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Research Article

L-algebraic system and its reflectivity

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Abstract

In this paper, with a continuous lattice L as the truth valued table, we first prove that the non-topological category L-**AlgSys** of L-algebraic systems can be embedded into the topological category of variety-based (A, L)-fuzzy algebraic closure spaces. Subsequently, we demonstrate that the Sierpinski L-algebraic system $(L, \mathcal{S}, \models_{\mathcal{S}})$ is an injective object in the category L-$\mathbf{AlgSys}_0$ of S_0-L-algebraic systems. Furthermore, we prove that L-$\mathbf{AlgSys}_0$ is epireflective in L-**AlgSys**, while the category L-**SobAlgSys** of sober L-algebraic systems is reflective in L-**AlgSys**. Finally, we consider the relationships between the category of L-algebraic closure spaces and that of strong L-algebraic systems, and between the category of continuous lattices and that of sober L-algebraic systems.

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1. Introduction

For emphasizing the relevant characteristics of program semantics, Vickers[20] introduced the notion of topological systems using the concept of satisfaction relation. It is well known that the category **TopSys** of topological systems, has ground category **Set** × **Loc**, where **Loc** is the category of locales (i.e., the opposite category of the category of frames). Crucially, **TopSys** exhibits significant differences compared to the category **Top** of topological spaces. Specifically, **Top** is topological over **Set**, whereas **TopSys** fails to be topological over its ground category **Set** × **Loc**. Consequently, **TopSys** lacks initial and final structures in general[4]. In the context of domain theory, topological concepts are employed to analyze program semantics. While one might naturally expect **TopSys** to exhibit topological behavior relative to its ground category, this intuition is invalid. However, Denniston et al. in [4, 5] demonstrated that **TopSys** can be embedded into the topological category **Loc-Top** of variable-basis topological spaces. This embedding suggests that **Loc-Top** may be regarded as a "topological completion" of **TopSys**. Furthermore, since **TopSys** is essentially algebraic over **Set** × **Loc**, a continuous map in **TopSys** is a homeomorphism if and only if it is an isomorphism in the ground **Set** × **Loc**[17, 18]. This property parallels the behavior of classical algebraic categories such as the category

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of compact Hausdorff topological spaces, where categorical isomorphisms coincide with topological homeomorphisms.

In recent years, Noor and Srivastava in [8] introduced a Sierpinski topological system and characterized injective objects in the category $\mathbf{TopSys}_0$ of T_0-topological systems. Additionally, they proved that both $\mathbf{TopSys}_0$ and the category **SobTopSys** of sober topological systems, are reflective subcategories of **TopSys**. Following the foundational work introduced by Zadeh in [23], numerous mathematical structures have been generalized within the framework of fuzzy set theory. Building on this development, Denniston et al. in [5] investigated lattice-valued topological systems, analyzing their algebraic properties and relationships to both point-set and point-free topology.

It is well-known that alongside topological closure spaces, algebraic closure spaces (also termed convex spaces [19]) constitute another fundamental class of closure spaces. There are great differences between topological closure spaces and algebraic closure spaces, for example, the way to generate these closure spaces via subbases. Furthermore, algebraic closure spaces are deeply connected to continuous lattices and algebraic lattices. In [14], an adjunction between the category of algebraic closure spaces and that of continuous lattices is established. In [22], Yao and Zhou demonstrated a dual equivalence between the category of sober fuzzy algebraic closure spaces and that of fuzzy algebraic lattices. Building on these results, the following questions arise:

- Can a notion analogous to topological systems be defined for algebraic closure spaces (termed algebraic systems)? If so, is the category of algebraic systems topological?
- What are the injective objects in the category of algebraic systems? How can reflectivity properties (e.g., reflective subcategories) be characterized?
- What categorical relationships exist among algebraic systems, algebraic closure spaces, and continuous lattices?

This paper addresses the aforementioned questions within the framework of many-valued logic. The contexts are organized as follows. Section 2 recalls some basic concepts about categories, L-algebraic closure spaces and L-algebraic systems. Section 3 demonstrates that the category of L-algebraic systems fails to be topological, yet admits an embedding into a topological category. Section 4 introduces the Sierpinski L-algebraic system $(L, \mathcal{S}, \models_{\mathcal{S}})$, and proves that $(L, \mathcal{S}, \models_{\mathcal{S}})$ is an injective object in the category of S_0-L-algebraic systems. Section 5 establishes the reflectivity of the categories of S_0-L-algebraic systems and of sober L-algebraic systems. Section 6 establishes the relationships between the categories of L-algebraic closure spaces and strong L-algebraic systems, and between the categories of continuous lattices and sober L-algebraic systems.

2. Preliminaries

In this section, we will review some basic concepts and results.

Let L be a complete lattice. The greatest element and the least element of L are denoted by $\top$ and $\bot$, respectively. For any $S \subseteq L$, write $\bigvee S$ and $\bigwedge S$ for the least upper bound (or the supremum) and the greatest lower bound (or the infimum) of S, respectively. In particular, $\bigvee \emptyset = \bot$ and $\bigwedge \emptyset = \top$. An *$L$-subset* on a set X is a map from X to L, and the family of all L-subsets on X will be denoted by L^X or $P_L X$, called the *L-power set* of X. We do not distinguish the element $a \in L$ and the constant map such that $a(x) = a$ for each $x \in X$. All algebraic operations on L can be extended to L^X in the pointwise way. That is, for each $u, v \in L^X$, $a \in L$ and $x \in X$,

(1) $(u \vee v)(x) = u(x) \vee v(x)$;

(2) $(u \wedge v)(x) = u(x) \wedge v(x)$ and $(a \wedge u)(x) = a \wedge u(x)$;

For a map $f : X \to Y$, we can define $f_L^{\rightarrow} : L^X \to L^Y$ and $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ by $f_L^{\rightarrow}(u)(y) = \bigvee_{f(x)=y} u(x)$ and $f_L^{\leftarrow}(v)(x) = v(f(x))$, respectively. Then $f_L^{\rightarrow}$ is the left adjoint of $f_L^{\leftarrow}$.

For each $a, b \in L$, a is called *way below* b, which is denoted by $a \ll b$, if for each directed subset $D \subseteq L$, $b \leq \bigvee D$ implies there exists $d \in D$ such that $a \leq d$. A *continuous lattice* L is a complete lattice such that $a = \bigvee \Downarrow a$ for each $a \in L$, where $\Downarrow a = \{b \in L \mid b \ll a\}$. A map $f : L \longrightarrow M$ between two continuous lattices is called *a continuous lattice homomorphism* if it preserves arbitrary infima and directed suprema and satisfies that $f(\bot) = \bot$.

More details about the order theory can be found in [3, 6]. In this paper, if there is no further statement, L is always a continuous lattice.

2.1. Categories

From now on, we shall give some requisite definitions and results of the category theory[1, 2, 12]. We hope that the readers are familiar with the basic concepts, such as the dual categories, the products of categories and so on.

