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Research Article

A collection of inequalities based on the Carleman integral inequality

Christophe Chesneau [1,*]

[1] *Department of Mathematics, LMNO, University of Caen-Normandie, 14032 Caen, France*

Abstract. This article presents a collection of new inequalities derived from the Carleman integral inequality. Several of these results incorporate auxiliary elements, such as secondary, reciprocal, and primitive functions, as well as Laplace transforms. Complete proofs are provided, making use of a variety of well-established inequalities and analytical techniques. Many of these methods are broadly applicable and can be adapted to other contexts involving integral inequalities.

Keywords. Carleman integral inequality, Laplace transform, reciprocal function, Young product inequality, Chebyshev integral inequality, Hölder integral inequality

1. Introduction

Integral inequalities are a well-known subject in mathematical analysis. Essentially, they aim to provide manageable bounds for integrals of functions under various conditions. They have concrete applications in differential equations, probability, and numerical analysis. Well-known examples include the Cauchy-Schwarz, Hölder, Minkowski, Grönwall, Young, Chebyshev, and Carleman integral inequalities. In this article, we focus on a general version of the Carleman integral inequality. To set the stage, we first present the original version below. Let $f:(0,+\infty)\rightarrow(0,+\infty)$ be a positive function. Then the original Carleman integral inequality ensures that

$$\int_0^{+\infty}\exp\left[\frac{1}{x}\int_0^x \ln[f(t)]dt\right]dx \leq e\int_0^{+\infty} f(t)dt,$$

where $e=\exp(1)\approx 2.71828$. For a modern approach to the Carleman integral inequality, see [1]. The more general version we will consider is defined over a finite integration interval. It states that, for any $a>0$, we have

$$\int_0^{a}\exp\left[\frac{1}{x}\int_0^x \ln[f(t)]dt\right]dx \leq e\int_0^{a} f(t)\left(1-\frac{t}{a}\right)dt. \tag{1.1}$$

It is also sometimes referred to as the Pólya-Knopp integral inequality (see [2, 3]). For the sake of completeness, we will provide its proof in the appendix, which is mainly adapted from [1]. Further significant results on Carleman-type integral inequalities can be found in [1–17].

In this article, we present a new collection of integral inequalities derived from Inequality (1.1). Our results extend the traditional framework in several ways. Some involve a secondary function, while others rely on reciprocal functions, primitives, or Laplace transforms. A few include parameters or monotonicity assumptions. Each

*Corresponding author
Email address: `christophe.chesneau@gmail.com` (Christophe Chesneau)
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result offers a unique viewpoint and makes use of classical tools such as logarithmic properties, Hölder integral inequality, Chebyshev integral inequality, or the Young product inequality. Several of the proposed inequalities appear to be new. Some extend existing results, while others appear to have no direct counterpart in the literature. Together, they demonstrate the versatility and adaptability of the Carleman framework to new analytical contexts. To maintain flexibility in the integrability conditions, we primarily consider a finite integration interval of the form $(0,a)$ with $a>0$. However, most of the results can be extended to the interval $(0,+\infty)$ by taking the limit as $a\to+\infty$.

The rest of the article is as follows: Section 2 presents the collection of integral inequalities. The proofs are given in Section 3. Section 4 provides a conclusion. The article concludes with an appendix offering a simple proof of Inequality (1.1).

2. Collection of integral inequalities

Our collection contains ten integral inequalities. It is presented below in order of increasing proof complexity (subjective). For simplicity, we refer to the Carleman integral inequality as the inequality in Inequality (1.1).

The result below provides an alternative perspective on a Carleman-type integral inequality involving a secondary function. The proof is based on a fundamental property of the logarithmic function and the Carleman integral inequality.

Proposition 2.1. *Let $a>0$ and $f,g:(0,a)\to(0,+\infty)$. Then we have*

$$\int_0^a \exp\left[\frac{1}{x}\int_0^x g(t)\ln[f(t)]dt\right]dx \le e\int_0^a f^{g(t)}(t)\left(1-\frac{t}{a}\right)dt.$$

If we take $g=1$, this result simplifies to the Carleman integral inequality.

The result below presents a product approach that still uses a secondary function. The proof remains based on a fundamental property of the logarithmic function and the Carleman integral inequality.

