Bisector Surfaces Through A Common Line Of Curvatures And Its Classifications

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Abstract

In this paper, we study the Bisector surface, which defines the line of curvature on a surface play an importance role. Firstly, the Bisector surface constructed by a point and a space curve given in Euclidean 3-space. Then, it is investigated that the necessary and sufficient condition for directrix curve of this surface to satisfy line of curvature. After this, we classify the Bisector surfaces.

Keywords: Bisector surface, ruled surface, the line of curvature, rational surface.

1. INTRODUCTION

The bisector surface is obtained points which are equidistant from any two objects. The distance is measured orthogonal to both objects. For an object in the plane or space, the medial surface is also much the same related to the bisector surface. Therefore, the medial surface can be defined as the set of interior points of the object which have the minimum distance, [4,15]. Additionally, the computation of the bisector often not easy, [15]. Then, this surface is often used in scientific research from the past with geometric properties of two curve, a curve with a point, a curve with surface and two surface bisectors in Euclidean 3- space.

For example, Horvath proved that all bisectors are topological images of a plane of the embedding Euclidean 3-space in [6] and Elber studied a new computational model for constructing a curve-surface and surface-surface bisectors in E^3 in [5].

Because Modern surface modeling systems contain the Ruled surface, this surface frequently used many areas such that simulation of the rigid body, design, production, motion analysis. For this reason, it has an important place in kinematical geometry and positional mechanisms in Euclidean 3-space. For instance, Brosius classified rank 2-vector bundles on a ruled surface and Onder and other authors viewed ruled surfaces Minkowski space, [2,11,14].

In this paper, we obtain a classification, which is the Bisector surface constructed by a point and a space curve given in Euclidean 3-space. Firstly, it is tersely summarized properties the basic concepts on surfaces. Then, we give general conditions to classify the Bisector surface. Finally, we give some

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examples and showed them with the figure in E^3.

2. PRELIMINARIES

A ruled surface is a surface swept out by a straight line moving along a curve β. The various positions of the generating line are called the rulings of the surface. Such a surface thus always has a parametrization in ruled form

$$X(u,v) = \beta(u) + v\delta(u), \qquad (1)$$

where we call β the base curve, δ the director curve. Alternatively, we may visualize δ as a vector field on β, [13].

The ruled surface (1) is developable iff

$$\left(\beta'(u), \delta(u), \delta'(u)\right) = 0. \qquad (2)$$

Then, the developable surface is cylinder iff

$$m(u) \times m'(u) = 0, \qquad (3)$$

where $\mathbf{T}(s)$ is a tangent vector of $\beta(u)$ and $m(u) = \delta(u) \times \mathbf{T}(s)$. The developable surface is a cone iff

$$w'(u) = 0, \qquad (4)$$

where $w(u)$ is the line of striction of surface (1). The developable surface is a tangent surface iff, [12],

$$w'(u) \mathsf{P} m(u). \qquad (5)$$

Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet frame along the curve α in the space E^3. For an arbitrary curve α with first and second curvature, κ and τ in the space E^3, the following Frenet-Serret formulae is given

$$\mathbf{T}' = \kappa\mathbf{N}, \qquad (6)$$

$$\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}, \qquad (7)$$

$$\mathbf{B}' = -\tau\mathbf{N}. \qquad (8)$$

Here, curvature functions are defined by $\kappa = \kappa(s) = \left\|\mathbf{T}'(s)\right\|$ and $\tau(s) = -\langle \mathbf{N}, \mathbf{B}' \rangle$. Torsion of the curve α is given by the aid of the mixed product, [14],

$$\tau = \frac{(\alpha', \alpha'', \alpha''')}{\kappa^2}. \qquad (9)$$

3. CLASSIFICATIONS OF THE BISECTOR SURFACES IN E³

In this paper, our goal is to find the necessary and sufficient condition for the directrix curve of this surface to satisfy line of curvature. In [16], to construct the Bisector surface obtained a point and a space curve, they gave the parametric form of the surface as follows;

$$B(s,t) = \left(b_x(s,t), b_y(s,t), b_z(s,t)\right). \qquad (10)$$

Here,

$$b_x(s,t) = h_1(s) + tq_x(s),$$
$$b_y(s,t) = h_2(s) + tq_y(s), \qquad (11)$$
$$b_z(s,t) = h_3(s) + tq_z(s),$$
$$q_i(s) = t_i^c(s) \times (c_i(s) - p_i)$$

