Journal of New Theory
dergipark.org.tr/tr/pub/jnt
Open Access

ISSN: 2149-1402 (51) (2025) 76-91 doi.org/10.53570/jnt.1700338

A Multi-Species Keller-Segel Chemotaxis-Competition Model: Global Existence, Boundedness, and Mass Persistence

Halil İbrahim Kurt[1], Mustafa Ekici[2]

Article Info
Received: 15 May 2025
Accepted: 18 Jun 2025
Published: 30 Jun 2025
Research Article

Abstract— This paper investigates the population dynamics of solutions to a parabolic-parabolic-elliptic type of multi-species Keller-Segel chemotaxis system under the Neumann boundary conditions in a smoothly bounded domain. It studies dynamical properties such as L^ρ-bounds, global existence, global boundedness, and combined mass persistence of solutions for the aforementioned system. Under certain specified parameter conditions, the paper shows that the system admits a unique global classical solution that remains uniformly bounded from above. Furthermore, it establishes that the entire population persists at all times; in other words, this study proves that any globally bounded classical solution maintains a positive lower mass bound.

Keywords — *Keller-Segel system, chemotaxis, global existence, global boundedness, mass persistence*

Mathematics Subject Classification (2020) 35K57, 92C17

1. Introduction

Chemotaxis is the process of directed movement of mobile organisms or cells in response to a chemical gradient. Keller and Segel [1, 2] first established a mathematical model to explain this phenomenon in the late 1970s. This phenomenon plays a key role in many biological processes, including population dynamics, immune cell migration, and tumor growth. In the aftermath of this period, many authors investigated various chemotaxis models from various perspectives, including local existence, uniqueness, finite time blow-up, global existence and boundedness, persistence, stability, and special solutions in various research publications, making significant contributions to the mentioned problems above. For further details, see [3–5].

Regarding these problems in more general frameworks, including two species with chemical signals, some variants of the model of (1.1) have also been researched in various ways. A comprehensive comparison exists between one-species and multi-species chemotaxis models, addressing their mathematical frameworks, biological implications, and essential dynamical characteristics, and they are applicable in symbiotic or competitive systems, predator-prey dynamics, and host-pathogen interactions. The main difference between one-species and two-species chemotaxis models lies in the number of interacting populations and their interaction with chemical signals. In simpler terms, in a one-species system,

[1]hkurt@ktu.edu.tr; [2]mustafa.ekici@comu.edu.tr (Corresponding Author)
[1]Department of Mathematics, Faculty of Science, Karadeniz Technical University, Trabzon, Türkiye
[2]Department of Mathematics and Science Education, Faculty of Education, Çanakkale Onsekiz Mart University, Çanakkale, Türkiye

the cell population reacts to a single stimuli, while in two-species models, the system comprises two populations that interact with one another and respond to a single chemical. It is well-known that multi-species chemotaxis models offer a more accurate representation of biological scenarios. Nevertheless, these models, while more realistic, working on those ones mathematically is quite challenging. It is also natural to regard competition and cooperation in chemotaxis models. In this context, the paper presents a model that incorporates three interacting populations responding to a single chemical, which allow for to study new challenges and areas of research, such as coexistence and extinction, provided that a globally bounded classical solution exists. In this regard, this article first studies global existence, global boundedness, and persistence of solutions within the following model, thereby providing a way for an exploration of the model's long-term behaviors. However, we leave open the topics associated with the large time behaviors to investigate somewhere else.

This paper aims to investigate a more realistic scenario in a biological environment by illustrating the interactions among three different cell populations as they react to one chemical. This is far more realistic compared to the previous works, revealing numerous intriguing dynamic scenarios within such chemotaxis systems. In this respect, this research paper analyzes the dynamical characteristics of the population as described by the subsequent parabolic-parabolic-parabolic-elliptic chemotaxis growth model involving strong logistic kinetics:

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla z) + u(h_1 - k_1 u) \\ v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla z) + v(h_2 - k_2 v) \\ w_t = \Delta v - \chi_3 \nabla \cdot (w \nabla z) + w(h_3 - k_3 w) \\ 0 = \Delta z - az + bu + cv + dw \end{cases} \tag{1.1}$$

with the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0$$

and the initial data $u_0(x) := u(0, x; u_0)$, $v_0(x) := v(0, x; u_0)$, and $w_0(x) := w(0, x; u_0)$ satisfying

$$u_0, v_0, w_0 \in C^0(\bar{S}) \quad \text{and} \quad u_0, v_0, w_0 \geq 0 \tag{1.2}$$

where $S \subseteq \mathbb{R}^n$ with $n \geq 1$ is a smooth bounded domain; $a, b, c, d > 0$, and $\chi_i, h_i, k_i > 0$, for $i \in \{1, 2, 3\}$. Moreover, assume that

$$k_1 > (n-2) \left\{ \frac{b\chi_1}{n} + \frac{(c+d)\chi_1}{n+2} + \frac{2b(\chi_2 + \chi_3)}{n(n+2)} \right\} \tag{1.3}$$

$$k_2 > (n-2) \left\{ \frac{(b+d)\chi_2}{n+2} + \frac{c\chi_2}{n} + \frac{2c(\chi_1 + \chi_3)}{n(n+2)} \right\} \tag{1.4}$$

and

$$k_3 > (n-2) \left\{ \frac{(b+c)\chi_3}{n+2} + \frac{2d(\chi_1 + \chi_2)}{n(n+2)} + \frac{d\chi_3}{n} \right\} \tag{1.5}$$

From a biological standpoint, the system described by (1.1) illustrates the evolution of three competing mobile species, namely u, v, and w, which are affected by a single chemical substance z. In this context, the mobile cells u, v, and z are attracted by the chemical substance z. In the framework of (1.1), the unknown functions $u(x,t)$, $v(x,t)$, and $w(x,t)$ represent the density of cells, while $z(x,t)$ indicates the concentration of the chemical signal at time t and space $x \in \Omega$; the cross-diffusion terms $-\chi_1 \nabla \cdot (u \nabla z)$, $-\chi_2 \nabla \cdot (v \nabla z)$, and $-\chi_3 \nabla \cdot (w \nabla z)$ reflect the chemotactic movement, where $\chi_1, \chi_2, \chi_3 > 0$ are the chemotactic sensitivity coefficients. The parameters $h_1, h_2, h_3 > 0$ represent the intrinsic growth rates, while the parameters $k_1, k_2, k_3 > 0$ indicate the self-limitation effects of the species u, v and w, respectively. Additionally, the parameters $a > 0$ indicate the degradation rate of chemical substance w; the parameters $b, c, d > 0$ denote the production rate of the mobile cells u, v, and w.

In the competitive scenario, all three species strive to generate stimuli to attract their rivals to gain dominance. Multi-species chemotaxis models have a great biological importance in real-world scenarios, as they provide insights into the movement of different cell types or organisms in response to chemical signals, particularly when interacting among various species or cell types. From the biological perspective, the model in (1.1) describes the evolution of three competitive species subject to one chemical substance. It is essential to highlight that the system represented by (1.1) is under investigation for the first time. It is particularly noteworthy that the system incorporates three species and one stimulus with regular sensitivity, which gives us the opportunity to compare and discuss these three distinct cell populations at the same time. Hence, we explore the interactions among all the species and their mutual influences on the dynamic properties of the system in (1.1). Throughout this study, investigate the L^ρ-boundedness, global existence, global boundedness, and combined mass persistence of solutions to the system in (1.1).

Various versions of the system in (1.1), such as one-species or multi-species and one-multi type chemical substance models, have been analyzed in many research papers so far. First, assume that $v(x,t) = w(t,x) = 0$. Then, the following system is obtained:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla z) + h_1 u - k_1 u^2 \\ 0 = \Delta z - az + bu \end{cases} \tag{1.6}$$

For the case $n \geq 2$ and $h_1 = k_1 = 0$, (1.6) has a finite-time blows-up in solutions of (1.1) under some restriction on the initial data, see [6–9]. For the case $a = b = 1$ and $h_1, k_1 > 0$, (1.6) has a bounded classical solution if $n < 2$ or $n \geq 3$ whenever $\chi_1 < \frac{k_1 n}{n-2}$ [10]. Moreover, the global existence and boundedness of this model was obtained at the critical point, which is $\chi_1 = \frac{k_1 n}{n-2}$ with $n \geq 3$ [11]. In addition, the mass persistence of solutions of (1.6) was first studied in [12], and it was shown that in any space dimensional setting, when S is a convex domain, all positive solutions to the model in (1.1) always persists as a whole, that is,

$$\int_S u \geq c_* > 0 \tag{1.7}$$

Recently, the convexity condition for the persistence of mass of solutions has been eliminated in [13] under the the following explicit conditions, which means (1.7) also holds for any domain $S \subseteq \mathbb{R}^n$ if

$$n \leq 2 \quad \text{or} \quad \chi \leq \frac{k_1}{b} \cdot \frac{n}{n-2} \ \text{ with } \ n \geq 3$$

For the other dynamical behaviors of solutions, including weak solutions, stability, and persistence, see [11, 14–26].