Let $F : \mathbf{A} \longrightarrow \mathbf{B}$ and $G : \mathbf{B} \longrightarrow \mathbf{A}$ be functors between categories. We say that F is *the left adjoint* to G (or G is *the right adjoint* to F), and write $F \dashv G$, if for each $\mathbf{A}$-object A and $\mathbf{B}$-object B, there is a bijection

$$\phi_{A,B} : \mathbf{B}(F(A), B) \longrightarrow \mathbf{A}(A, G(B))$$

that is natural in A and B. A functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ is said to be an *equivalence* of categories if there is a functor $G : \mathbf{B} \longrightarrow \mathbf{A}$ such that $G \circ F$ is naturally isomorphic to $id_{\mathbf{A}}$ and $F \circ G$ is naturally isomorphic to $id_{\mathbf{B}}$. A category $\mathbf{A}$ is said to be *equivalent* to $\mathbf{B}$ if there exists an equivalence functor $F : \mathbf{A} \longrightarrow \mathbf{B}$. $F : \mathbf{A} \longrightarrow \mathbf{B}$ is called an *embedding* if F is injective on morphisms (hence on objects).

Throughout the following section, without other statements, $(\mathbf{A}, U)$ denotes a concrete category over $\mathbf{X}$, and $\mathbf{S}$ denotes a subcategory of $\mathbf{A}$ which is always assumed to be isomorphism-closed.

Let $\mathcal{H}$ denote some class of $\mathbf{A}$-morphisms, then an $\mathbf{A}$-object A is called $\mathcal{H}$-*injective* if for each morphism $e : Y \longrightarrow Z$ in $\mathcal{H}$ and each $\mathbf{A}$-morphism $f : Y \longrightarrow X$, there exists an $\mathbf{A}$-morphism $g : Z \longrightarrow X$ such that $g \circ e = f$. A family $\{f_i : X \longrightarrow Y_i \mid i \in I\}$ of $\mathbf{A}$-morphisms is said to be *initial* if for each $\mathbf{A}$-object Z and each $\mathbf{X}$-morphism $g : |Z| \longrightarrow |X|$, $g : Z \longrightarrow X$ is an $\mathbf{A}$-morphism if $f_i \circ g : Z \longrightarrow Y_i$ is an $\mathbf{A}$-morphism for each $i \in I$. An $\mathbf{A}$-object S is called a *Sierpinski object* if for each $\mathbf{A}$-object X, the family of all $\mathbf{A}$-morphisms from X to S is initial. $\mathbf{A}$ is called *topological* over $\mathbf{X}$ w.r.t. the functor U if every U-structured source has a unique initial U-lift in $\mathbf{A}$.

An $\mathbf{A}$-morphism $f : X \longrightarrow Y$ is called an *epimorphism* if for each $\mathbf{A}$-object Z and $\mathbf{A}$-morphisms $g, h : Y \longrightarrow Z$, $g \circ f = h \circ f$ implies $g = h$. $\mathbf{S}$ is a *reflective* (*epireflective*) subcategory of $\mathbf{A}$ provided that for each $\mathbf{A}$-object Y, there exist an $\mathbf{S}$-object X_Y and an $\mathbf{A}$-morphism ($\mathbf{A}$-epimorphism) $r : Y \longrightarrow X_Y$ such that for each $\mathbf{S}$-object Z and $\mathbf{A}$-morphism $f : Y \longrightarrow Z$, there exists a unique $\mathbf{S}$-morphism $f^* : X_Y \longrightarrow Z$ with $f^* \circ r = f$.

2.2. L-algebraic systems

At the end of this section, we will introduce a generalization of L-algebraic closure space, which is called L-algebraic system.

Definition 2.1 ([7, 13])**.** A subset $\mathcal{C}$ of L^X is called an *L-closure structure* on X if it satisfies the following conditions:

(CS1) $\bot, \top \in \mathcal{C}$;

(CS2) For each $\{u_i \mid i \in I\} \subseteq \mathcal{C}$, $\bigwedge_{i \in I} u_i \in \mathcal{C}$.

The pair $(X, \mathcal{C})$ is called an *L-closure space.*

Definition 2.2 ([7,13])**.** An *L-algebraic closure structure* $\mathcal{C}$ on X is an L-closure structure that satisfies the following condition:

(CS3) For each directed family $\{v_i \mid i \in I\} \subseteq^d \mathcal{C}$, $\bigvee_{i\in I} v_i \in \mathcal{C}$.

The pair $(X, \mathcal{C})$ is said to be an *L-algebraic closure space.* An L-algebraic closure space is also called an *L-convex space* in [9,10,15,21].

Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ be two L-algebraic closure spaces and $f : X \longrightarrow Y$ be a map, then f is called *L-continuous* provided that $f_L^{\leftarrow}(v) \in \mathcal{C}_X$ for each $v \in \mathcal{C}_Y$. Let L-**Alg** denote the category of L-algebraic closure spaces and L-continuous maps.

Definition 2.3. Let X be a set and A be a continuous lattice. An L-relation $\models_A : X \times A \longrightarrow L$ is called an *L-satisfaction relation* on (X, A) if $\models_A$ satisfies:

(SR1) For each $x \in X$, $\models_A(x, \bot) = \bot$;

(SR2) For each $x \in X$ and $\{a_i \mid i \in I\} \subseteq A$, $\models_A(x, \bigwedge_{i\in I} a_i) = \bigwedge_{i\in I} \models_A(x, a_i)$;

(SR3) For each $x \in X$ and directed $\{d_i \mid i \in I\} \subseteq^d A$, $\models_A(x, \bigvee_{i\in I} d_i) = \bigvee_{i\in I} \models_A(x, d_i)$.

Remark 2.4. If $\models_A$ is an L-satisfaction relation on (X, A), then it holds that $\models_A(x, \top) = \top$ for every $x \in X$.

Definition 2.5. An *L-algebraic system* is an ordered triple $(X, A, \models)$, where $(X, A) \in |\mathbf{Set} \times \mathbf{Cnt}^{op}|$, **Cnt** is the category of continuous lattices and continuous lattice homomorphisms, and $\models_A$ is an L-satisfaction relation on (X, A).

Given L-algebraic systems $(X, A, \models_A)$ and $(Y, B, \models_B)$, the morphism $(f, \phi) : (X, A, \models_A) \longrightarrow (Y, B, \models_B)$ in $\mathbf{Set} \times \mathbf{Cnt}^{op}$ is called an *L-algebraic map* provided that $\models_A(x, \phi^{op}(b)) = \models_B(f(x), b)$ for each $x \in X$ and $b \in B$. Let L-**AlgSys** denote the category of L-algebraic systems and L-algebraic maps.

An L-algebraic map $(f, \phi) : (X, A, \models_A) \longrightarrow (Y, B, \models_B)$ between two L-algebraic systems is called an *embedding* provided that f is injective and ϕ^{op} is surjective, and it is called a *homeomorphism* provided that f is bijective and ϕ^{op} is a continuous lattice isomorphism (i.e., bijective continuous lattice homomorphism). It is routine to check that if $(f, \phi) : (X, A, \models_A) \longrightarrow (Y, B, \models_B)$ is a homeomorphism, then (f^{-1}, ϕ^{-1}) is an L-algebraic map. An L-algebraic system $(X, A, \models_A)$ is called S_0-separated provided that for each $x, y \in X$ with $x \neq y$, there exists $a \in A$ such that $\models_A(x, a) \neq \models_A(y, a)$. Let L-$\mathbf{AlgSys_0}$ denote the full subcategory of L-**AlgSys**, whose objects are S_0-L-algebraic systems. If the L-satisfaction relation $\models_A$ is understood, we may suppress the subscripts.

