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FOURIER METHOD FOR INVERSE COEFFICIENT EULER-BERNOULLI BEAM EQUATION

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Abstract. In this study, we find the inverse coefficient in the Euler-Bernoulli beam equation with over determination conditions. We show the existence, stability of the solution by iteration method.

1. Introduction

Mathematical modeling of sound wave distribution problems and also the vibration, buckling and dynamic behavior of various building elements widely used in nanotechnology are formulated with following Euler-Bernoulli beam equations

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}=f(x,t,u)$$

Due to the new and exceptionally its electronic and mechanical properties, carbon nanotubes are considered to be one of the most useful material in future. Nowadays, nanotubes are used as atomic force microscopy, nanofillers for nanomotors, nanobearings and nanosprings [1, 2, 3]. These elements are tackled by different boundary conditions depending on different loading conditions. Therefore, investigation used in the mathematical modeling of the structural components of nanomaterials continues to be a focus of interest amongst mathematicians.

In mathematics, the classical statement of Euler-Bernoulli beam equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}=0$$

is used for beam vibration equation.

As well as the homogeneous equation, quasilinear and non-linear equations can be handled in this case. Various problems for equations of this type were investigated and many results have been obtained in different ways. The practical advantages of remote sensing are what make the inverse problems important in [4, 6, 11].The

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investigation of various problems concerning 4 th order homogeneous, linear and quasi-linear equations has been one of the most attractive areas for mathematicians and engineers due to their importance in the solution of several engineering problems. The reader is refereed to [9, 10, 12] for some relevant previous work on linear and quasi-linear equations, applications.

The periodic boundary conditions are used many area [5]. Periodic boundary conditions are used in molecular dynamics simulations to avoid problems with boundary effects caused by finite size, and make the system more like an infinite one, at the cost of possible periodicity effects,heat transfer, life sciences,on lunar theory. A liquid, in the thermodynamic limit, would occupy an infinite volume. It is common experience that one can perfectly well obtain the thermodynamic properties of a material from a more modest sample. However, even a droplet has more atoms or molecules than one can possibly hope to introduce into ones computer simulation. Thus to simulate a bulk sample of liquid it is common practice to use a trick known as periodic boundary conditions [7, 8].

Let $T > 0$ be fixed number and denote by $\Gamma := \{0 < x < \pi,\ 0 < t < T\}$. Let $\{g(t), u(x,t)\}$ satisfying the following equations

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} - g(t)u = f(x,t,u), (x,t)\epsilon\ \Gamma \tag{1}$$

$$\begin{aligned} u(0,t) &= u(\pi,t), u_x(0,t) = u_x(\pi,t) \\ u_{xx}(0,t) &= u_{xx}(\pi,t), u_{xxx}(0,t) = u_{xxx}(\pi,t), t\epsilon\,[0,T] \end{aligned} \tag{2}$$

$$\begin{aligned} u(x,0) &= \varphi(x), \\ u_t(x,0) &= \psi(x)\ , x\epsilon\,[0,\pi] \end{aligned} \tag{3}$$

$$H(t) = \int_0^\pi u(x,t)dx, t\epsilon\,[0,T] \tag{4}$$

Here $\Gamma := \{0 < x < \pi,\ 0 < t < T\}$, $\varphi(x)\epsilon\,[0,\pi]$ and $f(x,t,u)\epsilon\ \overline{\Gamma} \times (-\infty,\infty)$.

Definition 1. $\{g(t), u(x,t)\} \in C[0,T] \times (C^{2,1}(\Gamma) \cap C^{1,0}(\overline{\Gamma}))$ *is called the classical solution .*

Definition 2. $w(x,t) \in C(\overline{\Gamma})$ *is referred test function that gives the following conditions:*

$w(T,x) = w_t(T,x) = 0$, $w(0,t) = w(\pi,t)$, $w_x(0,t) = w_x(\pi,t)$, $w_{xx}(0,t) = w_{xx}(\pi,t)$, $w_{xxx}(0,t) = w_{xxx}(\pi,t)$, $t\epsilon\,[0,T]$.