A selection of known results concerning similar models of (1.1) can be outlined as follows: Consider the subsequent two-species one chemoattractant Keller-Segel model

$$\begin{cases} u_t = d_1 \Delta u - \chi_1 \nabla \cdot (u \nabla z) + \mu_1 u(1 - u - a_1 v) \\ v_t = d_2 \Delta v - \chi_2 \nabla \cdot (u \nabla z) + \mu_2 v(1 - a_2 u - v) \\ 0 = d_3 \Delta z - \gamma z + \alpha u + \beta v \end{cases} \tag{1.8}$$

Tello and Winkler [27] established the global existence, boundedness, and stability of solutions of (1.8) under the conditions $d_3 = \alpha = \beta = 1$, $2(\chi_1 + \chi_2) + a_2 \mu_1 < \mu_2$, and $2(\chi_1 + \chi_2) + a_1 \mu_2 < \mu_1$.

The same results [28] were achieved under the relaxed conditions provided that $\frac{\chi_1}{\mu_1} < \min\left\{\frac{d_3}{2\alpha}, \frac{a_1 d_3}{\beta}\right\}$, $\frac{\chi_2}{\mu_2} < \min\left\{\frac{d_3}{2\beta}, \frac{a_2 d_3}{\alpha}\right\}$, and $a_1 a_2 d_3^2 < \left(d_3 - \frac{2\alpha\chi_1}{\mu_1}\right)\left(d_3 - \frac{2\beta\chi_2}{\mu_2}\right)$. The long-time behaviors of solutions to the system in (1.8) has been established in [29] provided that $a_1 > 1 > a_2$, $d_3 = \beta = 1$, $\frac{\chi_1}{\mu_1} \leq a_1$, $\frac{\chi_2}{\mu_2} \leq \frac{1}{2}$, and $\frac{\chi_1}{\mu_1} + \max\left\{\frac{\chi_2}{\mu_2}, \frac{a_2(\mu_2-\chi_2)}{\mu_2-2\chi_2}, \frac{(\alpha-a_2)\chi_2}{\mu_2-2\chi_2}\right\} < 1$. Afterward, in the general case, i.e., $a_1, a_2 > 0$, it was demonstrated in [30] that the system in (1.8) has a global bounded classical solutions if $n \leq 2$ or $n \geq 3$ with $\frac{\chi_1}{\mu_1} < \frac{d_3 n}{n-2}\min\left\{\frac{1}{\alpha}, \frac{a_1}{\beta}\right\}$ and $\frac{\chi_2}{\mu_2} < \frac{d_3 n}{n-2}\min\left\{\frac{1}{\beta}, \frac{a_2}{\alpha}\right\}$. This result was improved in [31] when $\alpha = \beta = \gamma = 1$ if $\frac{\chi_1}{\mu_1} + \frac{\chi_2}{\mu_2} < d_3$. Finally, in [32], the most general case for the arbitrary parameters, the system in (1.8) admits a bounded solution under the much milder suitable conditions on the parameters. For the existence, boundedness, long-term behavior of solutions, such as asymptotic stability, persistence, competitive exclusion, and coexistence, for similar models of (1.8), see [33–41].

The remainder of this paper is structured as follows: Section 2 focuses on presenting key estimates and L^ρ-bounds and discussing the global existence and boundedness of solutions to (1.1). Section 3 analyzes the persistence of the mass of globally bounded solutions to (1.1). The last section discusses the need for further research.

2. Preliminaries

This section aims to introduce several fundamental lemmas. Initially, it discusses the local existence and uniqueness of the solution to (1.1).