Example 2.6. Let $(X, \mathcal{C})$ be an L-algebraic closure space, then it is routine to check that $\mathcal{C}$ is a continuous lattice. Define $\models_{\mathcal{C}} : X \times \mathcal{C} \longrightarrow L$ by $\models_{\mathcal{C}}(x, u) = u(x)$ for each $u \in \mathcal{C}$ and $x \in X$, then $(X, \mathcal{C}, \models_{\mathcal{C}})$ is an L-algebraic system.

3. Embedding L-AlgSys into a topological category

In this section, we analyze the categorical behavior of L-**AlgSys**. Specifically, we demonstrate that it is not topological over its ground category $\mathbf{Set} \times \mathbf{Cnt}^{op}$. However, it can be embedded into the topological category $\mathbf{Cnt}^{op}$-**FAlg**.

Let $U : L$-$\mathbf{AlgSys} \longrightarrow \mathbf{Set} \times \mathbf{Cnt}^{op}$ be the forgetful functor. We want to show that there exists a U-structured source that does not have an initial U-lift. Let $X = \{\cdot\}$ be a single-point set, and let $A = \{\bot, \top\}$, and $B = \{\bot, b, \top\}$ with the order $\bot < b < \top$, then A and B are continuous lattices. Define $\phi : A \longrightarrow B$ by

$$\phi^{op}(\bot) = \phi^{op}(b) = \bot, \ \phi^{op}(\top) = \top,$$

then ϕ^{op} is a continuous lattice homomorphism. Define $\models_B : X \times B \longrightarrow L$ by

$$\models_B(\cdot, \bot) = \bot, \ \models_B(\cdot, b) = \models_B(\cdot, \top) = \top,$$

then $(X,B,\models_B)$ is an L-algebraic system. Thus, $((X,A),(id_X,\phi):(X,A)\longrightarrow U(X,B,\models_B))$ is a U-structured source. Suppose it has an initial U-lift $\models_A$, then $(id_X,\phi):(X,A,\models_A)\longrightarrow (X,B,\models_B)$ is an L-algebraic map. However, this leads to the contradiction

$$\bot=\models_A(\cdot,\bot)=\models_A(\cdot,\phi^{op}(b))=\models_B(id_X(\cdot),b)=\top.$$

That is to say this U-structured source lacks initial U-lift. Therefore, L-**AlgSys** is not a topological category over **Set** $\times$ **Cnt**^{op} with respect to the forgetful functor U.

Next, we construct a topological category by (A,L)-fuzzy algebraic closure spaces (or (A,L)-fuzzy convex spaces) in the sense of [11, 16].

Definition 3.1. The category **Cnt**^{op}-**FAlg** has ground category **Set** $\times$ **Cnt**^{op} and it is defined as follows:

- Objects: An object is an (A,L)-*fuzzy algebraic closure space* $(X,A,\mathcal{C})$, i.e., $(X,A)\in|\mathbf{Set}\times\mathbf{Cnt}^{op}|$, and the map $\mathcal{C}:A^X\longrightarrow L$ satisfies:
  (1) $\mathcal{C}(\bot)=\mathcal{C}(\top)=\top$;
  (2) $\bigwedge_{i\in I}\mathcal{C}(u_i)\le\mathcal{C}(\bigwedge_{i\in I}u_i)$ for each family $\{u_i\mid i\in I\}\subseteq A^X$;
  (3) $\bigwedge_{i\in I}\mathcal{C}(v_i)\le\mathcal{C}(\bigvee_{i\in I}v_i)$ for each directed family $\{v_i\mid i\in I\}\subseteq A^X$.
  In this case, $\mathcal{C}$ is called an (A,L)-*fuzzy algebraic closure structure.*
- Morphisms: An morphism is an L-*fuzzy continuous map* $(f,\phi):(X,A,\mathcal{C}_X)\longrightarrow (Y,B,\mathcal{C}_Y)$, i.e., (f,ϕ) is a **Set** $\times$ **Cnt**^{op}-morphism, and $\mathcal{C}_Y\le\mathcal{C}_X\circ(f,\phi)_L^{\leftarrow}$, where $(f,\phi)_L^{\leftarrow}:B^Y\longrightarrow A^X$ is defined by $(f,\phi)_L^{\leftarrow}(v)=\phi^{op}\circ v\circ f$ for each $v\in B^Y$.
- Composition, identities: As in **Set** $\times$ **Cnt**^{op}.

Let $\mathscr{S}:A^X\longrightarrow L$ be a map. The (A,L)-fuzzy algebraic closure structure $\mathcal{C}:A^X\longrightarrow L$ generated by $\mathscr{S}$ is defined by

$$\forall u\in A^X,\ \mathcal{C}(u)=\bigwedge\{\mathcal{D}(u)\mid \mathscr{S}\le\mathcal{D}\in\varrho\},$$

where ϱ denotes the set of all the (A,L)-fuzzy algebraic closure spaces. Then $\mathscr{S}$ is called a *subbase*[16] of $\mathcal{C}$.

Proposition 3.2. **Cnt**^{op}-**FAlg** *is topological over* **Set** $\times$ **Cnt**^{op} *with respect to the forgetful functor* $V:\mathbf{Cnt}^{op}\text{-}\mathbf{FAlg}\longrightarrow\mathbf{Set}\times\mathbf{Cnt}^{op}$.

Proof of Proposition 3.2. Suppose $((X,A),(f_i,\phi_i):(X,A)\longrightarrow V(X_i,A_i,\mathcal{C}_i))$ is a V-structure source. Define $\mathscr{S}:A^X\longrightarrow L$ by

$$\mathscr{S}(u)=\begin{cases}\bigvee\limits_{i\in I}\ \bigvee\limits_{(f_i,\phi_i)_L^{\leftarrow}(v)=u}\mathcal{C}_i(v) & \exists i\in I,\ ((f_i,\phi_i)_L^{\leftarrow})^{\leftarrow}(\{u\})\neq\emptyset\\ \bot & otherwise\end{cases}$$

for each $u\in A^X$. Let $\mathcal{C}$ be the (A,L)-fuzzy algebraic closure space generated by the subbase $\mathscr{S}$. For each $w\in A_i^{X_i}$, we have

$$\mathcal{C}((f_i,\phi_i)_L^{\leftarrow}(w))\ge\mathscr{S}((f_i,\phi_i)_L^{\leftarrow}(w))=\bigvee_{j\in I}\ \bigvee_{(f_j,\phi_j)_L^{\leftarrow}(v)=(f_i,\phi_i)_L^{\leftarrow}(w)}\mathcal{C}_j(v)\ge\mathcal{C}_i(w).$$

This implies that $(f_i,\phi_i):(X,A,\mathcal{C})\longrightarrow(X_i,A_i,\mathcal{C}_i)$ is L-fuzzy continuous. That is to say $\mathcal{C}$ is a V-lift.