Definition 3. $u(x,t) \in C(\overline{\Gamma})$ *is named generalized solution that gives the following equation:*

$$\int_0^T \int_0^\pi \left(\left\{ \frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} - g(t)w \right\} u - fw \right) dxdt - \int_0^\pi w(x,0)\psi(x)dx$$
$$+ \int_0^\pi w_t(x,0)\varphi(x)dx = 0$$

2. Solution of Euler Bernoulli Beam Equation

(C1): $H(t) \in C^1[0,T]$;
(C2): $\varphi(x) \in C^3[0,\pi]$, $\varphi(0) = \varphi(\pi)$, $\varphi'(0) = \varphi'(\pi)$, $\varphi''(0) = \varphi''(\pi), H(0) = \int_0^\pi \varphi(x)dx$,
(C3): $f(x,t,u)$ is provided the following conditions:

(1)

$$\left| \frac{\partial^{(n)} f(x,t,u)}{\partial x^n} - \frac{\partial^{(n)} f(x,t,\tilde{u})}{\partial x^n} \right| \leq b(x,t) \left| u - \tilde{u} \right| \;, n = 0,1,2,$$

where $b(x,t) \in L_2(\Gamma)$ is Fourier coefficient $(b(x,t) \geq 0)$,
(2) $f(x,t,u) \in C^3[0,\pi]$, $t \in [0,T]$,
(3) $f(x,t,u)|_{x=0} = f(x,t,u)|_{x=\pi}$, $f_x(0,t,u)|_{x=0} = f_x(\pi,t,u)|_{x=\pi}$, $f_{xx}(0,t,u)|_{x=0} = f_{xx}(\pi,t,u)|_{x=\pi}$.

By Fourier method,

$$\begin{aligned}
u_0 &= \varphi_0 + \psi_0 t + \frac{2}{\pi} \int_0^t \int_0^\pi (t-\tau) F(\xi,\tau,g,u) d\xi d\tau \\
u_{ck} &= \varphi_{ck} \cos(2k)^2 t + \frac{\psi_{ck}}{(2k)^2} \sin(2k)^2 t \\
&\quad + \frac{2}{\pi(2k)^2} \int_0^t \int_0^\pi F(\xi,\tau,g,u) \sin(2k)^2(t-\tau) \cos 2k\xi d\xi d\tau \\
u_{sk} &= \varphi_{sk} \cos(2k)^2 t + \frac{\psi_{sk}}{(2k)^2} \sin(2k)^2 t \\
&\quad + \frac{2}{\pi(2k)^2} \int_0^t \int_0^\pi F(\xi,\tau,g,u) \sin(2k)^2(t-\tau) \sin 2k\xi d\xi d\tau.
\end{aligned}$$

Let $F(x,t,g,u) = g(t)u(x,t) + f(x,t,u)$.

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\varphi_0 + \psi_0 t + \frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau)\Big(g(\tau)u(\xi,\tau) + f(\xi,\tau,u)\Big)\,d\xi d\tau\right] \\ &+ \sum_{k=1}^{\infty}\cos 2kx\left[\varphi_{ck}\cos(2k)^2 t + \frac{\psi_{ck}}{(2k)^2}\sin(2k)^2 t\right] \\ &+ \sum_{k=1}^{\infty}\frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi \Big(g(\tau)u(\xi,\tau) + f(\xi,\tau,u)\Big) \\ &\qquad\qquad \times \sin(2k)^2(t-\tau)\cos 2k\xi\cos 2kx d\xi d\tau \\ &+ \sum_{k=1}^{\infty}\sin 2kx\left[\varphi_{sk}\cos(2k)^2 t + \frac{\psi_{sk}}{(2k)^2}\sin(2k)^2 t\right] \\ &+ \sum_{k=1}^{\infty}\frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi \Big(g(\tau)u(\xi,\tau) + f(\xi,\tau,u)\Big) \\ &\qquad\qquad \times \sin(2k)^2(t-\tau)\sin 2k\xi\sin 2kx d\xi d\tau \end{aligned} \tag{5}$$

where

$$\begin{aligned} \varphi_0 &= \frac{2}{\pi}\int_0^\pi \varphi(\tau)d\tau, \varphi_{ck} = \frac{2}{\pi}\int_0^\pi \varphi(\tau)\cos 2k\tau d\tau, \varphi_{sk} = \frac{2}{\pi}\int_0^\pi \varphi(\tau)\sin 2k\tau d\tau, \\ \psi_0 &= \frac{2}{\pi}\int_0^\pi \psi(\tau)d\tau, \psi_{ck} = \frac{2}{\pi}\int_0^\pi \psi(\tau)\cos 2k\tau d\tau, \psi_{sk} = \frac{2}{\pi}\int_0^\pi \psi(\tau)\sin 2k\tau d\tau, \\ f_0(t,u) &= \frac{2}{\pi}\int_0^\pi f(\tau,t,u)d\tau, f_{ck}(t,u) = \frac{2}{\pi}\int_0^\pi f(\tau,t,u)\cos 2k\tau d\tau, \\ f_{sk}(t,u) &= \frac{2}{\pi}\int_0^\pi f(\tau,t,u)\sin 2k\tau d\tau, k = 1,2,3,... \end{aligned}$$