and q_i is components of direction vector,

$$h_1(s) = \frac{1}{J(s)} \begin{vmatrix} d_1(s) & y'(s) & z'(s) \\ d_2(s) & y_n(s) & z_n(s) \\ m(s) & y_b(s) & z_b(t) \end{vmatrix}, \qquad (12)$$

$$h_2(s) = \frac{1}{J(s)} \begin{vmatrix} x'(s) & d_1(s) & z'(s) \\ x_n(s) & d_2(s) & z_n(s) \\ x_b(s) & m(s) & z_b(s) \end{vmatrix}, \qquad (13)$$

$$h_3(s) = \frac{1}{J(s)} \begin{vmatrix} x'(s) & y'(s) & d_1(s) \\ x_n(s) & y_n(s) & d_2(s) \\ x_b(s) & y_b(s) & m(s) \end{vmatrix}, \qquad (14)$$

and

$$J(s) = \begin{vmatrix} x'(s) & y'(s) & z'(s) \\ x_n(s) & y_n(s) & z_n(s) \\ x_b(s) & y_b(s) & z_b(s) \end{vmatrix}, \qquad (15)$$

Firstly, we will show that it is the line of curvature of directrix curve of the Bisector surface and developable surface of the Bisector surface

Theorem 3.1. The directrix curve of the Bisector surface is a line of curvature if and only if

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$$\theta = -\int_{s_0}^{s} \tau ds + \theta_0, \qquad (16)$$

$$\frac{l_2}{\sqrt{l_2^2 + l_3^2}} = \sin\theta, \frac{l_3}{\sqrt{l_2^2 + l_3^2}} = -\cos\theta. \qquad (17)$$

Proof. Assume that the normal surface of $H(s)$ is

$$K(s,t) = H(s) + tL(s). \qquad (18)$$

Here,

$$L(s) = \mathbf{n}(s)\cos\theta(s) + \mathbf{b}(s)\sin\theta(s) \qquad (19)$$

and $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)\}$ is the Frenet frame of directrix curve of Bisector. Because the surface $K(s,t)$ is a developable surface. we can easily write the following equation,

$$(H_s, L, L_s) = 0, \qquad (20)$$

$$\Leftrightarrow \begin{array}{l}(\mathbf{t}, \mathbf{n}\cos\theta + \mathbf{b}\sin\theta, (-\theta_s\sin\theta - \tau\sin\theta)\mathbf{n} \\ + (\tau\cos\theta + \theta_s\cos\theta)\mathbf{b} = 0 \end{array} \qquad (21)$$

$$\Leftrightarrow \theta_s + \tau = 0, \qquad (22)$$

$$\Leftrightarrow \theta_s = -\tau \qquad (23)$$

From the last equation above, we can easily write

$$\theta = -\int_{s_0}^{s} \tau ds + \theta_0. \qquad (24)$$

On the other hand, suppose that curve $t_i^c(s) \times (c_i(s) - p_i)$ written by without loss of generality below form;

$$t_i^c(s) \times (c_i(s) - p_i) = l_1(s)\mathbf{t}(s) + l_2(s)\mathbf{n}(s) + l_3(s)\mathbf{b}(s). \qquad (25)$$

Here, $t_i^c(s) \times (c_i(s) - p_i)$ is a unit speed curve, $t_i^c(s) \times (c_i(s) - p_i) \neq 0$, and $l_1^2 + l_2^2 + l_3^2 = 1$. Then, the corresponding Bisector surface is

$$B(s,t) = H(s) + t(t_i^c(s) \times (c_i(s) - p_i)), \qquad (26)$$

$$l_1^2 + l_2^2 + l_3^2 = 1, t_i^c(s) \times (c_i(s) - p_i) \neq 0. \qquad (27)$$

Therefore, normal of the Bisector surface is

$$N(s,t_0) = \frac{-l_3(s)\mathbf{n}(s) + l_2(s)\mathbf{b}(s)}{\sqrt{l_2^2 + l_3^2}}. \qquad (28)$$

Because the directrix curve is a line of curvature, normal of Bisector surface $B(s,t)$ and normal of

curve $H(s)$ are parallel to each other. Reconsidering these equations (19) and (28). We can be express as follows

$$\frac{l_2}{\sqrt{l_2^2 + l_3^2}} = \sin\theta, \frac{l_3}{\sqrt{l_2^2 + l_3^2}} = -\cos\theta.$$

Then, the proof is complete.

Theorem 3.2. Assume that $B(s,t)$ is the developable Bisector surface with a common line of curvature. Bisector surface $B(s,t)$ is a cylinder if and only if $t_i^c(s) \times (c_i(s) - p_i)$ is a constant curve, and $a = 0$.

