}^n_+$ and p_t denotes the Poisson-type kernel given by

$$p_t(x) = p(x,t) = c_\nu \frac{t}{(t^2+|x|^2)^{\frac{n+\nu+1}{2}}} \quad \text{where} \quad c_\nu = \frac{2^{n+\nu}\Gamma\left(\frac{n+\nu+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

Recall that the Poisson-type integral $u_t(f)$, defined by $u_t(f) = (p_t \otimes f)(x)$, is a B-convolution type singular integral operator and satisfies the following properties:

Proposition 4.2. [20] Let $f \in S(\mathbb{R}^n_+)$ be a positive function and $p > 1$. Then,

i. $u_t(x) \leq C(t^2+|x|^2)^{\left(\frac{n-\nu+3}{2}\right)}$

ii. $\dfrac{\partial u}{\partial t}(x) \leq Ct^{n-\nu+4}$

iii. $\dfrac{\partial u}{\partial x_i}(x) \leq C(t^2+|x|^2)^{\left(\frac{n-\nu+4}{2}\right)}$, for all $1 \leq i \leq n$

iv. $(p_t * f)(x) \leq M_\nu f(x)$

where $M_\nu f$ is the B-maximal function.

Definition 4.3. [20] Let $f \in S(\mathbb{R}^n_+)$. Then, a g-function associated with the Laplace-Bessel differential operators is defined by

$$g(f)(x) = \left(\int_0^\infty |\nabla u_t(x)|^2\, t\ dt\right)^{1/2} \tag{4.1}$$

where $x \in \mathbb{R}^n_+$, u_t is the Poisson-type integral, and $|\nabla_t(x)|^2 = \left|\frac{\partial u}{\partial t}(x)\right|^2 + \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}(x)\right|^2$.

Theorem 4.4. [20] Let $1 < p < 2$ and ω be positive measurable weight function on $\mathbb{R}^n_+ \times [0,\infty)$ satisfying (3.1). Then, there exists a positive constant $C_{p,\nu}$ such that for all $f \in \mathcal{M}_{p,\omega,\nu}(\mathbb{R}^n_+)$,

$$\|g(f)\|_{\mathcal{M}_{p,\omega,\nu}} \leq C_{p,\nu}\|f\|_{\mathcal{M}_{p,\omega,\nu}}$$

The proof follows from the proof of Theorem 6.1 in [20].

5. Main Results

This section primarily employs similar techniques to those in [32, 33] to derive the following lemmas, which play a crucial role in establishing the boundedness of the Littlewood-Paley g-function. Afterward, it provides the main result, Theorem 5.5. This section considers the Littlewood-Paley g-function with the same kind of kernel as in (4.1). This leads to the following definition:

Definition 5.1. Let $\varepsilon > 0$ and φ be a fixed function satisfying the following properties:

i. $\int\limits_{\mathbb{R}^n_+} \varphi(x)x_n^\nu dx = 0$

ii. $|T^y\varphi(x)| \leq T^y|\varphi(x)| \leq C(1+|x|)^{-(n+\nu+1)}$

iii. If $2|y| < |x|$, then $|T^y\varphi(x) - \varphi(y)| \leq C|y|^\varepsilon(1+|x|)^{-(n+\nu+1+\varepsilon)}$

Here, $C > 0$ is a constant independent of x. Thus, the Littlewood-Paley g-function is defined by

$$g(f)(x) = \left(\int\limits_0^\infty |\varphi_t \otimes f(x)|^2 \frac{dt}{t}\right)^{1/2}$$

where φ_t is the dilation of φ given by $\varphi_t(x) = t^{n+\nu}\varphi(\frac{x}{t})$.

Lemma 5.2. Let $1 < p < \infty$ and $0 < D < 2^{n+\nu}$. Then, for any smooth function f for which the left-hand side is finite,

$$||M_\nu f||_{\mathcal{M}_{p,\varphi,\nu}} \leq C||M_\nu^\# f||_{\mathcal{M}_{p,\varphi,\nu}}$$

