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On Total Shear Curvature of Surfaces in E^{n+2}

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Abstract

In this study we consider the surfaces in E^{n+2}. First, we give preleminaries of second fundamental form and curvature properties of the surfaces. Further, we obtained some results related with the total shear curvature of the surfaces. Finally, we give an example of a surface in Euclidean 4-space E^4 with vanishing shear curvature.

Keywords: Shear tensor, Gaussian curvature, mean curvature, umbilical surface.

1. INTRODUCTION

The second fundamental tensor of a Riemannian submanifold plays an important role to characterize the total curvature and mean curvature of the submanifold. The shape operator of a submanifold is related with the second fundamental tensor. So, in order to characterize the umbilicity of the submanifolds we need to find the shape operator of the submanifold.

Recently, Cipriani at all. introduced the concept of total shear tensor and shear operator for a Riemannian submanifold [5].

For the case of surfaces in n-dimensional Euclidean spaces E^n, K. Enomoto defined a difference function F to characterize the umbilicity of the surfaces Euclidean spaces [7].

This paper is organized as follows: In section 2, we give some basic concepts of the second fundamental form and curvatures of the surfaces in E^n. In Section 3, we define the total shear curvature of a surface in (n+2)-dimensional Euclidean space E^{n+2} with respect to its total shear tensor. Further, we obtain some results of these surfaces. In the final section, we give an example of generalized spherical surfaces in E^4 which have vanishing total shear curvature.

2. BASIC CONCEPTS

Consider a surface $M \subset E^{n+2}$ given with the parametrization $X(u,v)$. The coefficients of the first fundamental form of M are given by

$$g_{11} = \langle X_u, X_u \rangle,\ g_{12} = \langle X_u, X_v \rangle,\ g_{22} = \langle X_v, X_v \rangle \quad (2.1)$$

where $X_u, X_v \in T_pM$. For the local vector fields $X_1 = X_u$, $X_2 = X_v$ tangent to M, the equation of Gauss is defined by

$$\tilde{\nabla}_{X_i} X_j = \nabla_{X_i} X_j + h(X_i, X_j),\ 1 \le i, j \le 2 \quad (2.2)$$

where ∇ and $\tilde{\nabla}$ are the covariant derivatives of M and E^{n+2}, respectively [4].

For any arbitrary orthonormal normal frame field $\{N_1, N_2, ..., N_n\}$ of M, the Weingarten equation of M becomes

$$\tilde{\nabla}_{X_j} N_\alpha = -A_{N_\alpha} X_j + \nabla^{\perp}_{X_j} N_\alpha,\ \ X_j \in \chi(M) \quad (2.3)$$

where $\nabla^{\perp}$ is the connection in the normal bundle. These two equations satisfy the following relations:

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$$\left\langle A_{N_\alpha} X_j, X_i \right\rangle = \left\langle h(X_i, X_j), N_\alpha \right\rangle = L_{ij}^\alpha,$$
$$1 \le i, j \le 2;\ 1 \le \alpha \le n. \quad (2.4)$$

From the equation (2.2) one can get the second fundamental form

$$h(X_i, X_j) = \sum_{\alpha=1}^{n} L_{ij}^\alpha N_\alpha,\ 1 \le i, j \le 2. \quad (2.5)$$

The shape shape operator matrix of the surface M with respect to N_α is given by (see, [2])

$$A_{N_\alpha} = \begin{pmatrix} \frac{L_{11}^\alpha}{g_{11}} & \frac{1}{\sqrt{g}}\left(L_{12}^\alpha - \frac{g_{12}}{g_{11}} L_{11}^\alpha\right) \\ \frac{1}{\sqrt{g}}\left(L_{12}^\alpha - \frac{g_{12}}{g_{11}} L_{11}^\alpha\right) & \frac{1}{g}\left(g_{11} L_{22}^\alpha - 2 g_{12} L_{12}^\alpha + \frac{g_{12}^2}{g_{11}} L_{11}^\alpha\right) \end{pmatrix} \quad (2.6)$$

Then the Gaussian curvature K of the surface M is given by

$$K = \sum_{\alpha=1}^{n} \det(A_{N_\alpha}) = \frac{1}{g} \sum_{\alpha=1}^{n} (L_{11}^\alpha L_{22}^\alpha - (L_{12}^\alpha)^2), \quad (2.7)$$

where $g = g_{11} g_{22} - g_{12}^2$ is the Riemannian metric on M.

Further, the mean curvature vector of the surface M is given by

$$\vec{H} = \frac{1}{2} \sum_{\alpha=1}^{n} tr(A_{N_\alpha}) N_\alpha$$
$$= \frac{1}{2g} \sum_{\alpha=1}^{n} (L_{11}^\alpha g_{22} + L_{22}^\alpha g_{11} - 2 L_{12}^\alpha g_{12}) N_\alpha. \quad (2.8)$$

A point p is said to be *umbilical* with respect to N_α if A_{N_α} is proportional to the identity transformation of T_pM. Consequently, M is called *totally umbilical* if M is umbilical at every point of M [3].