Furthermore, to prove the initiality, let $(g_i,\psi_i):(Y,B,\delta)\longrightarrow(X_i,A_i,\mathcal{C}_i)$ be L-fuzzy continuous maps, and (h,ψ) be a **Set** $\times$ **Cnt**^{op}-morphism with $(g_i,\psi_i)=(f_i,\phi_i)\circ(h,\psi)$. For each $u\in A^X$, if $((f_i,\phi_i)_L^{\leftarrow})^{\leftarrow}(\{u\})=\emptyset$ for any $i\in I$, then we have $\mathscr{S}(u)=\bot\le$

$\delta((h,\psi)_L^{\leftarrow}(u))$. If there exists $i\in I$ such that $((f_i,\phi_i)_L^{\leftarrow})^{\leftarrow}(\{u\})\neq\emptyset$, it holds that

$$\begin{aligned}\mathscr{S}(u) &= \bigvee_{i\in I}\bigvee_{(f_i,\phi_i)_L^{\leftarrow}(v)=u}\mathcal{C}_i(v)\\ &\leq \bigvee_{i\in I}\bigvee_{(f_i,\phi_i)_L^{\leftarrow}(v)=u}\delta((g_i,\psi_i)_L^{\leftarrow}(v))\\ &= \bigvee_{i\in I}\bigvee_{(f_i,\phi_i)_L^{\leftarrow}(v)=u}\delta((h,\psi)_L^{\leftarrow}((f_i,\phi_i)_L^{\leftarrow}(v)))\\ &= \delta((h,\psi)_L^{\leftarrow}(u)).\end{aligned}$$

Thus, we have $\mathscr{S}\leq\delta\circ(h,\psi)_L^{\leftarrow}$ for each $u\in A^X$. Since it is routine to check $\delta\circ(h,\psi)_L^{\leftarrow}$ is an (A,L)-fuzzy algebraic closure space, we obtain $\mathcal{C}\leq\delta\circ(h,\psi)_L^{\leftarrow}$ from the definition of $\mathcal{C}$. Therefore, $(h,\psi):(Y,B,\delta)\longrightarrow(X,A,\mathcal{C})$ is L-fuzzy continuous, and the V-lift is initial. Moreover, the uniqueness of the initial V-lift is obvious. This completes the proof. □

Given an L-algebraic system $(X,A,\models)$. Define $\mathcal{C}_{\models}:A^X\longrightarrow L$ by

$$\mathcal{C}_{\models}(u)=\begin{cases}\bigwedge_{x\in X}\models(x,u(x)) & u\neq\bot\\ \top & u=\bot\end{cases}$$

for each $u\in A^X$. It can be verified that $(X,A,\mathcal{C}_{\models})$ is an (A,L)-fuzzy algebraic closure space.

Proposition 3.3. *Let $(f,\phi):(X,A,\models_A)\longrightarrow(Y,B,\models_B)$ be an L-algebraic map in L-**AlgSys**. Then $(f,\phi):(X,A,\mathcal{C}_{\models_A})\longrightarrow(Y,B,\mathcal{C}_{\models_B})$ is an L-fuzzy continuous map in **Cnt**^{op}-**FAlg**.*

Proof of Proposition 3.3. For each $v\in B^Y$, if $(f,\phi)_L^{\leftarrow}(v)=\bot$, then $\mathcal{C}_{\models_A}((f,\phi)_L^{\leftarrow}(v))=\top\geq\mathcal{C}_{\models_B}(v)$. If $(f,\phi)_L^{\leftarrow}(v)\neq\bot$, then $v\neq\bot$. We derive

$$\begin{aligned}\mathcal{C}_{\models_B}(v) &= \bigwedge_{y\in Y}\models_B(y,v(y))\\ &\leq \bigwedge_{x\in X}\models_B(f(x),v(f(x)))\\ &= \bigwedge_{x\in X}\models_A(x,\phi^{op}(v(f(x))))\\ &= \bigwedge_{x\in X}\models_A(x,(f,\phi)_L^{\leftarrow}(v)(x))\\ &= \mathcal{C}_{\models_A}((f,\phi)_L^{\leftarrow}(v)).\end{aligned}$$

Therefore, $\mathcal{C}_{\models_B}\leq\mathcal{C}_{\models_A}\circ(f,\phi)_L^{\leftarrow}$, proving that (f,ϕ) is an L-fuzzy continuous map. □

From Proposition 3.3, we obtain a functor $F_{\models}:L$-**AlgSys** $\longrightarrow$ **Cnt**^{op}-**FAlg** defined explicitly as:

$$F_{\models}:\begin{cases}L\text{-}\mathbf{AlgSys}\longrightarrow\mathbf{Cnt}^{op}\text{-}\mathbf{FAlg}\\ (X,A,\models)\longmapsto(X,A,\mathcal{C}_{\models})\\ (f,\phi)\longmapsto(f,\phi)\end{cases}$$

Theorem 3.4. *L-**AlgSys** can be embedded into the topological category **Cnt**^{op}-**FAlg** w.r.t. the functor $F_{\models}$.*

Proof of Theorem 3.4. Let $(X,A)\in|\mathbf{Set}\times\mathbf{Cnt}^{op}|$. We only show that $\mathcal{C}_{\models_A}\neq\mathcal{C}_{\hat{\models}_A}$ when the L-satisfaction relations $\models_A$ and $\hat{\models}_A$ on (X,A) are different here. Since $\models_A\neq\hat{\models}_A$, there exist $x_0\in X$ and $a_0\in A$ such that $\models_A(x_0,a_0)\neq\hat{\models}_A(x_0,a_0)$. Put $u\in A^X$ by

$$u(x) = \begin{cases} a_0 & x = x_0 \\ \top & x \neq x_0 \end{cases}$$

for each $x \in X$. Then we have

$$\mathcal{C}_{\models_A}(u) = \bigwedge_{x\in X} \models_A(x, u(x)) = \models_A(x_0, a_0) \wedge (\bigwedge_{x\neq x_0} \models_A(x, \top)) = \models_A(x_0, a_0).$$

Similarly, $\mathcal{C}_{\hat{\models}_A}(u) = \hat{\models}_A(x_0, a_0)$. Thus, $\mathcal{C}_{\models_A} \neq \mathcal{C}_{\hat{\models}_A}$. □

4. The Sierpinski L-algebraic system

Consider the continuous lattice $\mathcal{S} = \{\perp_{\mathcal{S}}, \alpha, \top_{\mathcal{S}}\}$ with the partial order on $\mathcal{S}$ as $\perp_{\mathcal{S}} < \alpha < \top_{\mathcal{S}}$. Define an L-relation $\models_{\mathcal{S}} : L \times \mathcal{S} \longrightarrow L$ as follows: for each $t \in L$, $\models_{\mathcal{S}}(t, \perp_{\mathcal{S}}) = \perp$, $\models_{\mathcal{S}}(t, \alpha) = t$, $\models_{\mathcal{S}}(t, \top_{\mathcal{S}}) = \top$, then it is easy to proof that $\models_{\mathcal{S}}$ is an L-satisfaction relation on $(L, \mathcal{S})$. Thus, $(L, \mathcal{S}, \models_{\mathcal{S}})$ is an L-algebraic system. We call the L-algebraic system $(L, \mathcal{S}, \models_{\mathcal{S}})$ an *Sierpinski L-algebraic system.* Furthermore, since $\models_{\mathcal{S}}(a, \alpha) = a \neq b = \models_{\mathcal{S}}(b, \alpha)$ for each $a, b \in L$ with $a \neq b$, it follows that the Sierpinski L-algebraic system $(L, \mathcal{S}, \models_{\mathcal{S}})$ is S_0-separated.