Lemma 2.1. For all u_0 and v_0 satisfying (1.2), there exists a $T_{\max}(u_0, v_0, w_0) \in (0, \infty]$ such that the system described by (1.1) and (1.2) admit a classical solution on $(0, T_{\max})$ with initial conditions $u(0,x) = u_0(x)$, $v(0,x) = v_0(x)$, and $w(0,x) = w_0(x)$ satisfying

$$\lim_{t\to 0}\|u(t,\cdot) - u_0(\cdot)\|_{L^\infty(\bar{S})} = \lim_{t\to 0}\|v(t,\cdot) - v_0(\cdot)\|_{L^\infty(\bar{S})} = \lim_{t\to 0}\|w(t,\cdot) - w_0(\cdot)\|_{L^\infty(\bar{S})} = 0$$

where $u, v, w \in C((0, T_{\max}) \times \bar{S}) \cap C^{2,1}((0, T_{\max}) \times \bar{S}))$ and $z \in C^{2,0}((0, T_{\max}) \times \bar{S}))$. In addition, being $T_{\max}(u_0, v_0, w_0) < \infty$ also implies

$$\|u(t,\cdot) + v(t,\cdot) + w(t,\cdot)\|_{L^\infty(\bar{S})} = \infty \quad \text{as } \ t \to T_{\max}$$

The proof can be obtained from the similar operations of Theorem 2.1 in [10].

In the subsequent discussion, we establish upper bounds for the solutions that serve as a foundation for proving the main results herein. Note that we prove the following lemmas in the interval $t \in (0, T_{\max})$, for all $0 < t < T_{\max}(u_0, v_0, w_0) \in (0, \infty]$.

Lemma 2.2. The following hold:

i. Let $|S|$ be the Lebesgue measure of S. Then,

$$\int_S u \leq m_1 := \max\Big\{\frac{h_1}{k_1}|S|, \int_S u_0\Big\}$$

$$\int_S v \leq m_2 := \max\Big\{\frac{h_2}{k_2}|S|, \int_S v_0\Big\}$$

and

$$\int_S w \leq m_3 := \max\Big\{\frac{h_3}{k_3}|S|, \int_S v_0\Big\}$$

for all $t \in (0, T_{\max})$.

ii. Let $\xi > 1$. For all $\varepsilon > 0$, there are $C(\varepsilon, \xi, m_1) > 0, C(\varepsilon, \xi, m_2) > 0$, and $C(\varepsilon, \xi, m_3)$ such that

$$\int_S u^\xi \leq \varepsilon \int_S u^{\xi-2}|\nabla u|^2 + C(\varepsilon, \xi, m_1)$$
$$\int_S v^\xi \leq \varepsilon \int_S v^{\xi-2}|\nabla v|^2 + C(\varepsilon, \xi, m_2)$$

and

$$\int_S w^\xi \leq \varepsilon \int_S w^{\xi-2}|\nabla w|^2 + C(\varepsilon, \xi, m_3)$$

for all $t \in (0, T_{\max})$.

Proof. *i.* Integrating the first and second equalities in (1.1) and using Hölder inequality,

$$\frac{d}{dt}\int_S u = h_1 \int_S u - k_1 \int_S u^2 \leq h_1 \int_S u - \frac{k_1}{|S|}\Big(\int_S u\Big)^2$$
$$\frac{d}{dt}\int_S v = h_2 \int_S v - k_2 \int_S v^2 \leq h_2 \int_S v - \frac{k_2}{|S|}\Big(\int_S v\Big)^2$$

and

$$\frac{d}{dt}\int_S w = h_3 \int_S v - k_3 \int_S w^2 \leq h_3 \int_S w - \frac{k_3}{|S|}\Big(\int_S w\Big)^2$$

for all $t \in (0, T_{\max})$. Then, i follows from the ODE's comparison principle.

ii. By the Erhling type lemma, for $\varepsilon > 0$, $C(\varepsilon, \xi) > 0$ such that

$$\int_S u^\xi \leq \frac{4\varepsilon}{\xi^2}\int_S |\nabla u^{\frac{\xi}{2}}|^2 + C(\varepsilon,\xi)\Big(\int_S u\Big)^\xi \leq \varepsilon \int_S u^{\xi-2}|\nabla u|^2 + C(\varepsilon,\xi)(m_1)^\xi$$
$$\int_S v^\xi \leq \frac{4\varepsilon}{\xi^2}\int_S |\nabla v^{\frac{\xi}{2}}|^2 + C(\varepsilon,\xi)\Big(\int_S v\Big)^\xi \leq \varepsilon \int_S v^{\xi-2}|\nabla v|^2 + C(\varepsilon,\xi)(m_2)^\xi$$

and

$$\int_S w^\xi \leq \frac{4\varepsilon}{\xi^2}\int_S |\nabla w^{\frac{\xi}{2}}|^2 + C(\varepsilon,\xi)\Big(\int_S w\Big)^\xi \leq \varepsilon \int_S w^{\xi-2}|\nabla w|^2 + C(\varepsilon,\xi)(m_3)^\xi$$