Under the condition (C1)-(C3), differentiating (4), we obtain

$$\int_0^\pi u_{tt}(x,t)dx = H^{''}(t), 0 \le t \le T. \tag{6}$$

From (5) and (6), we have

$$g(t)=\frac{H^{''}(t)-\int\limits_{0}^{\pi} f(\xi,t,u)\,d\xi}{H(t)} \tag{7}$$

Let $\{u(t)\}=\{u_0(t),u_{ck}(t),u_{sk}(t),k=1,...,n\}$ is satisfied that

$$\max_{0\le t\le T}\frac{|u_0(t)|}{2}+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{ck}(t)|+\max_{0\le t\le T}|u_{sk}(t)|\right)<\infty,\ by\ \mathbf{B}.$$

$$\|u(t)\|=\max_{0\le t\le T}\frac{|u_0(t)|}{2}+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}|u_{ck}(t)|+\max_{0\le t\le T}|u_{sk}(t)|\right),$$

where $\mathbf{B}$ is Banach space.

Theorem 4. *Let the assumptions (C1)-(C3) be provided. Then the problem (1)-(4) has a unique solution.*

Proof. Iteration to equation (5), we get

$$u_0^{(N+1)}(t)=u_0^{(0)}(t) \tag{8}$$
$$+\frac{2}{\pi}\int\limits_0^t\int\limits_0^\pi (t-\tau)\left(g^{(N)}(\tau)u^{(N)}(\xi,\tau)+f(\xi,\tau,u^{(N)})\right)d\xi d\tau$$

$$u_{ck}^{(N+1)}(t)=u_{ck}^{(0)}(t)+\frac{2}{\pi(2k)^2}\int\limits_0^t\int\limits_0^\pi\left(g^{(N)}(\tau)u^{(N)}(\xi,\tau)+f(\xi,\tau,u^{(N)})\right)$$
$$\times\sin(2k)^2(t-\tau)\cos 2k\xi d\xi d\tau,$$

$$u_{sk}^{(N+1)}(t)=u_{sk}^{(0)}(t)$$
$$+\frac{2}{\pi(2k)^2}\int\limits_0^t\int\limits_0^\pi\left(g^{(N)}(\tau)u^{(N)}(\xi,\tau)+f(\xi,\tau,u^{(N)})\right)\sin(2k)^2(t-\tau)\sin 2k\xi\ d\xi d\tau,$$

$$\begin{aligned}
u_0^{(0)}(t) &= \varphi_0+\psi_0 t,\\
u_{ck}^{(0)}(t) &= \varphi_{ck}\cos(2k)^2t+\frac{\psi_{ck}}{(2k)^2}\sin(2k)^2t,\\
u_{sk}^{(0)}(t) &= \varphi_{sk}\cos(2k)^2t+\frac{\psi_{sk}}{(2k)^2}\sin(2k)^2t.
\end{aligned}$$

$$g^{(N+1)}(t)=\frac{H^{''}(t)-\int\limits_0^\pi f\left(\xi,t,u^{(N)}\right)d\xi}{H(t)}$$

From assumptions of the theorem, we have $u^{(0)}(t) \in \mathbf{B}$, $t \in [0,T]$.

$$u_0^{(1)}(t) = u_0^{(0)}(t) + \frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau)\left(g^{(0)}(\tau)u^{(0)}(\xi,\tau) + f(\xi,\tau,u^{(0)})\right)d\xi d\tau$$

After applying Cauchy inequalities, we have

$$\begin{aligned}\left|u_0^{(1)}(t)\right| \le\ & \left|u_0^{(0)}(t)\right| + \frac{2}{\pi}\left(\int_0^t\int_0^\pi (t-\tau)^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi \left(g^{(0)}(\tau)u^{(0)}(\xi,\tau)\right)^2 d\xi d\tau\right)^{\frac{1}{2}} \\ & +\frac{2}{\pi}\left(\int_0^t\int_0^\pi (t-\tau)^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi \left[f(\xi,\tau,u^{(0)}) - f(\xi,\tau,0)\right]^2 d\xi d\tau\right)^{\frac{1}{2}} \\ & +\frac{2}{\pi}\left(\int_0^t\int_0^\pi (t-\tau)^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t\int_0^\pi f^2(\xi,\tau,0) d\xi d\tau\right)^{\frac{1}{2}}\end{aligned}$$