PROOF. For any cube $Q = Q(x,r)$ in $\mathbb{R}^n_+$, $M_\nu(\chi_Q)(x) \in A_{1,\nu}$ by [34]. It must be noted that $M_\nu(\chi_Q)(x) \leq 1$. By Lemma 3.2, for $f \in \mathcal{M}_{p,\omega,\nu}(\mathbb{R}^n_+)$,

$$\begin{aligned}
\int\limits_Q T^y|M_\nu f(x)|^p(x)\, y_n^\nu\, dy &= \int\limits_{\mathbb{R}^n_+} T^y|M_\nu f(x)|^p(x)\ (\chi_Q)(x)\ y_n^\nu\, dy \\
&\leq \int\limits_{\mathbb{R}^n_+} T^y|M_\nu f(x)|^p(x) M_\nu(\chi_Q)(x)\ y_n^\nu\, dy \\
&= \int\limits_{\mathbb{R}^n_+} |M_\nu f(y)|^p\ T^y M_\nu(\chi_Q)(x)\ y_n^\nu\, dy \\
&\leq C \int\limits_{\mathbb{R}^n_+} |M_\nu^\# f(y)|^p T^y M_\nu(\chi_Q)(x)\ y_n^\nu\, dy \\
&\leq C \Bigg[\int\limits_Q |M_\nu^\# f(y)|^p T^y M_\nu(\chi_Q)(x) y_n^\nu dy \\
&\quad + \sum_{k=0}^\infty \int\limits_{2^{k+1}Q\setminus 2^kQ} |M_\nu^\# f(y)|^p T^y M_\nu(\chi_Q)(x) y_n^\nu\, dy\Bigg]
\end{aligned}$$

$$\leq C\left[\int_Q T^y|M_\nu^{\#}f(x)|^p\, y_n^\nu\, dy + \sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ} T^y|M_\nu^{\#}f(x)|^p\, \frac{|Q|_\nu}{|2^{k+1}Q|_\nu}\, y_n^\nu\, dy\right]$$

$$\leq C\left[\int_Q T^y|M_\nu^{\#}f(x)|^p\, y_n^\nu\, dy + \sum_{k=0}^{\infty}\int_{2^{k+1}Q} T^y|M_\nu^{\#}f(x)|^p\, 2^{-k(n+\nu)}\, y_n^\nu\, dy\right]$$

$$\leq C||T^y(M_\nu^{\#}(f))||^p_{\mathcal{M}_{p,\varphi,\nu}}\sum_{k=0}^{\infty}2^{-k(n+\nu)}\varphi(2^{k+1}r)$$

$$\leq C||T^y(M_\nu^{\#}(f))||^p_{\mathcal{M}_{p,\varphi,\nu}}\sum_{k=0}^{\infty}(2^{-(n+\nu)}D)^k\varphi(r)$$

$$\leq C||T^y(M_\nu^{\#})(f)||^p_{\mathcal{M}_{p,\varphi,\nu}}\varphi(r)$$

Thus,

$$\left(\frac{1}{\varphi(r)}\int_Q T^y|M_\nu f(x)|^p x_n^\nu dx\right)^{1/p} \leq C\left(\frac{1}{\varphi(r)}\int_Q T^y|M_\nu^{\#}f(x)|^p x_n^\nu dx\right)^{1/p}$$

Hence,

$$||M_\nu(f)||_{\mathcal{M}_{p,\varphi,\nu}} \leq C||M_\nu^{\#}(f)||_{\mathcal{M}_{p,\varphi,\nu}}$$

□

Theorem 5.3. [35] Let T be a convolution-type singular integral operator. Then, there exists a constant $C > 0$ for $\omega \in A_{1,\nu}$, for $0 < p < 1$, and for every ball Q such that

$$\int_Q |Tf|^p\omega(x)x_n^\nu\, dx \leq C(n,p,[\omega])_{A_{1,\nu}}\omega(E)^{1-p}\left(\int_{\mathbb{R}_+^n}|f(x)|\omega(x)x_n^\nu\, dx\right)^p$$

Lemma 5.4. Let $0 < \delta < 1$. Then, there exists a constant $C > 0$ only depending on δ such that

$$M_\nu^{\#}(g(f))(x) \leq CM_\nu(f)(x) \tag{5.1}$$

where

$$M_\nu^{\#}(g(f))(x) = \left(\sup_{x\in Q}\inf_{c\in\mathbb{R}}\frac{1}{|Q|_\nu}\int_Q \Big|T^y|g(f)|^\delta - |c|^\delta\Big|y_n^\nu\, dy\right)^{\frac{1}{\delta}}$$