Proposition 4.1. *Given an L-algebraic system $(X, A, \models)$. For each $a \in A$, define $f_a : X \longrightarrow L$ by $f_a(x) = \models(x, a)$, and define $\phi_a^{op} : \mathcal{S} \longrightarrow A$ as follows: $\phi_a^{op}(\perp_{\mathcal{S}}) = \perp$, $\phi_a^{op}(\alpha) = a$, $\phi_a^{op}(\top_{\mathcal{S}}) = \top$, then*

(1) $(f_a, \phi_a) : (X, A, \models) \longrightarrow (L, \mathcal{S}, \models_{\mathcal{S}})$ is an L-algebraic map;

(2) For each L-algebraic map $(g, \psi) : (X, A, \models) \longrightarrow (L, \mathcal{S}, \models_{\mathcal{S}})$, there exists $a \in \psi^{op}(\alpha)$ such that $(g, \psi) = (f_a, \phi_a)$.

Proof of Proposition 4.1. (1) It is easy to see that ϕ_a^{op} is a continuous lattice homomorphism. Moreover, it holds that

$$\models(x, \phi_a^{op}(\alpha)) = \models(x, a) = f_a(x) = \models_{\mathcal{S}}(f_a(x), \alpha)$$

for each $x \in X$. Similarly, we have $\models(x, \phi_a^{op}(\perp_{\mathcal{S}})) = \models_{\mathcal{S}}(f_a(x), \perp_{\mathcal{S}})$, $\models(x, \phi_a^{op}(\top_{\mathcal{S}})) = \models_{\mathcal{S}}(f_a(x), \top_{\mathcal{S}})$.

(2) For each L-algebraic map $(g, \psi) : (X, A, \models) \longrightarrow (L, \mathcal{S} \models_{\mathcal{S}})$ and $a = \psi^{op}(\alpha)$, we have $\psi^{op} = \phi_a^{op}$. For each $x \in X$, since

$$f(x) = \models(x, a) = \models(x, \psi^{op}(\alpha)) = \models_{\mathcal{S}}(g(x), \alpha) = g(x),$$

it follows that $g = f_a$. □

Proposition 4.2. $(L, \mathcal{S}, \models_{\mathcal{S}})$ *is a Sierpinski object in the category L-$\mathbf{AlgSys_0}$.*

Proof of Proposition 4.2. Let $(X, A, \models_A)$ and $(Y, B, \models_B)$ be two L-algebraic systems, and $(g, \psi) : (Y, B) \longrightarrow (X, A)$ be a map. Let $\mathcal{A} = \{(f, \phi) \mid (f, \phi) : (X, A, \models_A) \longrightarrow (L, \mathcal{S}, \models_{\mathcal{S}})$ *is an L-algebraic map*$\}$, then it suffices to check that $\mathcal{A}$ is an initial family. Suppose $(g, \psi) \circ (f, \phi) : (Y, B, \models_B) \longrightarrow (L, \mathcal{S}, \models_{\mathcal{S}})$ is an L-algebraic map. Obverse that

$$\begin{aligned} \models_A(g(y), a) &= \models_A(g(y), \phi_a^{op}(\alpha)) \\ &= \models_{\mathcal{S}}(f_a(g(y)), \alpha) \\ &= \models_B(y, (\phi_a \circ \psi)^{op}(\alpha)) \\ &= \models_B(y, \psi^{op}(\phi_a^{op}(\alpha))) \\ &= \models_B(y, \psi^{op}(a)) \end{aligned}$$

for each $y \in Y$ and $a \in A$, we obtain that $(g, \psi) : (Y, B, \models_B) \longrightarrow (X, A, \models_A)$ is an L-algebraic map. □

Proposition 4.3. *Let (f,ϕ) and (g,ψ) be two L-algebraic maps between S_0-L-algebraic systems $(X,A,\models_A)$ and $(Y,B,\models_B)$ with $(f,\phi)\neq(g,\psi)$, then there exists an L-algebraic map $(h,\theta):(Y,B,\models_B)\longrightarrow(L,\mathcal{S},\models_{\mathcal{S}})$ such that $h\circ f\neq h\circ g$.*

Proof of Proposition 4.3. Since $f\neq g$, then there exists $x\in X$ such that $f(x)\neq g(x)$. As $(Y,B,\models_B)$ is S_0-separated, it follows that there exists $b\in B$ such that $\models_B(f(x),b)\neq\models_B(g(x),b)$. Define L-algebraic map (h_b,θ_b) as Proposition 4.1, then

$$(h_b\circ f)(x)=\models_B(f(x),b)\neq\models_B(g(x),b)=(h_B\circ g)(x).$$

That is to say there exists an L-algebraic map (h_b,θ_b) such that $h_b\circ f\neq h_b\circ g$. □

In the following, let $\mathcal{H}$ be the class of all embeddings in the category L-$\mathbf{AlgSys}_0$, and we write injective in place of $\mathcal{H}$-injective.

Theorem 4.4. *$(L,\mathcal{S},\models_{\mathcal{S}})$ is an injective object in the category L-$\mathbf{AlgSys_0}$.*

Proof of Theorem 4.4. Let $(X,A,\models_A)$ and $(Y,B,\models_B)$ be two S_0-L-algebraic systems, and $(f,\phi):(Y,B,\models_B)\longrightarrow(X,A,\models_A)$ be an embedding. For each L-algebraic map $(g,\psi):(Y,B,\models_B)\longrightarrow(L,\mathcal{S},\models_{\mathcal{S}})$, observe that $\psi^{op}(\alpha)=b\in B$ and $\phi^{op}:A\longrightarrow B$ is a surjection, we obtain that there exists $a\in A$ such that $\phi^{op}(a)=b$. Then $(f_a,\phi_a):(X,A,\models_A)\longrightarrow(L,\mathcal{S},\models_{\mathcal{S}})$ defined as Proposition 4.1 is an L-algebraic map.

Furthermore, since

$$\begin{aligned}f_a\circ f(y)&=\models_A(f(y),a)\\&=\models_B(y,\phi^{op}(a))\\&=\models_B(y,b)\\&=\models_B(y,\psi^{op}(\alpha))\\&=\models_{\mathcal{S}}(g(y),\alpha)\\&=g(y)\end{aligned}$$

for each $y\in Y$, and

$$\phi^{op}\circ\phi_a^{op}(\alpha)=\phi^{op}(a)=b=\psi^{op}(\alpha),$$

it holds that $(f_a,\phi_a)\circ(f,\phi)=(g,\psi)$. □

5. The reflectivity of the L-algebraic systems

Given an L-algebraic system $(X,A,\models)$, define a binary relation $\sim$ on X as follows:

$$x\sim y\Longleftrightarrow\models(x,a)=\models(y,a)(\forall a\in A).$$

Then $\sim$ is an equivalence relation on X. Let $\tilde{X}=X/\sim$, and define the L-relation $\tilde{\models}:\tilde{X}\times A\longrightarrow L$ by $\tilde{\models}(\tilde{x},a)=\models(x,a)$ for each $\tilde{x}\in\tilde{X}$ and $a\in A$. For each $\tilde{x}\neq\tilde{y}$, we have $x\nsim y$, then there exists $a\in A$ such that $\models(x,a)\neq\models(y,a)$, thus $\tilde{\models}(\tilde{x},a)\neq\tilde{\models}(\tilde{y},a)$. That is to say that $(\tilde{X},A,\tilde{\models})$ is an S_0-L-algebraic system.