3. THE TOTAL SHEAR CURVATURE OF THE SURFACES IN E^{n+2}

Let M be a smooth surface in (n+2)-dimensional Euclidean space E^{n+2}. The second fundamental tensor ϕ on T_pM is defined as follows; chose an orthonormal frame $\{N_1, N_2, ..., N_n\}$ of $T_p^\perp M$ and for each α, $1 \le \alpha \le n$ define maps

$$\phi_\alpha : T_pM \times T_pM \to T_p^\perp M$$

$$\phi_\alpha(X_i) = A_{N_\alpha}(X_i) - \left\langle \vec{H}, N_\alpha \right\rangle X_i,\ i = 1,2. \quad (3.1)$$

The second fundamental tensor ϕ is given by (see, [6])

$$\phi(X_i, X_j) = \sum_{\alpha=1}^{n} \left\langle \phi_\alpha(X_i), X_j \right\rangle N_\alpha. \quad (3.2)$$

Substituting (2.4), (2.5), (2.8) and (3.1) into (3.2) we obtain the following result.

Proposition 3.1. Let M be a local surface in E^{n+2} given with the regular patch $X(u,v)$. For the orthonormal frame $\{N_1, N_2, ..., N_n\}$ of $T_p^\perp M$ the second fundamental tensor ϕ is

$$\phi(X_i, X_j) = \sum (L_{ij}^\alpha - g_{ij} H_\alpha) N_\alpha \quad (3.3)$$

where H_α is the α^{th} mean curvature of the surface M defined by

$$H_\alpha = \frac{1}{2g}(L_{11}^\alpha g_{22} + L_{22}^\alpha g_{11} - 2 L_{12}^\alpha g_{12}). \quad (3.4)$$

Furthermore, the total shear tensor $\tilde{h}$ is defined as:

$$\tilde{h}(X_i, X_j) = h(X_i, X_j) - g_{ij}\vec{H},\ X_i, X_j \in \chi(M) \quad (3.5)$$

where g is the induced metric and $\vec{H}$ is the mean curvature vector of M. The shear operator associated to $N_\alpha \in \chi^\perp(M)$ is defined by:

$$\tilde{A}_{N_\alpha} = A_{N_\alpha} - tr\tilde{A}_{N_\alpha} I \quad (3.6)$$

where I denotes the identity operator [5].

The total shear tensor and shear operators are obviously related by

$$\left\langle \tilde{A}_{N_\alpha} X_i, X_j \right\rangle = \left\langle \tilde{h}(X_i, X_j), N_\alpha \right\rangle;$$
$$\forall X_i, X_j \in \chi(M),\ \forall N_\alpha \in \chi^\perp(M). \quad (3.7)$$

Substituting (2.5) and (2.8) into (3.5) we get the following result.

Proposition 3.2. Let M be a smooth surface in (n+2)-dimensional Euclidean space E^{n+2}. For the orthonormal frame $\{N_1, N_2, ..., N_n\}$ of $T_p^\perp M$ the total shear tensor $\tilde{h}$ is given by

$$\tilde{h}(X_i, X_j) = \sum_{\alpha=1}^{n} (L_{ij}^\alpha - g_{ij} H_\alpha) N_\alpha. \quad (3.8)$$

As a consequence of previous propositions we obtain the following result.

Corollary 3.3. Let M be a local surface in (n+2)-dimensional Euclidean space E^{n+2} with the regular patch $X(u,v)$. For the orthonormal frame $\{N_1, N_2, ..., N_n\}$ of $T_p^{\perp}M$ the total shear tensor $\widetilde{h}$ coincides with the second fundamental tensor ϕ, i.e.,

$$\phi(X_i, X_j) = \widetilde{h}(X_i, X_j). \tag{3.9}$$

By the use of total shear tensor $\widetilde{h}$ defined in (3.5) it is possible to define total shear curvature $\widetilde{K}$ of M as follows;

Definition 3.4. Let M be a local surface in (n+2)-dimensional Euclidean space E^{n+2}. The *total shear curvature* $\widetilde{K}$ of M at point p is defined by:

$$\widetilde{K}(p) = \frac{1}{g}\sum_{\alpha=1}^{n}\left\{\begin{array}{l}\left\langle \widetilde{h}(X_u, X_u), \widetilde{h}(X_v, X_v)\right\rangle \\ -\left\langle \widetilde{h}(X_u, X_v), \widetilde{h}(X_u, X_v)\right\rangle\end{array}\right\}. \tag{3.10}$$

The following result shows that the total shear curvature $\widetilde{K}$ is related with the Gaussian curvature K and the mean curvature $H = \left\|\vec{H}\right\|$ of M.