Proposition 2.2. *Let $a>0$ and $f,g:(0,a)\to(0,+\infty)$. Then we have*

$$\int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[f(t)]dt\right]\exp\left[\frac{1}{x}\int_0^x \ln[g(t)]dt\right]dx \le e\int_0^a f(t)g(t)\left(1-\frac{t}{a}\right)dt.$$

If we take $g=1$, this result simplifies to the Carleman integral inequality.

A Carleman-type integral inequality using the reciprocal of the main function is presented below. The proof is based on a logarithmic inequality and the Carleman integral inequality.

Proposition 2.3. *Let $a>0$ and $f:(0,a)\to(0,+\infty)$. Then we have*

$$\int_0^a \exp\left[-\frac{1}{x}\int_0^x \frac{1}{f(t)}dt\right]dx \le \int_0^a f(t)\left(1-\frac{t}{a}\right)dt.$$

This result thus goes beyond the standard logarithmic form.

The proposition below concerns a sophisticated Carleman-type integral inequality involving a secondary function, and subject to monotonic assumptions. The proof is based on the Chebyshev integral inequality and the Carleman integral inequality.

Proposition 2.4. *Let $a>0$ and $f,g:(0,a)\to(0,+\infty)$ be monotonic with $\ln(f)$ and g having the same monotonicity (so these functions are synchronous). Then we have*

$$\int_0^a \exp\left\{\frac{1}{x^2}\left[\int_0^x \ln[f(t)]dt\right]\left[\int_0^x g(t)dt\right]\right\}dx \le e\int_0^a f^{g(t)}(t)\left(1-\frac{t}{a}\right)dt.$$

If we take $g = 1$, this result simplifies to the Carleman integral inequality.

To the best of our knowledge, the proposition below is the first to present a Carleman-type integral inequality via the Laplace transform. The proof is based on an exponential development and the Carleman integral inequality.

Proposition 2.5. *Let $a > 0$ and $f : (0,a) \to (0,+\infty)$. Then, for any $s > 0$, we have*

$$\mathscr{L}\left(\exp\left[\frac{1}{\cdot}\int_0^{\cdot}\ln[f(t)]dt\right]\right)\left(\frac{s}{2}\right) \leq e\mathscr{L}\left[f(\cdot)\left(1-\frac{\cdot}{a}\right)\right](s),$$

where $\mathscr{L}(h)(s)$ denotes the Laplace transform of a function $h : (0,a) \to (0,+\infty)$ at s, defined by

$$\mathscr{L}(h)(s) = \int_0^a h(t)e^{-st}dt.$$

The proposition below considers a Carleman-type integral inequality with a secondary function and an adjustable parameter. The proof is based on the Hölder integral inequality and the Carleman integral inequality.

Proposition 2.6. *Let $a > 0$, $p > 1$, $q = p/(p-1)$, $f, g : (0,a) \to (0,+\infty)$. Then, for any $s > 0$, we have*

$$\int_0^a g(x)\exp\left[\frac{1}{px}\int_0^x \ln[f(t)e^{-st}]dt\right]dx \leq e^{1/p}\left[\int_0^a f(t)\left(1-\frac{t}{a}\right)dt\right]^{1/p}\left[\int_0^a g^q(t)dt\right]^{1/q}.$$

If we take $g = 1$ and $p \to 1$, this result simplifies to the Carleman integral inequality.

Another Carleman-type integral inequality is investigated in the result below. This inequality is considered with a secondary function in its primitive form and subject to boundedness conditions. The proof is based on a change of variables, the Carleman integral inequality, and a fundamental inequality.

Proposition 2.7. *Let $a > 0$, $f, g : (0,a) \to (0,+\infty)$ and $G(x) = \int_0^x g(t)dt$ such that $G(a) \leq a$ and there exists a constant $\kappa > 0$ such that, for any $x \in (0,a)$, $g(x) \geq \kappa$. Then we have*

$$\int_0^a \exp\left[\frac{1}{G(x)}\int_0^{G(x)}\ln[f(t)]dt\right]dx \leq \frac{e}{\kappa}\int_0^{G(a)} f(t)\left(1-\frac{t}{a}\right)dt.$$

A modification of the previous proposition is given below, relaxing the boundedness assumption on g. The proof is based on several changes of variables and the Carleman integral inequality.