After applying Lipschitz inequalities and taking the maximum of both sides of the last inequalities consecutively, we get

$$\begin{aligned}\max_{0\le t\le T}\left|u_0^{(1)}(t)\right| \le\ & |\varphi_0| + T|\psi_0| + 2\sqrt{\frac{T^3}{3\pi}}\left\|u^{(0)}(t)\right\|_B\left\|g^{(0)}(t)\right\|_{C[0,T]} \\ & +2\sqrt{\frac{T^3}{3\pi}}\left\|u^{(0)}(t)\right\|_B\|b(x,t)\|_{L_2(\Gamma)} + 2\sqrt{\frac{T^3}{3\pi}}\|f(x,t,0)\|_{L_2(\Gamma)}.\end{aligned}$$

$$\begin{aligned}u_{ck}^{(1)}(t) =\ & u_{ck}^{(0)}(t) + \frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi \left(g^{(0)}(\tau)u^{(0)}(\xi,\tau) + f(\xi,\tau,u^{(0)})\right) \\ & \times\sin(2k)^2(t-\tau)\cos 2k\xi d\xi d\tau\end{aligned}$$

After applying Cauchy inequalities,we have

$$\begin{aligned}\left|u_{ck}^{(1)}(t)\right| \le\ & \left|u_{ck}^{(0)}(t)\right| + \frac{2}{\pi(2k)^2}\left(\int_0^t\int_0^\pi \left(\sin(2k)^2(t-\tau)\right)^2 d\tau\right)^{\frac{1}{2}} \\ & \times\left(\int_0^t\int_0^\pi \left(g^{(0)}(\tau)u^{(0)}(\xi,\tau)\cos 2k\xi\right)^2 d\xi d\tau\right)^{\frac{1}{2}}\end{aligned}$$

$$+\frac{2}{\pi(2k)^2}\left(\int_0^t\int_0^\pi(\sin(2k)^2(t-\tau))^2d\tau\right)^{\frac{1}{2}}$$
$$\times\left(\int_0^t\int_0^\pi\left[f(\xi,\tau,u^{(0)})-f(\xi,\tau,0)\cos 2k\xi\right]^2d\xi d\tau\right)^{\frac{1}{2}}$$
$$+\frac{2}{\pi}(2k)^2\left(\int_0^t\int_0^\pi(\sin(2k)^2(t-\tau))^2d\tau\right)^{\frac{1}{2}}$$
$$\times\left(\int_0^t\int_0^\pi f^2(\xi,\tau,0)\cos^2 2k\xi d\xi d\tau\right)^{\frac{1}{2}}$$

Taking sum of both of side with respect to k $\left(k=\overline{1,\infty}\right)$ the last equations, we have

$$\begin{aligned}\left|u_{ck}^{(1)}(t)\right| \leq\ & \sum_{k=1}^{\infty}\left|u_{ck}^{(0)}(t)\right|+\frac{\sqrt{T}}{2\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\int_0^t\int_0^\pi\left(g^{(0)}(\tau)u^{(0)}(\xi,\tau)\cos 2k\xi\right)^2d\xi d\tau\right)^{\frac{1}{2}}\\ &+\frac{\sqrt{T}}{2\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\int_0^t\int_0^\pi\left[f(\xi,\tau,u^{(0)})-f(\xi,\tau,0)\cos 2k\xi\right]^2d\xi d\tau\right)^{\frac{1}{2}}\\ &+\frac{\sqrt{T}}{2\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\int_0^t\int_0^\pi f^2(\xi,\tau,0)\cos^2 2k\xi d\xi d\tau\right)^{\frac{1}{2}}\end{aligned}$$

Applying Hölder inequalities,we obtain

$$\begin{aligned}\left|u_{ck}^{(1)}(t)\right| \leq\ & \sum_{k=1}^{\infty}\left|u_{ck}^{(0)}(t)\right|+\frac{\sqrt{T}}{2\pi}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\\ &\times\left(\sum_{k=1}^{\infty}\int_0^t\int_0^\pi\left(g^{(0)}(\tau)u^{(0)}(\xi,\tau)\cos 2k\xi\right)^2d\xi d\tau\right)^{\frac{1}{2}}\\ &+\frac{\sqrt{T}}{2\pi}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\int_0^t\int_0^\pi\left[f(\xi,\tau,u^{(0)})-f(\xi,\tau,0)\cos 2k\xi\right]^2d\xi d\tau\right)^{\frac{1}{2}}\\ &+\frac{\sqrt{T}}{2\pi}\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty}\int_0^t\int_0^\pi f^2(\xi,\tau,0)\cos^2 2k\xi d\xi d\tau\right)^{\frac{1}{2}}\end{aligned}$$

where $\left(\sum\limits_{k=1}^{\infty} \frac{1}{k^2}\right)^{\frac{1}{2}} = \frac{\pi^2}{6}$.