Theorem 5.1. *The category L-$\mathbf{AlgSys_0}$ is an epireflective subcategory of the category L-$\mathbf{AlgSys}$.*

Proof of Theorem 5.1. Let $(X,A,\models_A)$ be an L-algebraic system, $(\tilde{X},A,\tilde{\models}_A)$ and $(Y,B,\models_B)$ be S_0-L-algebraic systems, and $(f,\phi):(X,A,\models_A)\longrightarrow(Y,B,\models_B)$ be an L-algebraic map. Define $q_X:X\longrightarrow\tilde{X}$ by $q_X(x)=\tilde{x}$ for each $x\in X$. Since

$$\models_A(x,id_A(a))=\models_A(x,a)=\tilde{\models}_A(\tilde{x},a)=\tilde{\models}_A(q_X(x),a)$$

for each $x\in X$ and $a\in A$, it follows that $(q_X,id_A^{op}):(X,A,\models_A)\longrightarrow(\tilde{X},A,\tilde{\models}_A)$ is an L-algebraic map. Furthermore, it is routine to check that (q_X,id_A^{op}) is an epimorphism.

Define $f^*:\tilde{X}\longrightarrow Y$ by $f^*(\tilde{x})=f(x)$ and $\phi^*=\phi$. For each $x_1\sim x_2$, suppose that $f(x_1)\neq f(x_2)$. Since $(Y,B,\models_B)$ is S_0-separated, there exists $b\in B$ such that

$\models_B(f(x_1), b) \neq \models_B(f(x_2), b)$. Thus, we have $\models_A(x_1, \phi^{op}(b)) \neq \models_A(x_2, \phi^{op}(b))$. This contradicts with $x_1 \sim x_2$. That is to say that the definition of f^* is well-defined.

Furthermore, since

$$\tilde{\models}_A(\tilde{x}, (\phi^*)^{op}(b)) = \tilde{\models}_A(\tilde{x}, \phi^{op}(b)) = \models_A(x, \phi^{op}(b)) = \models_B(f(x), b) = \models_B(f^*(\tilde{x}), b)$$

and

$$(f^*, \phi^*) \circ (q_X, id_A^{op})(x, a) = (f^*, \phi^*)(\tilde{x}, a) = (f, \phi)(x, a)$$

for each $x \in X$, $\tilde{x} \in \tilde{X}$, $a \in A$ and $b \in B$, it follows that $(f^*, \phi^*) : (\tilde{X}, A, \tilde{\models}_A) \longrightarrow (Y, B, \models_B)$ is an L-algebraic map, and it satisfies $(f^*, \phi^*) \circ (q_X, id_A^{op}) = (f, \phi)$. The unqueness of (f^*, ϕ^*) is obvious. □

For each L-algebraic system $(X, A, \models)$, let

$$pt(A) = \{p : A \longrightarrow L \mid p \textit{ is a continuous lattice homomorphism}\}.$$

Define $\eta_X : X \longrightarrow pt(A)$ by

$$\eta_X(x)(a) = \models(x, a)$$

for each $x \in X$ and $a \in A$. Then it is routine to check that the definition of η_X is reasonable.

Definition 5.2. An L-algebraic system $(X, A, \models)$ is called *sober* provided that $\eta_X : X \longrightarrow pt(A)$ is a bijection.

Let L-**SobAlgSys** denote the category of sober L-algebraic systems and L-algebraic maps. It is worth to mention that an L-algebraic system $(X, A, \models)$ is S_0 if and only if $\eta_X : X \longrightarrow pt(A)$ is injective. Thus, a sober L-algebraic system must be an S_0-L-algebraic system. Furthermore, the Sierpinski L-algebraic system $(L, \mathcal{S}, \models_{\mathcal{S}})$ is sober.

Proposition 5.3. *Let $(X, A, \models)$ be an L-algebraic system. Define $\models_{pt(A)} : pt(A) \times A \longrightarrow L$ by $\models_{pt(A)}(p, a) = p(a)$ for each $p \in pt(A)$ and $a \in A$, then:*

(1) $\models_{pt(A)}$ is an L-satisfaction relation;

(2) $(pt(A), A, \models_{pt(A)})$ is a sober L-algebraic system;

(3) $(\eta_X, id_A^{op}) : (X, A, \models) \longrightarrow (pt(A), A, \models_{pt(A)})$ is an L-algebraic map;

(4) $(X, A, \models)$ is sober iff $(\eta_X, id_A^{op}) : (X, A, \models) \longrightarrow (pt(A), A, \models_{pt(A)})$ is a homeomorphism.

Proof of Proposition 5.3. (1) It suffices to check that $\models_{pt(A)}$ satisfies (SR1)-(SR3).

(SR1) For each $p \in pt(A)$, $\models_{pt(A)}(p, \bot) = p(\bot) = \bot$.

(SR2) For each $p \in pt(A)$ and $\{a_i \mid i \in I\} \subseteq A$, it holds that

$$\models_{pt(A)}(p, \bigwedge_{i\in I} a_i) = p(\bigwedge_{i\in I} a_i) = \bigwedge_{i\in I} p(a_i) = \bigwedge_{i\in I} \models_{pt(A)}(p, a_i).$$

(SR3) For each $p \in pt(A)$ and directed $\{d_i \mid i \in I\} \subseteq^d A$, it follows that

$$\models_{pt(A)}(p, \bigvee_{i\in I} d_i) = p(\bigvee_{i\in I} d_i) = \bigvee_{i\in I} p(d_i) = \bigvee_{i\in I} \models_{pt(A)}(p, d_i).$$

(2) Since $\eta_{pt(A)}(p)(a) = \models_{pt(A)}(p, a) = p(a)$ for each $p \in pt(A)$ and $a \in A$, we obtain that $\eta_{pt(A)} : pt(A) \longrightarrow pt(A)$ is a bijection. Thus, $(pt(A), A, \models_{pt(A)})$ is sober.

(3) For each $x \in X$ and $a \in A$, observe that $\models(x, id_A(a)) = \models(x, a) = \eta_X(x)(a) = \models_{pt(A)}(\eta_X(x), a)$, it follows that $(\eta_X, id_A^{op}) : (X, A, \models) \longrightarrow (pt(A), A, \models_{pt(A)})$ is an L-algebraic map.

(4) It is immediate from the definitions. □

Theorem 5.4. *The category L-**SobAlgSys** is a reflective subcategory of the category L-**AlgSys**.*

Proof of Theorem 5.4. Let $(X, A, \models_A)$ be an L-algebraic system, and $(Y, B, \models_B)$ be a sober L-algebraic system. For each L-algebraic maps $(\eta_X, id_A^{op}) : (X, A, \models_A) \longrightarrow (pt(A), A, \models_{pt(A)})$ and $(f, \phi) : (X, A, \models_A) \longrightarrow (Y, B, \models_B)$, it suffices to check that there exists an unique L-algebraic map $(f^*, \phi^*) : (pt(A), A, \models_{pt(A)}) \longrightarrow (Y, B, \models_B)$ such that the following diagram is commutative:

$$\begin{array}{ccc} (X, A, \models_A) & \xrightarrow{(\eta_X, id_A^{op})} & (pt(A), A, \models_{pt(A)}) \\ & \searrow^{(f,\phi)} & \downarrow {\scriptstyle (f^*,\phi^*)} \\ & & (Y, B, \models_B) \end{array}$$

Step 1: Since $(Y, B, \models_B)$ is sober and $p \circ \phi^{op} \in pt(B)$ for each $p \in pt(A)$, it follows that there exists an unique $y \in Y$ such that $\eta_Y(y) = p \circ \phi^{op}$. Define $f^* : pt(A) \longrightarrow B$ by $f^*(p) = y$ and $\phi^* = \phi$, then we have

$$\begin{aligned} \models_{pt(A)}(p, (\phi^*)^{op}(b)) &= \models_{pt(A)}(p, \phi^{op}(b)) \\ &= p(\phi^{op}(b)) \\ &= \eta_Y(y)(b) \\ &= \eta_Y(f^*(p))(b) \\ &= \models_B(f^*(p), b) \end{aligned}$$

for each $b \in B$. That is to say $(f^*, \phi^*) : (pt(A), A, \models_{pt(A)}) \longrightarrow (Y, B, \models_B)$ is an L-algebraic map.