Proposition 2.8. *Let $a > 0$, $f, g : (0,a) \to (0,+\infty)$ and $G(x) = \int_0^x g(t)dt$. Then we have*

$$\int_0^a g(x)\exp\left[\frac{1}{G(x)}\int_0^x g(t)\ln[f(t)]dt\right]dx \leq e\int_0^a f(t)\left[1-\frac{G(t)}{G(a)}\right]g(t)dt.$$

If we take $g = 1$, this result simplifies to the Carleman integral inequality.

The proposition below considers a Carleman-type integral inequality with a secondary function and an adjustable parameter. The proof is based on the Young product inequality, the Hölder integral inequality and the Carleman integral inequality.

Proposition 2.9. *Let $a > 0$, $p > 1$, $q = p/(p-1)$, and $f, g : (0,a) \to (1,+\infty)$. Then we have*

$$\int_0^a \exp\left[\frac{1}{x}\int_0^x \{\ln[f(t)]\}^{1/p}\{\ln[g(t)]\}^{1/q}dt\right]dx$$
$$\leq e\left[\int_0^a f(t)\left(1-\frac{t}{a}\right)dt\right]^{1/p}\left[\int_0^a g(t)\left(1-\frac{t}{a}\right)dt\right]^{1/q}.$$

Note that f and g must be greater than one to validate the expressions $\{\ln[f(t)]\}^{1/p}$ and $\{\ln[g(t)]\}^{1/q}$. In the case of $a \to +\infty$, this result is not interesting because the integral of the functions necessarily diverges due to the boundedness assumption.

A generalization of the previous proposition is given below. The proof is based on the generalized Young product inequality, the generalized Hölder integral inequality and the Carleman integral inequality.

Proposition 2.10. *Let $a > 0$, $n \in \mathbb{N}\backslash\{0\}$, $p_1, \ldots, p_n > 1$ such that $\sum_{i=1}^{n} 1/p_i = 1$, and $f_1, \ldots, f_n : (0,a) \to (1,+\infty)$. Then we have*

$$\int_0^a \exp\left[\frac{1}{x}\int_0^x \prod_{i=1}^{n} \{\ln[f_i(t)]\}^{1/p_i} dt\right] dx \leq e \prod_{i=1}^{n} \left[\int_0^a f_i(t)\left(1-\frac{t}{a}\right) dt\right]^{1/p_i}.$$

If we take $n = 2$, this result reduces to Proposition 2.9.

To the best of our knowledge, these inequalities are a novel addition to the literature. The rest of the article is devoted to proving all the propositions.

3. Proofs

We recall that all the integrals involved are supposed to converge, and that the Carleman integral inequality applied to the function f indicates that in Inequality (1.1).

Proof. (Proof of Proposition 2.1) Using a basic property of the logarithmic function and the Carleman integral inequality applied to f^g, we get

$$\int_0^a \exp\left[\frac{1}{x}\int_0^x g(t)\ln[f(t)]dt\right] dx = \int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[f^{g(t)}(t)]dt\right] dx$$
$$\leq e\int_0^a f^{g(t)}(t)\left(1-\frac{t}{a}\right) dt.$$

This ends the proof of Proposition 2.1. □

Proof. (Proof of Proposition 2.2) Using basic properties of the exponential and logarithmic functions, and the Carleman integral inequality applied to fg, we obtain

$$\int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[f(t)]dt\right] \exp\left[\frac{1}{x}\int_0^x \ln[g(t)]dt\right] dx$$
$$= \int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[f(t)]dt + \frac{1}{x}\int_0^x \ln[g(t)]dt\right] dx$$
$$= \int_0^a \exp\left[\frac{1}{x}\int_0^x \{\ln[f(t)] + \ln[g(t)]\}dt\right] dx$$
$$= \int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[f(t)g(t)]dt\right] dx$$
$$\leq e\int_0^a f(t)g(t)\left(1-\frac{t}{a}\right) dt.$$

This ends the proof of Proposition 2.2. □

Proof. (Proof of Proposition 2.3) We propose two proofs on the same mathematical basis.