Proof. Let $x \in \mathbb{R}_+^n$ and Q be a cube containing x. To obtain (5.1), it suffices to show that

$$\left(\frac{1}{|Q|_\nu}\int_Q \Big||T^y[g(f)]|^\delta - |c|^\delta\Big|\, y_n^\nu\, dy\right)^{\frac{1}{\delta}} \leq CM_\nu f(x)$$

for some constant c to be determined. Using the inequality $\Big||u|^\delta - |v|^\delta\Big| \leq |u-v|^\delta$ such that $0 < \delta < 1$, define

$$\Big(g(f)\Big)_Q = \frac{1}{|Q|_\nu}\int_Q T^y\Big(g(f)\Big)(x)y_n^\nu dy$$

Denote f as $f = f_1 + f_2$, where $f_1 = \lambda_{2Q}$. We will show that $c = \Big(g(f_2)\Big)_Q$ satisfies the required inequality. By the linearity of the Littlewood-Paley operator $g(f)$,

$$\begin{aligned}
\mathcal{M}^{\#}_{\nu,\delta}([g(f)])(x) &:= \left(\sup_Q \mathcal{M}^{\#}_{\nu}\Big([g(f)]^{\delta}\Big)\right)^{\frac{1}{\delta}} \\
&= \left(\frac{1}{|Q|_\nu}\int_Q \left|\left|T^y\Big(g(f)\Big)(x)\right|^{\delta} - |c|^{\delta}\right| y_n^{\nu}\, dy\right)^{\frac{1}{\delta}} \\
&\leq C\left[\left(\frac{1}{|Q|_\nu}\int_Q \left|g\Big(T^y f_1(x)\Big)\right|^{\delta} y_n^{\nu}\, dy\right)^{\frac{1}{\delta}} + \left(\frac{1}{|Q|_\nu}\int_Q \left|g\Big(T^y f_2(x)\Big) - c\right|^{\delta} y_n^{\nu}\, dy\right)^{\frac{1}{\delta}}\right] \\
&:= C(I_1 + I_2)
\end{aligned}$$

First, we show the estimate I_1. For $0 < \delta < 1$, applying Theorem 5.3 (Kolmogorov's estimate of Bessel type),

$$I_1 = \left(\frac{1}{|Q|_\nu}\int_Q \left|g\Big(T^y f(x)\Big)\right|^{\delta} y_n^{\nu}\, dy\right)^{\frac{1}{\delta}} \leq \frac{1}{|Q|_\nu}|Q|_\nu^{1-\delta}\left[\left(\int_Q |T^y f_1(x)|\, y_n^{\nu}\, dy\right)^{\delta}\right]^{\frac{1}{\delta}} \leq CM_\nu f(x)$$

For the estimate of I_2, if $|x-y| > 2r$, by the Jensen inequality and Fubini's theorem for integrals,

$$\begin{aligned}
I_2 &= \frac{1}{|Q|_\nu}\int_Q \left|T^y\left(g(f_2)(x)\right) - \Big(g(f_2)\Big)_Q\right| y_n^{\nu}\, dy \\
&= \frac{1}{|Q|_\nu}\int_Q \left|T^y\Big(gf_2(x)\Big) - \frac{1}{|Q|_\nu}\int_Q T^y\Big(g(f_2)(x)\Big) z_n^{\nu}\, dz\right| y_n^{\nu}\, dy \\
&\leq \frac{1}{|Q|_\nu}\int_Q \left(\frac{1}{|Q|_\nu}\int_Q \left|g\Big(T^y f_2(x)\Big) - gf_2(y)\right| z_n^{\nu}\, dz\right) y_n^{\nu}\, dy \\
&\leq \frac{1}{|Q|_\nu}\int_Q \frac{1}{|Q|_\nu}\left(\int_Q \left|\int_{\mathbb{R}^n_+} \varphi(\tau)T^{\tau}\Big(T^y f_2(x)\Big)\tau_n^{\nu}\, d\tau - \int_{\mathbb{R}^n_+} \varphi(\tau)T^{\tau} f_2(x)\tau_n^{\nu}\, d\tau\right| z_n^{\nu}\, dz\right) y_n^{\nu}\, dy \\
&\leq \frac{1}{|Q|_\nu}\int_Q \frac{1}{|Q|_\nu}\left(\int_Q \left|\int_{\mathbb{R}^n_+} \varphi(\tau)T^{y}T^{\tau} f_2(x)\tau_n^{\nu}\, d\tau - \int_{\mathbb{R}^n_+} \varphi(\tau)T^{\tau} f_2(x)\tau_n^{\nu}\, d\tau\right| z_n^{\nu}\, dz\right) y_n^{\nu}\, dy \\
&\leq \frac{1}{|Q|_\nu}\int_Q \frac{1}{|Q|_\nu}\left(\int_Q \left|\int_{\mathbb{R}^n_+} T^{y}\varphi(\tau)T^{\tau} f_2(x)\tau_n^{\nu}\, d\tau - \int_{\mathbb{R}^n_+} \varphi(\tau)T^{\tau} f_2(x)\tau_n^{\nu}\, d\tau\right| z_n^{\nu}\, dz\right) y_n^{\nu}\, dy
\end{aligned}$$