Applying Bessel, Lipshitz conditions consecutively, we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} \max_{0\le t\le T}\left|u_{ck}^{(1)}(t)\right| \quad &\le \quad \sum_{k=1}^{\infty}|\varphi_{ck}| + \frac{\pi^2}{24}\sum_{k=1}^{\infty}|\psi_{ck}| \\ &+\frac{\pi\sqrt{T}}{12}\left\|u^{(0)}(t)\right\|_B \left\|g^{(0)}(t)\right\|_{C[0,T]} \\ &+\frac{\pi\sqrt{T}}{12}\left\|u^{(0)}(t)\right\|_B \left\|b(x,t)\right\|_{L_2(\Gamma)} + \frac{\pi\sqrt{T}}{12}\left\|f(x,t,0)\right\|_{L_2(\Gamma)},\end{aligned}$$

and by following the same approaches, we have

$$\begin{aligned}\sum_{k=1}^{\infty} \max_{0\le t\le T}\left|u_{sk}^{(1)}(t)\right| \quad &\le \quad \sum_{k=1}^{\infty}|\varphi_{sk}| + \frac{\pi^2}{24}\sum_{k=1}^{\infty}|\psi_{sk}| \\ &+\frac{\pi\sqrt{T}}{12}\left\|u^{(0)}(t)\right\|_B \left\|g^{(0)}(t)\right\|_{C[0,T]} \\ &+\frac{\pi\sqrt{T}}{12}\left\|u^{(0)}(t)\right\|_B \left\|b(x,t)\right\|_{L_2(\Gamma)} + \frac{\pi\sqrt{T}}{12}\left\|f(x,t,0)\right\|_{L_2(\Gamma)},\end{aligned}$$

$$\begin{aligned}\left\|u^{(1)}(t)\right\|_{\mathbf{B}} \quad &= \quad \max_{0\le t\le T}\frac{\left|u_0^{(1)}(t)\right|}{2} + \sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{ck}^{(1)}(t)\right| + \max_{0\le t\le T}\left|u_{sk}^{(1)}(t)\right|\right) \\ &\le \quad \frac{|\varphi_0|}{2} + \sum_{k=1}^{\infty}(|\varphi_{ck}| + |\varphi_{sk}|) + \frac{\pi^2}{24}\sum_{k=1}^{\infty}(|\psi_{ck}| + |\psi_{sk}|) \\ &+(2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6})\left\|u^{(0)}(t)\right\|_B \left\|g^{(0)}(t)\right\|_{C[0,T]} \\ &+(2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6})\left\|u^{(0)}(t)\right\|_B \left\|b(x,t)\right\|_{L_2(\Gamma)} \\ &+(2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6})\left\|f(x,t,0)\right\|_{L_2(\Gamma)}.\end{aligned}$$

From assumptions of the theorem, we have $u^{(1)}(t) \in \mathbf{B}$.

For the step N,

$$\begin{aligned}\left\|u^{(N+1)}(t)\right\|_{\mathbf{B}} &= \max_{0\le t\le T}\frac{\left|u_0^{(N)}(t)\right|}{2}+\sum_{k=1}^{\infty}\left(\max_{0\le t\le T}\left|u_{ck}^{(N)}(t)\right|+\max_{0\le t\le T}\left|u_{sk}^{(N)}(t)\right|\right)\\ &\le \frac{|\varphi_0|}{2}+\sum_{k=1}^{\infty}(|\varphi_{ck}|+|\varphi_{sk}|)+\frac{\pi^2}{24}\sum_{k=1}^{\infty}(|\psi_{ck}|+|\psi_{sk}|)\\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u^{(N)}(t)\right\|_B\left\|p^{(N)}(t)\right\|_{C[0,T]}\\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u^{(N)}(t)\right\|_B\|b(x,t)\|_{L_2(\Gamma)}\\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\|f(x,t,0)\|_{L_2(\Gamma)}.\end{aligned}$$

According to $u^{(N)}(t)\in\mathbf{B}$ and assumptions of the theorem, we get $u^{(N+1)}(t)\in\mathbf{B}$,

$$\{u(t)\}=\{u_0(t),u_{ck}(t),u_{sk}(t),\ k=1,2,...\}\in\mathbf{B}.$$

$$g^{(1)}(t)=\frac{H''(t)-\int_0^{\pi}f\left(\xi,t,u^{(0)}\right)d\xi}{H(t)}$$

After applying Cauchy inequalities, we have

$$\left|g^{(1)}(t)\right|\le\frac{\left|H''(t)\right|}{|H(t)|}+\frac{\left(\int_0^{\pi}d\xi\right)^{\frac{1}{2}}\left(\int_0^{\pi}\left(\left[f\left(\xi,t,u^{(0)}\right)-f\left(\xi,t,0\right)\right]\right)^2d\xi\right)^{\frac{1}{2}}}{|H(t)|}$$