Step 2: $(f^*, \phi^*) \circ (\eta_X, id_A^{op}) = (f, \phi)$. In fact, since

$$\begin{aligned} \eta_Y(f(x)) &= \models_B(f(x), b) \\ &= \models_A(x, \phi^{op}(b)) \\ &= \eta_X(x)(\phi^{op}(b)) \\ &= \models_{pt(A)}(\eta_X(x), \phi^{op}(b)) \\ &= \models_{pt(A)}(\eta_X(x), (\phi^*)^{op}(b)) \\ &= \models_B(f^*(\eta_X(x)), b) \\ &= \eta_Y(f^*(\eta_X(x)))(b) \end{aligned}$$

for each $x \in X$ and $b \in B$, and η_Y is a bijection, it holds that $f^*(\eta_X(x)) = f(x)$ for each $x \in X$. Furthermore, we have $\phi^* \circ id_A^{op} = \phi$.

Step 3: The uniqueness of ϕ^* is obvious. Suppose that there is an L-algebraic map $(\hat{f}, \phi^*)$ such that $(\hat{f}, \phi^*) \circ (\eta_X, id_A^{op}) = (f, \phi)$, and $\hat{f} \neq f^*$, then there exists $x \in X$ such that $\hat{f}(\eta_X(x)) \neq f^*(\eta_X(x))$. Since $(Y, B, \models_B)$ is an S_0-L-algebraic system, there exists $b \in B$ such that $\models_B(\hat{f}(\eta_X(x)), b) \neq \models_B(f^*(\eta_X(x)), b)$. Then we have $\models_{pt(A)}(\eta_X(x), \phi^{op}(b)) \neq \models_{pt(A)}(\eta_X(x), \phi^{op}(b))$, this is a contradiction. □

6. The relationships between the categories Cnt, L-Alg and L-AlgSys

In this section, we will show that there are natural relationships between the categories of L-algebraic closure spaces and strong L-algebraic systems, and between the categories of continuous lattices and sober L-algebraic systems.

Definition 6.1. A *strong L-satisfaction relation* $\models : X \times A \longrightarrow L$ on (X, A) is an L-satisfaction relation that satisfies:

(SR4) For each $a, b \in A$, if $\models(x, a) \leq \models(x, b)$ holds for any $x \in X$, then $a \leq b$.

Definition 6.2. A *strong L-algebraic system* is an ordered triple $(X, A, \models)$, where $(X, A) \in |\mathbf{Set} \times \mathbf{Cnt}^{op}|$ and $\models$ is a strong L-satisfaction relation on (X, A).

Let $\mathbf{S}L$-$\mathbf{AlgSys}$ denote the category of strong L-algebraic systems and L-algebraic maps, which is a full subcategory of L-$\mathbf{AlgSys}$.

Remark 6.3. For each L-algebraic closure space $(X, \mathcal{C})$, it is routine to check $(X, \mathcal{C}, \models_{\mathcal{C}})$ define as Example 2.6 is a strong L-algebraic system. For each L-continuous map $f : (X, \mathcal{C}_X) \longrightarrow (Y, \mathcal{C}_Y)$ between L-algebraic closure spaces, since

$$\models_{\mathcal{C}_X}(x, f_L^{\leftarrow}(v)) = f_L^{\leftarrow}(v)(x) = v(f(x)) = \models_{\mathcal{C}_Y}(f(x), v)$$

for each $x \in X$ and $v \in \mathcal{C}_Y$, it follows that $(f, (f_L^{\leftarrow})^{op}) : (X, \mathcal{C}_X, \models_{\mathcal{C}_X}) \longrightarrow (Y, \mathcal{C}_Y, \models_{\mathcal{C}_Y})$ is an L-algebraic map. Then we obtain a functor between categories L-$\mathbf{Alg}$ and $\mathbf{S}L$-$\mathbf{AlgSys}$ defined as follows:

$$F : \begin{cases} L\text{-}\mathbf{Alg} & \longrightarrow \quad \mathbf{S}L\text{-}\mathbf{AlgSys} \\ (X, \mathcal{C}) & \longmapsto \quad (X, \mathcal{C}, \models_{\mathcal{C}}) \\ f & \longmapsto \quad (f, (f_L^{\leftarrow})^{op}) \end{cases}$$

Proposition 6.4. *Let $(X, A, \models)$ be a strong L-algebraic system. Define $ext : A \longrightarrow L^X$ by $ext(a)(x) = \models(x, a)$, then $(X, (ext)^{\rightarrow}(A))$ is an L-algebraic closure space.*

Proof of Proposition 6.4. We only check that $(X, (ext)^{\rightarrow}(A))$ satisfies (CS3) here. Let $\{ext(d_i) \mid d_i \in A, i \in I\} \subseteq^d (ext)^{\rightarrow}(A)$ be directed, we firstly show that the family $\{d_i \mid i \in I\}$ is directed. For each d_i, d_j, there exists $ext(d_k)$ such that $ext(d_i) \leq ext(d_k)$ and $ext(d_j) \leq ext(d_k)$, then $\models(x, d_i) \leq \models(x, d_k)$ and $\models(x, d_j) \leq \models(x, d_k)$ hold for all $x \in X$. Thus, $d_i \leq d_k$ and $d_j \leq d_k$. That is to say $\{d_i \mid i \in I\}$ is directed. Furthermore, observe that

$$\bigvee_{i \in I} ext(d_i)(x) = \bigvee_{i \in I} \models(x, d_i) = \models(x, \bigvee_{i \in I} d_i) = ext(\bigvee_{i \in I} d_i)(x),$$

it follows that $\bigvee_{i \in I} ext(d_i) \in (ext)^{\rightarrow}(A)$. □

Proposition 6.5. *Let $(f, \phi) : (X, A, \models_A) \longrightarrow (Y, B, \models_B)$ be an L-algebraic map between strong L-algebraic systems, then $f : (X, (ext)^{\rightarrow}(A)) \longrightarrow (Y, (ext)^{\rightarrow}(B))$ is an L-continuous map.*

Proof of Proposition 6.5. For each $x \in X$ and $b \in B$, we have

$$\begin{aligned} f_L^{\leftarrow}(ext(b))(x) &= ext(b)(f(x)) \\ &= \models_B(f(x), b) \\ &= \models_A(x, \phi^{op}(b)) \\ &= ext(\phi^{op}(b))(x). \end{aligned}$$

□

From Proposition 6.4 and Proposition 6.5, we obtain the functor G :$\mathbf{S}L$-$\mathbf{AlgSys}$$\longrightarrow$$L$-$\mathbf{Alg}$ defined as follows:

$$G : \begin{cases} \mathbf{S}L\text{-}\mathbf{AlgSys} & \longrightarrow \quad L\text{-}\mathbf{Alg} \\ (X, A, \models) & \longmapsto \quad (X, (ext)^{\rightarrow}(A)) \\ (f, \phi) & \longmapsto \quad f \end{cases}$$

Theorem 6.6. *G is both the right adjoint and left inverse to F, i.e., $F \dashv G$ and $G \circ F = id_{L-\mathbf{Alg}}$.*

Proof of Theorem 6.6. Step 1: It suffices to check that there is a natural bijection between $\mathbf{S}L\text{-}\mathbf{AlgSys}(F(X,\mathcal{C}),(Y,B,\models_B))$ and $L\text{-}\mathbf{Alg}((X,\mathcal{C}),G(Y,B,\models_B))$ for each L-algebraic closure space $(X,\mathcal{C})$ and strong L-algebraic system $(Y,B,\models_B)$.