$$
\leq \frac{1}{|Q|_\nu} \int\limits_Q \frac{1}{|Q|_\nu} \left(\left| \int\limits_Q \left[T^y \varphi(\tau) - \varphi(\tau) \right] T^\tau f_2(x) \tau_n^\nu \, d\tau \right| z_n^\nu \, dz \right) y_n^\nu \, dy
$$

$$
\leq \frac{1}{|Q|_\nu^2} \int\limits_Q z_n^\nu \, dz \int\limits_Q \left[\int\limits_Q \left| T^y \varphi(\tau) - \varphi(\tau) \right| y_n^\nu \, dy \right] T^\tau f_2(x) \tau_n^\nu \, d\tau
$$

$$
\leq C \frac{1}{|Q|_\tau} \int\limits_Q T^\tau |f_2(x)| \tau_n^\nu \, d\tau \leq C M_\nu f(x)
$$

□

Theorem 5.5. Let $1 \leq p < \infty$ and $\omega \in A_{1,\nu}(\mathbb{R}_+^n)$. Then, there exists a positive constant $C > 0$ such that

$$
||g(f)||_{\mathcal{M}_{p,\omega,\nu}} \leq C \|f\|_{\mathcal{M}_{p,\omega,\nu}}
$$

The proof can be easily observed from Lemmas 5.2 and 5.4.

6. Conclusion

This paper presents a Fefferman-Stein type boundedness result for the Littlewood-Paley g-operator on generalized B-Morrey spaces by utilizing B-sharp maximal functions. The importance and fundamental difference of this paper lie in its use of different transformations (generalized transformations) of the obtained results. Future studies can extend this work to encompass multilinear analogues of the results presented here. B-Sharp maximal function estimates for multilinear singular integrals and their commutators can be constructed. In addition, their boundedness properties can be investigated on generalized B-Morrey spaces.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the second author's undergraduate thesis supervised by the first author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

References

[1] C. B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Transactions of American Mathematical Society 43 (1938) 126–166.

[2] F. Chiarenza, M. Frasca, *Morrey spaces and Hardy-Littlewood maximal function*, Rendiconti del Seminario Matematico della Università di Padova 7 (1987) 273–279.

[3] V. S. Guliyev, *Integral operators on function spaces on the homogeneous groups and on domains in* $\mathbb{R}^n$, Doctoral Dissertation Steklov Mathematical Institute (1994) Moscow.

[4] V. S. Guliyev, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, Journal of Inequalities and Applications 2009 (2009) 1–20.

[5] Y. Sawano, *A thought on generalized Morrey spaces*, Journal of The Indonesian Mathematical Society 25 (3) (2019) 210–281.

[6] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces*, Mathematische Nachrichten 166 (1994) 95–10.

[7] J. E. Littlewood, R. E. A. C. Paley, *Theorems on Fourier series and power series*, Journal of the London Mathematical Society 6 (1931) 230–233.

[8] J. E. Littlewood, R. E. A. C. Paley, *Theorems on Fourier series and power series (II)*, Proceedings of the London Mathematical Society 42 (1) (1936) 52–89.

[9] J. E. Littlewood, R. E. A. C. Paley, *Theorems on Fourier series and power series (III)*, Proceedings of the London Mathematical Society 43 (1937) 105–126.

[10] A. Zygmund, Trigonometric series, 3rd Edition, Cambridge University Press, 2002.

[11] E. M. Stein, Topics in harmonic analysis related to the Littlewood-Paley Theory, Princeton University Press, 1970.

[12] E. M. Stein, S. Wainger, *Problems in harmonic analysis related to curvature*, Bulletin of the American Mathematical Society 84 (1978) 1239–1295.

[13] E. M. Stein, *The development of square functions in the work of A. Zygmund*, Bulletin of the American Mathematical Society 7 (1982) 359–376.