- *Proof 1.* Using the well-known logarithmic inequality $1 - 1/t \leq \ln(t)$ for $t > 0$, we get

$$1-\frac{1}{x}\int_0^x \frac{1}{f(t)}dt = \frac{1}{x}\int_0^x \left[1-\frac{1}{f(t)}\right] dt \leq \frac{1}{x}\int_0^x \ln[f(t)]dt.$$

This and the Carleman integral inequality imply that

$$\begin{aligned}
&\int_0^a \exp\left[-\frac{1}{x}\int_0^x \frac{1}{f(t)}dt\right]dx = \int_0^a \exp\left\{\left[1-\frac{1}{x}\int_0^x \frac{1}{f(t)}dt\right]-1\right\}dx \\
&\leq \int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[f(t)]dt - 1\right]dx = e^{-1}\int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[f(t)]dt\right]dx \\
&\leq e^{-1}e\int_0^a f(t)\left(1-\frac{t}{a}\right)dt = \int_0^a f(t)\left(1-\frac{t}{a}\right)dt.
\end{aligned}$$

- *Proof 2*. We can write

$$\int_0^a \exp\left[-\frac{1}{x}\int_0^x \frac{1}{f(t)}dt\right]dx = \int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[f_\star(t)]dt\right]dx,$$

where $f_\star(t) = \exp[-1/f(t)]$. Applying the Carleman integral inequality and using the well-known logarithmic inequality $1-1/t \leq \ln(t)$ for $t > 0$, we obtain

$$\begin{aligned}
&\int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[f_\star(t)]dt\right]dx \leq e\int_0^a f_\star(t)\left(1-\frac{t}{a}\right)dt \\
&= e\int_0^a \exp\left[-\frac{1}{f(t)}\right]\left(1-\frac{t}{a}\right)dt = \int_0^a \exp\left[1-\frac{1}{f(t)}\right]\left(1-\frac{t}{a}\right)dt \\
&\leq \int_0^a \exp\{\ln[f(t)]\}\left(1-\frac{t}{a}\right)dt = \int_0^a f(t)\left(1-\frac{t}{a}\right)dt.
\end{aligned}$$

This concludes the proof of Proposition 2.3. □

Proof. (Proof of Proposition 2.4) Since $\ln(f)$ and g have the same monotonicity, the Chebyshev integral inequality gives

$$\begin{aligned}
&\frac{1}{x^2}\left[\int_0^x \ln[f(t)]dt\right]\left[\int_0^x g(t)dt\right] = \left[\frac{1}{x}\int_0^x \ln[f(t)]dt\right]\left[\frac{1}{x}\int_0^x g(t)dt\right] \\
&\leq \frac{1}{x}\int_0^x g(t)\ln[f(t)]dt.
\end{aligned}$$

We thus derive

$$\int_0^a \exp\left\{\frac{1}{x^2}\left[\int_0^x \ln[f(t)]dt\right]\left[\int_0^x g(t)dt\right]\right\}dx \leq \int_0^a \exp\left[\frac{1}{x}\int_0^x g(t)\ln[f(t)]dt\right]dx. \tag{3.1}$$

It follows from Proposition 2.1 that

$$\int_0^a \exp\left[\frac{1}{x}\int_0^x g(t)\ln[f(t)]dt\right]dx \leq e\int_0^a f^{g(t)}(t)\left(1-\frac{t}{a}\right)dt. \tag{3.2}$$

Joining Equations (3.1) and (3.2), we get

$$\int_0^a \exp\left\{\frac{1}{x^2}\left[\int_0^x \ln[f(t)]dt\right]\left[\int_0^x g(t)dt\right]\right\}dx \leq e\int_0^a f^{g(t)}(t)\left(1-\frac{t}{a}\right)dt.$$

This concludes the proof of Proposition 2.4. □

Proof. (Proof of Proposition 2.5) Applying the Carleman integral inequality to $fe^{-s\cdot}$, we obtain

$$\int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[f(t)e^{-st}]dt\right]dx \leq e\int_0^a f(t)\left(1-\frac{t}{a}\right)e^{-st}dt = e\mathscr{L}\left[f(\cdot)\left(1-\frac{\cdot}{a}\right)\right](s). \tag{3.3}$$