After applying Lipschitz inequalities and taking the maximum of both sides of the last inequalities consecutively, we get

$$\begin{aligned}\left\|g^{(1)}(t)\right\|_{C[0,T]} &\le\left|\frac{H''(t)}{H(t)}\right|+\frac{\sqrt{\pi}}{|H(t)|}\left\|u^{(0)}(t)\right\|_B\|b(x,t)\|_{L_2(\Gamma)}\\ &+\frac{\sqrt{\pi}}{|H(t)|}\|f(x,t,0)\|_{L_2(\Gamma)}.\end{aligned}$$

From assumptions of the theorem, we have $g^{(1)}(t)\in\mathbf{B}$.

For the step N,

$$\begin{aligned}\left\|g^{(N+1)}(t)\right\|_{C[0,T]} &\le\left|\frac{H''(t)}{H(t)}\right|+\frac{\sqrt{\pi}}{|H(t)|}\left\|u^{(N)}(t)\right\|_B\|b(x,t)\|_{L_2(\Gamma)}\\ &+\frac{\sqrt{\pi}}{|H(t)|}\|f(x,t,0)\|_{L_2(\Gamma)}.\end{aligned}$$

According to $g^{(N)}(t) \in \mathbf{B}$ and assumptions of the theorem, we get $g^{(N+1)}(t) \in \mathbf{B}$. For $N \to \infty$, $u^{(N+1)}(t)$, $g^{(N+1)}$ are converged.

After applying Cauchy, Bessel,Lipschitz, Hölder inequalities consecutively,we obtain

$$\begin{aligned}\left\|u^{(1)}(t)-u^{(0)}(t)\right\|_B &\leq (2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u^{(0)}(t)\right\|_B\left\|g^{(0)}(t)\right\|_{C[0,T]}\\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u^{(0)}(t)\right\|_B\|b(x,t)\|_{L_2(\Gamma)}\\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\|f(x,t,0)\|_{L_2(\Gamma)}.\end{aligned}$$

$$\begin{aligned}A &= (2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u^{(0)}(t)\right\|_B\left\|g^{(0)}(t)\right\|_{C[0,T]}+\\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u^{(0)}(t)\right\|_B\|b(x,t)\|_{L_2(\Gamma)}+M).\end{aligned}$$

$$\left\|g^{(1)}(t)-g^{(0)}(t)\right\|_{C[0,T]} \leq \frac{\sqrt{\pi}}{|H(t)|}\left\|u^{(1)}(t)-u^{(0)}\right\|_B\|b(x,t)\|_{L_2(\Gamma)}.$$

$$\begin{aligned}&\left\|u^{(2)}(t)-u^{(1)}(t)\right\|_B\\ \leq\ &(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u^{(2)}(t)\right\|_{C[0,T]}\left\|g^{(2)}(t)-g^{(1)}(t)\right\|_{C[0,T]} \qquad (9)\\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|g^{(1)}(t)\right\|_B\left\|u^{(2)}-u^{(1)}\right\|_B\\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u^{(1)}-u^{(0)}\right\|_B\|b(x,t)\|_{L_2(\Gamma)}.\end{aligned}$$

$$\left\|g^{(2)}(t)-g^{(1)}(t)\right\|_{C[0,T]} \leq \frac{\sqrt{\pi}}{|H(t)|}\left\|u^{(1)}(t)-u^{(0)}\right\|_B\|b(x,t)\|_{L_2(\Gamma)}. \qquad (10)$$

Using (10),we obtain

$$\begin{aligned}\left\|u^{(2)}(t)-u^{(1)}(t)\right\|_B &\leq (2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\frac{\sqrt{\pi}}{|H(t)|}\left\|u^{(1)}(t)-u^{(0)}\right\|_B\|b(x,t)\|_{L_2(\Gamma)}\\ &\times\left\|u^{(2)}(t)\right\|_{C[0,T]}\\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|g^{(1)}(t)\right\|_B\left\|u^{(2)}-u^{(1)}\right\|_B\\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u^{(1)}-u^{(0)}\right\|_B\|b(x,t)\|_{L_2(\Gamma)}.\end{aligned}$$

$$\left\|u^{(2)}(t)-u^{(1)}(t)\right\|_B \leq (2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\frac{\sqrt{\pi}A}{|H(t)|\,D}\left\|b(x,t)\right\|_{L_2(\Gamma)}.$$

where

$$\begin{aligned} D &= 1-(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\frac{\sqrt{\pi}}{|H(t)|}\left\|b(x,t)\right\|_{L_2(\Gamma)}\left\|u^{(2)}(t)\right\|_{C[0,T]} \\ &\quad -(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|g^{(1)}(t)\right\|_B. \end{aligned}$$