Proof. We suppose that the Bisector surface $B(s,t)$ is a developable surface. Then, it is easily seen that

$$\det\left(H_s(s), t_i^c(s) \times (c_i(s) - p_i), \frac{d}{ds}t_i^c(s) \times (c_i(s) - p_i)\right) = 0. \qquad (29)$$

From the last equation above, $H_s(s)$, $t_i^c(s) \times (c_i(s) - p_i)$, $\frac{d}{ds}t_i^c(s) \times (c_i(s) - p_i)$ are linearly dependent. That is, we get

$$a(s)H(s) + b(s)\left(t_i^c(s) \times (c_i(s) - p_i)\right)$$

$$+ c(s)\left(\frac{d}{ds}t_i^c \times (c_i(s) - p_i)\right) = 0, \qquad (30)$$

If $a(s) = 0$, then

$$\left\langle t_i^c(s) \times (c_i(s) - p_i), \frac{d}{ds}t_i^c(s) \times (c_i(s) - p_i)\right\rangle = 0. \qquad (31)$$

That is, $t_i^c(s) \times (c_i(s) - p_i)$ is a constant curve. Thus, the Bisector surface $B(s,t)$ is a cylinder.

Remark 3.3. We assume that $t_i^c(s)$ and $(c_i(s) - p_i)$ are linearly dependent. Because $c_i(s)$ is regular, and so $c_i(s)$ and $(c_i(s) - p_i)$ parallel to each other. Then, we can seen that the point p_i is on tangent of curve $c_i(s)$ for all s. That is, we are obtain a degererate case.

Theorem 3.4. Assume that $B(s,t)$ is the developable Bisector surface with a common line of

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curvature. Bisector surface $B(s,t)$ is a cone if and only if

$$a \neq 0, \qquad (32)$$

$$\xi(s) = \mu_s(s). \qquad (33)$$

Proof. We suppose that the coefficient of curve $H_s(s)$ in eq. (30) is not zero. That is, $a \neq 0$. Then we can write

$$H_s(s) = \xi(s)Q(s) + d(s)Q_s(s), \qquad (34)$$

where

$$\xi(s) = -b/a, d(s) = -c/a. \qquad (35)$$

On the other hand, we can write from eq. (26)

$$H(s) = \kappa(s) + \mu(s)Q(s). \qquad (36)$$

Here, if

$$\mu(s) = \frac{\langle H(s), Q_s(s) \rangle}{\|Q_s(s)\|^2}, \qquad (37)$$

then $\kappa(s)$ is a striction curve. Taking derivative eq. (36), it is easily seen that

$$\kappa(s) = (\xi(s) - \mu(s))Q(s). \qquad (38)$$

From the last equation above, we assume that $\xi(s) = \mu(s)$. We get $\kappa_s(s) = 0$. That is $\kappa(s) = \kappa_0$. Rearranging eqs. (26) and (36), we get

$$B(s,t) = \kappa_0 + (\mu(s) + t)Q(s). \qquad (39)$$

That is, the Bisector surface $B(s,t)$ is a cone.

Theorem 3.5. Assume that $B(s,t)$ is the developable Bisector surface with a common line of curvature. Bisector surface $B(s,t)$ is a tangential developable if and only if

$$a \neq 0, \qquad (40)$$

$$\xi(s) \neq \mu_s(s). \qquad (41)$$

Proof. The proof is clear that theorem 3.3.

Example 3.6. Let us consider a fixed point and a regular unit speed curve parametrized by s in E^3 by

$$b = (1,1,1), \qquad (42)$$

$$a(s) = (\cos s, 0, \sin s). \qquad (43)$$

One calculates tangent of this curve and the direction vector, respectively, as the following

$$t_a(s) = (-\sin s, 0, \cos s), \qquad (44)$$

$$P(s) = (\cos s, 1 - \cos s - \sin s, \sin s). \qquad (45)$$

Then, the figure of the Bisector surface, $\pi/8 \leq s \leq \pi/2 \ 1 \leq t \leq 5$, is

Figure 1. Bisector surface

Example 3.7. Let us consider a fixed point and a regular unit speed curve parametrized by s in E^3 by

$$w = (1,0,1), \qquad (46)$$

$$q(s) = (0, \cos s, \sin s). \qquad (47)$$

One calculate the direction vector, respectively, as the following

$$P(s) = (\sin s - 1, -\cos s, -\sin s). \qquad (48)$$

Then, the figure of the Bisector surface is

Figure 2. Bisector Ruled surface

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