Using standard properties of the logarithmic function and integration calculus, we can express the left-hand side term as follows:

$$\begin{aligned}
&\int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[f(t)e^{-st}]dt\right]dx = \int_0^a \exp\left[\frac{1}{x}\int_0^x \{\ln(e^{-st})+\ln[f(t)]\}dt\right]dx \\
&= \int_0^a \exp\left[\frac{1}{x}\int_0^x \{-st+\ln[f(t)]\}dt\right]dx \\
&= \int_0^a \exp\left[-s\frac{1}{x}\int_0^x tdt + \frac{1}{x}\int_0^x \ln[f(t)]\}dt\right]dx \\
&= \int_0^a \exp\left[-s\frac{x}{2} + \frac{1}{x}\int_0^x \ln[f(t)]\}dt\right]dx \\
&= \int_0^a e^{-sx/2}\exp\left[\frac{1}{x}\int_0^x \ln[f(t)]\}dt\right]dx \\
&= \mathscr{L}\left(\exp\left[\frac{1}{\cdot}\int_0^{\cdot} \ln[f(t)]\}dt\right]\right)\left(\frac{s}{2}\right).
\end{aligned} \tag{3.4}$$

Joining Equations (3.3) and (3.4), we obtain

$$\mathscr{L}\left(\exp\left[\frac{1}{\cdot}\int_0^{\cdot} \ln[f(t)]\}dt\right]\right)\left(\frac{s}{2}\right) \le e\mathscr{L}\left[f(\cdot)\left(1-\frac{\cdot}{a}\right)\right](s).$$

This concludes the proof of Proposition 2.5. □

Proof. (Proof of Proposition 2.6) It follows from the Hölder integral inequality and the Carleman integral inequality that

$$\begin{aligned}
&\int_0^a g(x)\exp\left[\frac{1}{px}\int_0^x \ln[f(t)e^{-st}]dt\right]dx \\
&\le \left[\int_0^a \exp\left[\frac{p}{px}\int_0^x \ln[f(t)e^{-st}]dt\right]dx\right]^{1/p}\left[\int_0^a g^q(t)dt\right]^{1/q} \\
&= \left[\int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[f(t)e^{-st}]dt\right]dx\right]^{1/p}\left[\int_0^a g^q(t)dt\right]^{1/q} \\
&\le \left\{e\left[\int_0^a f(t)\left(1-\frac{t}{a}\right)dt\right]\right\}^{1/p}\left[\int_0^a g^q(t)dt\right]^{1/q} \\
&= e^{1/p}\left[\int_0^a f(t)\left(1-\frac{t}{a}\right)dt\right]^{1/p}\left[\int_0^a g^q(t)dt\right]^{1/q}.
\end{aligned}$$

This ends the proof of Proposition 2.6. □

Proof. (Proof of Proposition 2.7) Using the change of variables $x = G^{-1}(u)$ with $G(0) = 0$, we obtain

$$\begin{aligned}
&\int_0^a \exp\left[\frac{1}{G(x)}\int_0^{G(x)} \ln[f(t)]dt\right]dx \\
&= \int_0^{G(a)} \exp\left[\frac{1}{G[G^{-1}(u)]}\int_0^{G[G^{-1}(u)]} \ln[f(t)]dt\right]\frac{1}{g[G^{-1}(u)]}du \\
&= \int_0^{G(a)} \exp\left[\frac{1}{u}\int_0^u \ln[f(t)]dt\right]\frac{1}{g[G^{-1}(u)]}du.
\end{aligned} \tag{3.5}$$

Using the boundedness assumption on g and applying the Carleman integral inequality, we obtain

$$\begin{aligned}
&\int_0^{G(a)} \exp\left[\frac{1}{u}\int_0^u \ln[f(t)]dt\right]\frac{1}{g[G^{-1}(u)]}du \\
&\le \frac{1}{\kappa}\int_0^{G(a)} \exp\left[\frac{1}{u}\int_0^u \ln[f(t)]dt\right]du \\
&\le \frac{1}{\kappa}\times e\int_0^{G(a)} f(t)\left(1-\frac{t}{a}\right)dt = \frac{e}{\kappa}\int_0^{G(a)} f(t)\left(1-\frac{t}{a}\right)dt.
\end{aligned} \tag{3.6}$$