Given an L-algebraic map $(f,\phi):(X,\mathcal{C},\models_{\mathcal{C}})\longrightarrow(Y,B,\models_B)$, define $\overline{(f,\phi)}=f$. Since

$$\begin{aligned} f_L^{\leftarrow}(ext(b))(x) &= ext(b)(f(x)) \\ &= \models_B(f(x),b) \\ &= \models_{\mathcal{C}}(x,\phi^{op}(b)) \\ &= \phi^{op}(b)(x) \end{aligned}$$

for each $x\in X$ and $b\in B$, it follows that $f_L^{\leftarrow}(ext(b))=\phi^{op}(b)\in\mathcal{C}$, that is to say $\overline{(f,\phi)}:(X,\mathcal{C})\longrightarrow(Y,(ext)^{\rightarrow}(B))$ is an L-continuous map.

Given an L-continuous map $g:(X,\mathcal{C})\longrightarrow(Y,(ext)^{\rightarrow}(B))$, define $\overline{g}=(g,\psi)$, where $\psi^{op}=g_L^{\leftarrow}\circ ext$. It is easy to see that $\psi^{op}(b)\in\mathcal{C}$ for each $b\in B$ and ψ^{op} is a continuous lattice homomorphism. Moreover, since

$$\begin{aligned} \models_B(g(x),b) &= ext(b)(g(x)) \\ &= g_L^{\leftarrow}(ext(b))(x) \\ &= \psi^{op}(b)(x) \\ &= \models_{\mathcal{C}}(x,\psi^{op}(b)) \end{aligned}$$

for each $x\in X$ and $b\in B$, we obtain that $\overline{g}:(X,\mathcal{C},\models_{\mathcal{C}})\longrightarrow(Y,B,\models_B)$ is an L-algebraic map.

Furthermore, since $\overline{\overline{(f,\phi)}}=(f,\psi)$, where $\psi^{op}=f_L^{\leftarrow}\circ ext$, and

$$\begin{aligned} \psi^{op}(b)(x) &= f_L^{\leftarrow}(ext(b))(x) \\ &= ext(b)(f(x)) \\ &= \models_B(f(x),b) \\ &= \models_{\mathcal{C}}(x,\phi^{op}(b)) \\ &= \phi^{op}(b)(x), \end{aligned}$$

this immediately forces $(f,\phi)=(f,\psi)$. The proof of $\overline{\overline{g}}=g$ is straightforward.

Step 2: Let $(X,\mathcal{C})$ be an L-algebraic closure space. Since $ext(u)(x)=\models_{\mathcal{C}}(x,u)=u(x)$ for each $u\in\mathcal{C}$ and $x\in X$, we have $(ext)^{\rightarrow}(\mathcal{C})=\mathcal{C}$. Thus, $G(F(X,\mathcal{C}))=G(X,\mathcal{C},\models_{\mathcal{C}})=(X,(ext)^{\rightarrow}(\mathcal{C}))=(X,\mathcal{C})$. □

Corollary 6.7. L-**Alg** *is a coreflective subcategory of* **S**L-**AlgSys**.

Remark 6.8. Define the functors $\Omega: L\text{-}\mathbf{AlgSys}\longrightarrow\mathbf{Cnt}^{op}$ and $Pt:\mathbf{Cnt}^{op}\longrightarrow L\text{-}\mathbf{AlgSys}$ as follows:

$$\Omega:\begin{cases} L\text{-}\mathbf{AlgSys} & \longrightarrow & \mathbf{Cnt}^{op} \\ (X,A,\models) & \longmapsto & A \\ (f,\phi) & \longmapsto & \phi \end{cases} \qquad Pt:\begin{cases} \mathbf{Cnt}^{op} & \longrightarrow & L\text{-}\mathbf{AlgSys} \\ A & \longmapsto & (pt(A),A,\models_{pt(A)}) \\ f & \longmapsto & (pt(f),f) \end{cases}$$

where $pt(f)(p)=p\circ f^{op}$ for each $p\in pt(A)$. Then it is routine to check that Ω is both the left adjoint and left inverse to Pt, i.e., $\Omega\dashv Pt$ and $\Omega\circ Pt=id_{\mathbf{Cnt}^{op}}$. Thus, $\mathbf{Cnt}^{op}$ is a reflective subcategory of L-**AlgSys**. Moreover, since $(pt(A),A,\models_{pt(A)})$ is a sober L-algebraic system, the adjoint $\Omega\dashv Pt$ can be restricted to an adjoint between the categories $\mathbf{Cnt}^{op}$ and L-**SobAlgSys**. Observe that the natural transformations $\eta: id_{L-\mathbf{SobAlgSys}}\Longrightarrow Pt\circ\Omega$ and $\varepsilon:\Omega\circ Pt\Longrightarrow id_{\mathbf{Cnt}^{op}}$ are natural isomorphisms, it follows that the categories **Cnt** and L-**SobAlgSys** are dually equivalent to each other.

7. Conclusions

In this paper, we first showed the non-topological category L-**AlgSys** can be embedded into the topological category $\mathbf{Cnt}^{op}$-**FAlg**. Then we studied the Sierpinski L-algebraic system, L-algebraic system and its reflectivity. Furthermore, the categorical relationships between the categories L-**Alg** and **S**L-**AlgSys**, and between the categories **Cnt** and L-**SobAlgSys** were considered. However, there are still some problems worth discussing:

- We know that each T_0-topological system can be embedded into a product of copies of the Sierpinski topological systems[8], where the "product" is defined by the coproduct of frames. What is the coproduct of continuous lattices? Can each S_0-algebraic system be embedded into a product of copies of the Sierpinski L-algebraic systems?
- Building on the framework of variety-based L-topological systems, the category $\mathbf{Cnt}^{op}$-**AlgSys** of variety-based L-algebraic systems can defined as follows: An object is a quadruple $(X, A, L, \models)$, where $(X, A, L) \in |\mathbf{Set} \times (\mathbf{Cnt}^{op})^2|$, and $\models\, : X \times A \longrightarrow L$ is an L-satisfaction relation on (X, A); A morphism is a $\mathbf{Set} \times (\mathbf{Cnt}^{op})^2$-morphism $(f, \phi, \psi) : (X, A, L, \models_1) \longrightarrow (Y, B, M, \models_2)$ that satisfies $\models_1(x, \phi^{op}(b)) = \psi^{op}(\models_2(f(x), b))$ for each $x \in X$ and $b \in B$. As demonstrated by the example in Section 3 of this article, we can also prove that $\mathbf{Cnt}^{op}$-**AlgSys** is not topological. Can it also be embedded into other topological categories related to fuzzy algebraic closure spaces?

We think these questions are worth further discussion.

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