For the step N,we get

$$\left\|g^{(N+1)}(t)-g^{(N)}(t)\right\|_{C[0,T]} \leq \frac{\sqrt{\pi}}{|H(t)|}\left\|u^{(N)}(t)-u^{(N-1)}\right\|_B\left\|b(x,t)\right\|_{L_2(\Gamma)}.$$

$$\left\|u^{(N+1)}(t)-u^{(N)}(t)\right\|_B \leq (2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\frac{\sqrt{\pi}A}{|H(t)|\,D\sqrt{N!}}\left\|b(x,t)\right\|^N_{L_2(\Gamma)}.$$

$u^{(N+1)} \to u^{(N)}$, $N\to\infty$, then $g^{(N+1)} \to g^{(N)}$, $N\to\infty$.

Let us show that

$$\lim_{N\to\infty} u^{(N+1)}(t)=u(t),\ \ \lim_{N\to\infty} g^{(N+1)}(t)=g(t).$$

$$\begin{aligned} &\left\|u-u^{(N+1)}\right\|_B \\ \leq\ & (2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|g(t)-g^{(N+1)}(t)\right\|_{C[0,T]}\left\|u^{(N+1)}(t)\right\|_B \\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u-u^{(N+1)}\right\|_B\left\|g^{(N+1)}(t)\right\|_{C[0,T]} \\ &+(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\left\|u-u^{(N+1)}\right\|_B\left\|b(x,t)\right\|_{L_2(\Gamma)} \\ &+\left(\frac{2\sqrt{3T}+\pi}{2\sqrt{3\pi}}\right)\left\|u^{(N+1)}-u^{(N)}\right\|_B\left\|b(x,t)\right\|_{L_2(\Gamma)}, \end{aligned} \tag{11}$$

$$\left\|g(t)-g^{(N+1)}(t)\right\|_{C[0,T]} \leq \frac{\sqrt{\pi}}{|H(t)|}\left\|u-u^{(N+1)}\right\|_B\left\|b(x,t)\right\|_{L_2(\Gamma)}.$$

Let us consider (10) in (11) and applying Gronwall's inequality to (11), we yield

$$\begin{aligned} \left\|u(t)-u^{(N+1)}(t)\right\|^2_B \leq\ & \\ & 2\left[(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})\frac{1}{\sqrt{N!}}\left\|b(x,t)\right\|_{L_2(\Gamma)}\right]^2 \\ & \times\exp 2(2\sqrt{\frac{T^3}{3\pi}}+\frac{\pi\sqrt{T}}{6})^2\left(\left\|b(x,t))\right\|^{N+1}_{L_2(\Gamma)}\right)^2. \end{aligned} \tag{12}$$

$u^{(N+1)} \to u,\ g^{(N+1)} \to g,\ N \to \infty.$

For the uniqueness, let (u, g), (v, h) are two solution of (1)-(4). After applying Cauchy, Bessel,Lipschitz, Hölder inequalities consecutively to $|u(t) - v(t)|$ and $|g(t) - h(t)|$, we obtain

$$\begin{aligned} \|u(t) - v(t)\|_B &\le (2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6}) \|g(t) - h(t)\|_{C[0,T]} \|u(t)\|_B \\ &\quad + (2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6}) \left(\int_0^t \int_0^\pi b^2(\xi, \tau) |u(\tau) - v(\tau)|^2 d\xi d\tau \right)^{\frac{1}{2}}, \end{aligned}$$

$$\|g(t) - h(t)\|_{C[0,T]} \le \frac{\sqrt{\pi}}{|H(t)|} \|u(t) - v(t)\|_B \|b(x,t)\|_{L_2(\Gamma)},$$

$$\|u(t) - v(t)\|_B \le (2\sqrt{\frac{T^3}{3\pi}} + \frac{\pi\sqrt{T}}{6}) \left(\int_0^t \int_0^\pi b^2(\xi, \tau) |u(\tau) - v(\tau)|^2 d\xi d\tau \right)^{\frac{1}{2}}, \quad (13)$$

If (13) inequalities is applied Gronwall, then we get $u(t) = v(t)$ .Hence we have $g(t) = h(t)$. The proof is completed. □

3. Stability of the solution (g, u)

Theorem 5. *Let the assumptions (C1)-(C3) be provided, the solution (g, u) of the problem (1)-(4) depends continuously upon the data φ, H.*