Joining Equations (3.5) and (3.6), we obtain

$$\int_0^a \exp\left[\frac{1}{G(x)}\int_0^{G(x)} \ln[f(t)]dt\right]dx \le \frac{e}{\kappa}\int_0^{G(a)} f(t)\left(1-\frac{t}{a}\right)dt.$$

This ends the proof of Proposition 2.7. □

Proof. (Proof of Proposition 2.8) Using the change of variables $t = G^{-1}(u)$ with $G(0) = 0$, then the change of variables $x = G^{-1}(v)$ with $G(0) = 0$, we obtain

$$\begin{aligned}
&\int_0^a g(x)\exp\left[\frac{1}{G(x)}\int_0^x g(t)\ln[f(t)]dt\right]dx \\
&= \int_0^a g(x)\exp\left[\frac{1}{G(x)}\int_0^{G(x)} g[G^{-1}(u)]\ln\{f[G^{-1}(u)]\}\frac{1}{g[G^{-1}(u)]}du\right]dx \\
&= \int_0^a g(x)\exp\left[\frac{1}{G(x)}\int_0^{G(x)} \ln\{f[G^{-1}(u)]\}du\right]dx \\
&= \int_0^{G(a)} g[G^{-1}(v)]\exp\left[\frac{1}{G[G^{-1}(v)]}\int_0^{G[G^{-1}(v)]} \ln\{f[G^{-1}(u)]\}du\right]\frac{1}{g[G^{-1}(v)]}dv \\
&= \int_0^{G(a)} \exp\left[\frac{1}{v}\int_0^v \ln\{f[G^{-1}(u)]\}du\right]dv. \qquad (3.7)
\end{aligned}$$

Applying the Carleman integral inequality to $f[G^{-1}]$ on the interval $(0, G(a))$, we get

$$\int_0^{G(a)} \exp\left[\frac{1}{v}\int_0^v \ln\{f[G^{-1}(u)]\}du\right]dv \le e\int_0^{G(a)} f[G^{-1}(u)]\left(1-\frac{u}{G(a)}\right)du. \qquad (3.8)$$

In this last integral, making the change of variables $u = G(w)$ with $G(0) = 0$, we obtain

$$\begin{aligned}
&\int_0^{G(a)} f[G^{-1}(u)]\left(1-\frac{u}{G(a)}\right)du = \int_0^a f\{G^{-1}[G(w)]\}\left(1-\frac{G(w)}{G(a)}\right)g(w)dw \\
&= \int_0^a f(w)\left(1-\frac{G(w)}{G(a)}\right)g(w)dw. \qquad (3.9)
\end{aligned}$$

Joining Equations (3.7), (3.8) and (3.9) and uniformizing the notation, we get

$$\int_0^a g(x)\exp\left[\frac{1}{G(x)}\int_0^x g(t)\ln[f(t)]dt\right]dx \le e\int_0^a f(t)\left(1-\frac{G(t)}{G(a)}\right)g(t)dt.$$

This ends the proof of Proposition 2.8. □

Proof. (Proof of Proposition 2.9) Applying the Young product inequality, we get

$$\{\ln[f(t)]\}^{1/p}\{\ln[g(t)]\}^{1/q} \le \frac{1}{p}\ln[f(t)] + \frac{1}{q}\ln[g(t)],$$

so that

$$\begin{aligned}
&\exp\left[\frac{1}{x}\int_0^x \{\ln[f(t)]\}^{1/p}\{\ln[g(t)]\}^{1/q}dt\right] \\
&\le \exp\left[\frac{1}{x}\int_0^x \left[\frac{1}{p}\ln[f(t)] + \frac{1}{q}\ln[g(t)]\right]dt\right] \\
&= \exp\left[\frac{1}{xp}\int_0^x \ln[f(t)]dt\right]\exp\left[\frac{1}{xq}\int_0^x \ln[g(t)]dt\right].
\end{aligned}$$

We thus derive

$$\begin{aligned}
&\int_0^a \exp\left[\frac{1}{x}\int_0^x \{\ln[f(t)]\}^{1/p}\{\ln[g(t)]\}^{1/q}dt\right]dx \\
&\le \int_0^a \exp\left[\frac{1}{xp}\int_0^x \ln[f(t)]dt\right]\exp\left[\frac{1}{xq}\int_0^x \ln[g(t)]dt\right]dx. \qquad (3.10)
\end{aligned}$$