Lemma 3.5. Let M be a local surface in (n+2)-dimensional Euclidean space E^{n+2} with total shear curvature $\widetilde{K}$. Then

$$\widetilde{K} = K - H^2 \tag{3.11}$$

holds.

Proof. Let us assume that M be a smooth surface in E^{n+2}. Then, from (3.8) and (3.10) we have

$$\begin{aligned}\widetilde{K} &= \frac{1}{g}\sum_{\alpha=1}^{n}(L_{11}^{\alpha} - g_{11}H_{\alpha})(L_{22}^{\alpha} - g_{22}H_{\alpha}) - (L_{12}^{\alpha} - g_{12}H_{\alpha})^2 \\ &= \frac{1}{g}\sum_{\alpha=1}^{n}(L_{11}^{\alpha}L_{22}^{\alpha} - (L_{12}^{\alpha})^2) + \frac{1}{g}\sum_{\alpha=1}^{n}(g_{11}g_{22} - (g_{12})^2)H_{\alpha}^{\ 2} \\ &- \frac{1}{g}\sum_{\alpha=1}^{n}(L_{11}^{\alpha}g_{22} + L_{22}^{\alpha}g_{11} - 2L_{12}^{\alpha}g_{12})H_{\alpha}.\end{aligned}$$

Furthermore, by the use of (2.7) and (2.8) with (3.4) we get the result.

Remark. In [7] K. Enomoto defined a curvature function *F(p)* on M by

$$F(p) = (H^2 - K)(p) \tag{3.12}$$

for $p \in M$. It is to see that, the total shear curvature $\widetilde{K}$ coincides with the Enomoto curvature function F, i.e. $\widetilde{K}(p) = -F(p)$.

We obtain the following result;

Theorem 3.6. Let M be a local surface in (n+2)-dimensional Euclidean space E^{n+2}. Then for every $p \in M$ the total shear curvature $\widetilde{K}$ is identically zero if and only if M is a totally umbilical surface in E^{n+2}.

Proof. Let $\{N_1, N_2, ..., N_n\}$ be an orthonormal frame of $T_p^{\perp}M$. Using (2.7) and (2.8) the Enomoto curvature function becomes

$$F(p) = \left\|\vec{H}^2(p) - K(p)\right\| = \frac{1}{4}\sum_{\alpha=1}^{n-2}\left\{(tr(A_{N_\alpha}))^2 - 4\det(A_{N_\alpha})\right\}.$$

So, using elementary linear algebra, we can see that

$$(tr(A_{N_\alpha}))^2 - 4\det(A_{N_\alpha}) \geq 0,$$

and the equality holds if and only if A_{N_α} is proportional to the identity transformation [7]. This completes the proof of the theorem.

4. SPHERICAL SURFACES WITH VANISHING TOTAL SHEAR CURVATURE

Let M^2 be a local surface given with the regular patch (radius vector) $E^n \subset E^{n+1}$;

$$M^2 : X(u,v) = \varphi(u) + \lambda\cos\left(\frac{u}{c}\right)\rho(v) \tag{4.1}$$

where the vector function

$$\varphi(u) = (f_1(u), ..., f_n(u); 0, ..., 0)$$

given with

$$\left\|\varphi'(u)\right\|^2 = 1 - \frac{\lambda^2}{c^2}\sin^2\left(\frac{u}{c}\right)$$

and generates a generalized spherical curve with radius vector

$$\gamma(u) = \varphi(u) + \lambda\cos\left(\frac{u}{c}\right)e_{n+1} \tag{4.2}$$

and the vector function

$$\rho(v) = (0, ..., 0; g_1(v), ..., g_m(v))$$

given with $\left\|\rho(v)\right\| = 1$, $\left\|\rho'(v)\right\| = 1$. So, the surface M^2 is obtained by rotating the generalized spherical curve γ along the spherical curve ρ which is called generalized spherical surface in E^{n+m}.

For $n=2$ and $m=2$, the radius vector (4.2) satisfying the indicated properties described the generalized spherical surface given with the radius vector

$$X(u,v)=(f_1(u),f_2(u),\lambda\cos\left(\frac{u}{c}\right)\cos v,\lambda\cos\left(\frac{u}{c}\right)\sin v) \quad (4.3)$$

where

$$f_1(u)=\int\sqrt{1-\frac{\lambda^2}{c^2}\sin^2\left(\frac{u}{c}\right)}\cos\alpha(u)du,$$
$$f_2(u)=\int\sqrt{1-\frac{\lambda^2}{c^2}\sin^2\left(\frac{u}{c}\right)}\sin\alpha(u)du \quad (4.4)$$

are differentiable functions. This surface called a *generalized spherical surface* of first kind.