Proof. Suppose $\Psi = \{\varphi,\ H,\ f\}$, $\overline{\Psi} = \left\{\overline{\varphi},\ \overline{H},\ f\right\}$ and positive constants M_i, $i = 1, 2$ such that

$$\|H\|_{C^1[0,T]} \le M_1, \left\|\overline{H}\right\|_{C^1[0,T]} \le M_1,\ \|\varphi\|_{C^3[0,\pi]} \le M_2, \|\overline{\varphi}\|_{C^3[0,\pi]} \le M_2.$$

Let $\|\Psi\| = (\|H\|_{C^1[0,T]} + \|\varphi\|_{C^3[0,\pi]} + \|f\|_{C^{3,0}(\overline{\Gamma})})$. Let (g, u) and $(\overline{g}, \overline{u})$ are solution.

$$\begin{aligned} u - \overline{u} &= \frac{(\varphi_0 - \overline{\varphi_0})}{2} + \frac{\left(\psi_0 - \overline{\psi_0}\right) t}{2} + \sum_{k=1}^{\infty} \cos 2kx\ \ (\varphi_{ck} - \overline{\varphi_{ck}}) \cos(2k)^2 t \\ &\quad + \sum_{k=1}^{\infty} \sin 2kx\, (\varphi_{sk} - \overline{\varphi_{sk}}) \sin(2k)^2 t \\ &\quad + \frac{1}{2} \left(\frac{2}{\pi} \int_0^t \int_0^\pi (t - \tau) \left(g(\tau) - \overline{g(\tau)}\right) u(\tau) d\xi d\tau \right) \end{aligned}$$

$$+\frac{1}{2}\left(\frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau)\left(u(\tau)-\overline{u(\tau)}\right)p(\tau)d\xi d\tau\right)$$

$$+\frac{1}{2}\left(\frac{2}{\pi}\int_0^t\int_0^\pi (t-\tau)\left[f(\xi,\tau,u(\xi,\tau))-f(\xi,\tau,\overline{u}(\xi,\tau))\right]d\xi d\tau\right)$$

$$+\sum_{k=1}^{\infty}\frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi \left(g(\tau)-\overline{g(\tau)}\right)u(\tau)\sin(2k)^2(t-\tau)\cos 2k\xi d\xi d\tau$$

$$\sum_{k=1}^{\infty}\frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi \left(u(\tau)-\overline{u(\tau)}\right)p(\tau)\sin(2k)^2(t-\tau)\cos 2k\xi d\xi d\tau$$

$$+\sum_{k=1}^{\infty}\frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi \Big[f(\xi,\tau,u(\xi,\tau))-f(\xi,\tau,\overline{u}(\xi,\tau))\Big]$$
$$\times\sin(2k)^2(t-\tau)\cos 2k\xi d\xi d\tau$$

$$+\sum_{k=1}^{\infty}\frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi \left(p(\tau)-\overline{p(\tau)}\right)u(\tau)\sin(2k)^2(t-\tau)\sin 2k\xi d\xi d\tau$$

$$+\sum_{k=1}^{\infty}\frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi \left(u(\tau)-\overline{u(\tau)}\right)p(\tau)\sin(2k)^2(t-\tau)\sin 2k\xi d\xi d\tau$$

$$+\sum_{k=1}^{\infty}\frac{2}{\pi(2k)^2}\int_0^t\int_0^\pi \left[f(\xi,\tau,u(\xi,\tau))-f(\xi,\tau,\overline{u}(\xi,\tau))\right]\sin(2k)^2(t-\tau)\cos 2k\xi d\xi d\tau,$$

$$|u-\overline{u}|^2 \leq 2M_3^2\left\|\Psi-\overline{\Psi}\right\|^2 \times\exp 2M_4^2\left(\int_0^t\int_0^\pi b^2(\xi,\tau)d\xi d\tau\right). \tag{14}$$

For $\Psi\to\overline{\Psi}$ then $u\to\overline{u}$. Hence $g\to\overline{g}$. □

4. Conclusion

The inverse problem regarding the simultaneously identification of the time-dependent coefficient and the temperature distribution in Euler-Bernoulli beam equations with periodic boundary and integral overdetermination conditions has been considered. In the article, the conditions for the existence, uniqueness and continuous dependence upon the data of this inverse problem have been established.

This provides the insight the modeling of problems with periodic boundary conditions. Periodic boundary conditions are more difficult than local boundary conditions for the inverse coefficient problems. This work advances our understanding of the use of the Fourier method of separation of variables in the investigation of inverse problems for Euler-Bernoulli beam equations. The author plan to consider various inverse problems in future studies, since the method discussed has a wide range of applications.

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