Using the Hölder integral inequality, we have

$$\begin{aligned}
&\int_0^a \exp\left[\frac{1}{xp}\int_0^x \ln[f(t)]dt\right]\exp\left[\frac{1}{xq}\int_0^x \ln[g(t)]dt\right]dx \\
&\leq \left[\int_0^a \exp\left[\frac{p}{xp}\int_0^x \ln[f(t)]dt\right]dx\right]^{1/p}\left[\int_0^a \exp\left[\frac{q}{xq}\int_0^x \ln[g(t)]dt\right]dx\right]^{1/q} \\
&= \left[\int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[f(t)]dt\right]dx\right]^{1/p}\left[\int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[g(t)]dt\right]dx\right]^{1/q}. \qquad (3.11)
\end{aligned}$$

Applying the Carleman integral inequality to f and g, and using $1/p+1/q=1$, we obtain

$$\begin{aligned}
&\left[\int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[f(t)]dt\right]dx\right]^{1/p}\left[\int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[g(t)]dt\right]dx\right]^{1/q} \\
&\leq \left[e\int_0^a f(t)\left(1-\frac{t}{a}\right)dt\right]^{1/p}\left[e\int_0^a g(t)\left(1-\frac{t}{a}\right)dt\right]^{1/q} \\
&= e\left[\int_0^a f(t)\left(1-\frac{t}{a}\right)dt\right]^{1/p}\left[\int_0^a g(t)\left(1-\frac{t}{a}\right)dt\right]^{1/q}. \qquad (3.12)
\end{aligned}$$

Joining Equations (3.10), (3.11) and (3.12), we get

$$\begin{aligned}
&\int_0^a \exp\left[\frac{1}{x}\int_0^x \{\ln[f(t)]\}^{1/p}\{\ln[g(t)]\}^{1/q}dt\right]dx \\
&\leq e\left[\int_0^a f(t)\left(1-\frac{t}{a}\right)dt\right]^{1/p}\left[\int_0^a g(t)\left(1-\frac{t}{a}\right)dt\right]^{1/q}.
\end{aligned}$$

This ends the proof of Proposition 2.9. □

Proof. (Proof of Proposition 2.10) Applying the generalized Young product inequality, we get

$$\prod_{i=1}^n \{\ln[f_i(t)]\}^{1/p_i} \leq \sum_{i=1}^n \frac{1}{p_i}\ln[f_i(t)],$$

so that

$$\begin{aligned}
&\exp\left[\frac{1}{x}\int_0^x \prod_{i=1}^n \{\ln[f_i(t)]\}^{1/p_i}dt\right] \leq \exp\left[\frac{1}{x}\int_0^x \left[\sum_{i=1}^n \frac{1}{p_i}\ln[f_i(t)]\right]dt\right] \\
&\leq \exp\left[\sum_{i=1}^n \frac{1}{p_i}\left[\frac{1}{x}\int_0^x \ln[f_i(t)]dt\right]\right] = \prod_{i=1}^n \left[\exp\left[\frac{1}{x}\int_0^x \ln[f_i(t)]dt\right]\right]^{1/p_i}. \qquad (3.13)
\end{aligned}$$

Using the generalized Hölder integral inequality (see [18, 19]), we have

$$\begin{aligned}
&\int_0^a \left\{\prod_{i=1}^n \left[\exp\left[\frac{1}{x}\int_0^x \ln[f_i(t)]dt\right]\right]^{1/p_i}\right\}dx \\
&\leq \prod_{i=1}^n \left\{\int_0^a \left[\exp\left[\frac{1}{x}\int_0^x \ln[f_i(t)]dt\right]\right]dx\right\}^{1/p_i}. \qquad (3.14)
\end{aligned}$$