The tangent space of M^2 is spanned by the vector fields:

$$X_u=(f_1'(u),f_2'(u),-\frac{\lambda}{c}\sin\left(\frac{u}{c}\right)\cos v,-\frac{\lambda}{c}\sin\left(\frac{u}{c}\right)\sin v),$$
$$X_v=(0,0,-\lambda\cos\left(\frac{u}{c}\right)\sin v,\lambda\cos\left(\frac{u}{c}\right)\cos v)$$

and hence the coefficients of the first fundamental form of M^2 are

$$g_{11}=1,\ g_{12}=0,\ g_{22}=\lambda^2\cos^2\left(\frac{u}{c}\right). \quad (4.5)$$

We calculate the second partial derivatives of *X(u,v):*

$$X_{uu}=(f_1'',f_2'',-\frac{\lambda}{c^2}\cos\left(\frac{u}{c}\right)\cos v,-\frac{\lambda}{c^2}\cos\left(\frac{u}{c}\right)\sin v),$$
$$X_{uv}=(0,0,\frac{\lambda}{c}\sin\left(\frac{u}{c}\right)\sin v,-\frac{\lambda}{c}\sin\left(\frac{u}{c}\right)\cos v), \quad (4.6)$$
$$X_{vv}=(0,0,-\lambda\cos\left(\frac{u}{c}\right)\cos v,-\lambda\cos\left(\frac{u}{c}\right)\sin v).$$

Let us consider the following orthonormal normal frame field of M^2:

$$N_1=\frac{1}{\kappa_\gamma}(f_1'',f_2'',-\frac{\lambda}{c^2}\cos\left(\frac{u}{c}\right)\cos v,-\frac{\lambda}{c^2}\cos\left(\frac{u}{c}\right)\sin v), \quad (4.7)$$
$$N_2=\frac{1}{\kappa_\gamma}(-\frac{\lambda f_2'}{c^2}\cos\left(\frac{u}{c}\right)+\frac{\lambda f_2''}{c}\sin\left(\frac{u}{c}\right),-\frac{\lambda f_1''}{c}\sin\left(\frac{u}{c}\right)+\frac{\lambda f_1'}{c^2}\cos\left(\frac{u}{c}\right),$$
$$(f_1'f_2''-f_2'f_1'')\cos v,(f_1'f_2''-f_2'f_1'')\sin v)$$

where

$$\kappa_\gamma=\sqrt{(f_1'')^2+(f_2'')^2+\frac{\lambda^2}{c^4}\cos^2\left(\frac{u}{c}\right)} \quad (4.8)$$

is the curvature of the profile curve γ.

Using (4.6) and (4.7) we can calculate L_{ij}^{α} as follows;

$$L_{11}^1=\kappa_\gamma,\ L_{12}^1=L_{12}^2=L_{11}^2=0,$$
$$L_{22}^1=\frac{\lambda^2\cos^2\left(\frac{u}{c}\right)}{c^2\kappa_\gamma}, \quad (4.9)$$
$$L_{22}^2=-\frac{\lambda\cos\left(\frac{u}{c}\right)\kappa_1}{\kappa_\gamma}$$

where

$$\kappa_1=f_1'f_2''-f_2'f_1'' \quad (4.10)$$

is the curvature of the projection of the curve γ on the Oe_1e_2 - plane [1].

As a consequence of (2.7), (2.8), (4.9) and (3.11) it is easy to show that the total shear curvature $\widetilde{K}$ is identically zero if and only if

$$\left(g_{22}L_{11}^1-L_{22}^1\right)^2+\left(L_{22}^2\right)^2=0 \quad (4.11)$$

holds.

We get the following result.

Theorem 4.1. Let M^2 be a generalized spherical surface given the parametrization (4.1). If for every $p\in M$ the total shear curvature $\widetilde{K}$ is identically zero then

$$f_1(u)=\cos\alpha\int\cos\left(\frac{u}{c}\right)du,$$
$$f_2(u)=\sin\alpha\int\cos\left(\frac{u}{c}\right)du \quad (4.12)$$

holds, where $\alpha(u)$ is a constant function.

Proof. Suppose that the total shear curvature $\widetilde{K}$ of the generalized spherical surface surface M^2 vanishes identically, then by the use of (4.9) and (4.5) with (4.11) the following equalities hold:

$$\kappa_1(u)=0 \text{ and } \kappa_\gamma=\frac{1}{c}. \quad (4.13)$$

Consequently, the equation (4.10) and (4.4), (4.13) imply that $\alpha(u)$ is a constant function. Furthermore, the equation (4.8) and $\kappa_\gamma=\frac{1}{c}$ imply that $\lambda=c$. Substituting these values into (4.4) we get the result.

Remark. Since $\kappa_1(u)=0$ then the spherical surface given with the parametrization (4.12) lies in 3-dimensional Euclidean space E^3.

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