[14] M. H. Taibleson, *Harmonic analysis on n-dimensional vector spaces over local fields*, Mathematische Annalen 187 (1970) 259–271.

[15] A. Uchiyama, *Characterization of* $H^p(\mathbb{R}^n)$ *in terms of generalized Littlewood-Paley g-functions*, Studia Mathematica 81 (1985) 135–158.

[16] S. L. Wang, *Boundedness of the Littlewood-Paley g-function on* $Lip_\alpha(\mathbb{R}^n)$ $(0 < \alpha < 1)$, Illinois Journal of Mathematics 33 (1989) 531–541.

[17] A. Akbulut, V. S. Guliyev, M. Dziri, *Weighted norm inequalities for the g-Littlewood-Paley operators associated with Laplace-Bessel differential operators*, Mathematical Inequalities and Applications 17 (1) (2014) 317–333.

[18] D. Chen, H. Huang, *Sharp function estimates and boundedness for Toeplitz-type operators associated with general fractional integral operators*, Open Mathematics 19 (2021) 1554–1566.

[19] A. K. Lerner, *Sharp weighted norm inequalities for Littlewood-Paley operators and singular integrals*, Advances in Mathematics 226 (2011) 3912–3926.

[20] A. Akbulut, M. Dziri, I. Ekincioglu, *Maximal function and fractional integral associated with Laplace-Bessel differential operators on generalized Morrey spaces*, Transactions Issue Mathematics. National Academy of Sciences of Azerbaijan. Series of Physical-Technical & Mathematical Sciences 42 (1) (2022) 8–25.

[21] J. García-Cuerva, J. R. De Francia, Weighted norm inequalities and related topics, Elsevier B.V., 1985.

[22] B. M. Levitan, *Bessel function expansions in series and Fourier integrals*, Uspekhi Matematicheskikh Nauk 6 (2) (1951) 102–143.

[23] I. A. Kipriyanov, *Fourier–Bessel transforms and imbedding theorems for weight classes*, Trudy Matematicheskogo Instituta imeni V. A. Steklova 89 (1967) 130–213.

[24] L. N. Lyakhov, *Multipliers of the mixed Fourier-Bessel transform*, Proceedings of the Steklov Institute of Mathematics 214 (1997) 234–249.

[25] A. Serbetci, I. Ekincioglu, *On boundedness of Riesz potential generated by generalized translate operator on Ba spaces*, Czechoslovak Mathematical Journal 54 (3) (2004) 579–589.

[26] E. L. Shishkina, S. M. Sitnik, Transmutations, singular and fractional differential equations with applications to mathematical physics, Elsevier, 2020.

[27] I. Ekincioglu, J. J. Hasanov, C. Keskin, *On the boundedness of B-Riesz potential and its commutators on generalized weighted B-Morrey spaces*, Hacettepe Journal of Mathematics and Statistics 53 (2) (2023) 321–332.

[28] V.S. Guliyev, *Sobolev theorems for anisotropic Riesz-Bessel potentials on Morrey-Bessel spaces*, Doklady Akademii Nauk 367 (2) (1999) 155–156.

[29] C. Fefferman, E. M. Stein, *H^p spaces of several variables*, Acta Mathematica 129 (1972) 137–193.

[30] I. Ekincioglu, C. Keskin, A. B. Ozyurt Serim, *New estimates for the B-Riesz transform*, Filomat 39 (11) (2025) 3729–3740.

[31] C. Keskin, I. Ekincioglu, V. S. Guliyev, *Characterizations of Hardy spaces associated with Laplace-Bessel operators*, Analysis and Mathematical Physics 9 (4) (2019) 2281–2310.

[32] G, Sheng, H. Chuangxia, L. Lanzhe, *Sharp maximal function estimates and boundedness for commutator related to generalized fractional integral operator*, Analele Universitatii de Vest, Timişoara Seria Matematica-Informatica L 2 (2012) 97–115.

[33] Q. Xue, X. Peng, K. Yabuta, *On the theory of multilinear Littlewood-Paley g-function*, Journal of the Mathematical Society of Japan 67 (2) (2015) 535–559.

[34] R. R. Coifman, R. Rochberg, *Another characterization of BMO*, Proceedings of the American Mathematical Society 79 (2) (1980) 249–254.

[35] I. Ekincioglu, E. Kaya, *Bessel type Kolmogorov inequalities on weighted Lebesgue spaces*, Applicable Analysis 100 (8) (2021) 1634–1643.