Applying the Carleman integral inequality to $f_1,\ldots,f_n$ and using $\sum_{i=1}^n 1/p_i = 1$, we obtain

$$\begin{aligned}
&\prod_{i=1}^n \left\{\int_0^a \left[\exp\left[\frac{1}{x}\int_0^x \ln[f_i(t)]dt\right]\right]dx\right\}^{1/p_i} \\
&\leq \prod_{i=1}^n \left[e\int_0^a f_i(t)\left(1-\frac{t}{a}\right)dt\right]^{1/p_i} = e^{\sum_{i=1}^n 1/p_i}\prod_{i=1}^n \left[\int_0^a f_i(t)\left(1-\frac{t}{a}\right)dt\right]^{1/p_i} \\
&= e\prod_{i=1}^n \left[\int_0^a f_i(t)\left(1-\frac{t}{a}\right)dt\right]^{1/p_i}. \qquad (3.15)
\end{aligned}$$

Joining Equations (3.13), (3.14) and (3.15), we get

$$\int_0^a \exp\left[\frac{1}{x}\int_0^x \prod_{i=1}^n \{\ln[f_i(t)]\}^{1/p_i}dt\right]dx \leq e\prod_{i=1}^n \left[\int_0^a f_i(t)\left(1-\frac{t}{a}\right)dt\right]^{1/p_i}.$$

This ends the proof of Proposition 2.10. □

4. Conclusion

This collection of inequalities builds on the Carleman integral inequality framework by introducing new forms and techniques. From a theoretical perspective, the results demonstrate the versatility of Carleman-type integral inequalities when combined with auxiliary components, including functions, parameters and transforms. In practice, the methods employed in the proofs can be applied to solve a wider range of inequalities. These findings could lay the basis for further research in analysis and related fields.

Appendix: Proof of the Carleman integral inequality

A simple proof of the Carleman integral inequality given in Inequality (1.1) is developed below. It is a slight modification to that of the Carleman integral inequality given in [1]. We provide it mainly for the sake of completeness.

Using a basic property of the logarithmic function, we obtain

$$\begin{aligned}\int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[f(t)]dt\right]dx &= \int_0^a \exp\left[\frac{1}{x}\int_0^x \{\ln[tf(t)]dt-\ln(t)\}\right]dx\\ &= \int_0^a \exp\left[-\frac{1}{x}\int_0^x \ln(t)dx+\frac{1}{x}\int_0^x \ln[tf(t)]dt\right]dx\\ &= \int_0^a \exp\left[-\frac{1}{x}\int_0^x \ln(t)dx\right]\exp\left[\frac{1}{x}\int_0^x \ln[tf(t)]dt\right]dx. \end{aligned} \tag{4.1}$$

An integral calculus gives

$$-\frac{1}{x}\int_0^x \ln(t)dt = -\frac{1}{x}\left[t\ln(t)-t\right]_{t=0}^{t=x} = -\ln(x)+1. \tag{4.2}$$

Applying the Jensen integral inequality, we get

$$\exp\left[\frac{1}{x}\int_0^x \ln[tf(t)]dt\right] \le \frac{1}{x}\int_0^x \exp\left[\ln[tf(t)]\right]dt = \frac{1}{x}\int_0^x tf(t)dt. \tag{4.3}$$

Joining Equations (4.2) and (4.3), integrating with respect to $x \in (0,+\infty)$, exchanging the order of integration by the Fubini-Tonelli integral formula and using a basic integral calculus, we obtain

$$\begin{aligned}&\int_0^a \exp\left[-\frac{1}{x}\int_0^x \ln(t)dx\right]\exp\left[\frac{1}{x}\int_0^x \ln[tf(t)]dt\right]dx\\ &\le \int_0^a \exp\left[-\ln(x)+1\right]\times\left[\frac{1}{x}\int_0^x tf(t)dt\right]dx = e\int_0^a \frac{1}{x^2}\left[\int_0^x tf(t)dt\right]dx\\ &= e\int_0^a tf(t)\left[\int_t^a \frac{1}{x^2}dx\right]dt = e\int_0^a tf(t)\left[-\frac{1}{x}\right]_t^a dt\\ &= e\int_0^a tf(t)\times\left(\frac{1}{t}-\frac{1}{a}\right)dt = e\int_0^a f(t)\left(1-\frac{t}{a}\right)dt. \end{aligned} \tag{4.4}$$

Joining Equations (4.1) and (4.4), we get

$$\int_0^a \exp\left[\frac{1}{x}\int_0^x \ln[f(t)]dt\right]dx \le e\int_0^a f(t)\left(1-\frac{t}{a}\right)dt.$$

The elegant proof of the Carleman integral inequality is at an end, as is the article.

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