for all $t \in (0, T_{\max})$. Then, ii follows. □

Lemma 2.3. Let $\xi > 0$. Then, for all $t \in (0, T_{\max})$,

$$\int_S u^{\xi-1}\nabla u \cdot \nabla z \leq \left[\frac{b}{\xi} + \frac{c}{\xi+1} + \frac{d}{\xi+1}\right]\int_S u^{\xi+1} + \frac{c}{\xi(\xi+1)}\int_S v^{\xi+1} + \frac{d}{\xi(\xi+1)}\int_S w^{\xi+1}$$
$$\int_S v^{\xi-1}\nabla v \cdot \nabla z \leq \frac{b}{\xi(\xi+1)}\int_S u^{\xi+1} + \left[\frac{b}{\xi+1} + \frac{c}{\xi} + \frac{d}{\xi+1}\right]\int_S v^{\xi+1} + \frac{d}{\xi(\xi+1)}\int_S w^{\xi+1}$$

and

$$\int_S w^{\xi-1}\nabla w \cdot \nabla z \leq \frac{b}{\xi(\xi+1)}\int_S u^{\xi+1} + \frac{c}{\xi(\xi+1)}\int_S v^{\xi+1} + \left[\frac{b}{\xi+1} + \frac{c}{\xi+1} + \frac{d}{\xi}\right]\int_S w^{\xi+1}$$

Proof. By multiplying the third equality in (1.1) by $u^{\xi-1}$ and integrating by parts over S,

$$\int_S u^{\xi-1}\cdot(\Delta z - az + bu + cv + dw) = 0$$

which gives by Young's inequality

$$\xi\int_S u^{\xi-1}\nabla u\cdot\nabla z + a\int_S zu^\xi = b\int_S u^{\xi+1} + c\int_S vu^\xi + d\int_S wu^\xi$$
$$\leq b\int_S u^{\xi+1} + \frac{c}{\xi+1}\int_S v^{\xi+1} + \frac{c\xi}{\xi+1}\int_S u^{\xi+1} + \frac{d}{\xi+1}\int_S w^{\xi+1} + \frac{d\xi}{\xi+1}\int_S u^{\xi+1}$$

for all $t \in (0, T_{\max})$. Similarly,

$$\xi \int_S v^{\xi-1}\nabla v \cdot \nabla z + a\int_S zv^\xi = b\int_S uv^\xi + c\int_S v^{\xi+1} + d\int_S wv^\xi$$
$$\leq \frac{b}{\xi+1}\int_S u^{\xi+1} + \frac{b\xi}{\xi+1}\int_S v^{\xi+1} + c\int_S v^{\xi+1} + \frac{d}{\xi+1}\int_S w^{\xi+1} + \frac{d\xi}{\xi+1}\int_S v^{\xi+1}$$

and

$$\xi \int_S w^{\xi-1}\nabla w \cdot \nabla z + a\int_S zw^\xi = b\int_S uw^\xi + c\int_S vw^\xi + d\int_S w^{\xi+1}$$
$$\leq \frac{b}{\xi+1}\int_S u^{\xi+1} + \frac{b\xi}{\xi+1}\int_S w^{\xi+1} + \frac{c}{\xi+1}\int_S v^{\xi+1} + \frac{c\xi}{\xi+1}\int_S w^{\xi+1} + d\int_S w^{\xi+1}$$

for all $t \in (0, T_{\max})$. $\square$

The subsequent lemma represents a significant estimate for the L^ρ-bounds of $u + v$.

Lemma 2.4. Assume that (1.3)-(1.5) holds. Then for all k_1, k_2 and k_3, there is a $\xi := \xi(k_1, k_2, k_3) > 1$ such that

$$\int_S u^\xi + \int_S v^\xi + \int_S w^\xi \leq C$$

for all $t \in (0, T_{\max})$.

Proof. Multiplying the first equality in (1.1) by $u^{\xi-1}$ and integrating it over S,

$$\begin{aligned}\frac{1}{\xi}\cdot\frac{d}{dt}\int_S u^\xi &= \int_S u^{\xi-1}\Delta u - \chi_1\int_S u^{\xi-1}\nabla\cdot(u\nabla z) + h_1\int_S u^\xi - k_1\int_S u^{\xi+1} \\ &= -(\xi-1)\int_S u^{\xi-2}|\nabla u|^2 + (\xi-1)\chi_1\int_S u^{\xi-1}\nabla u\cdot\nabla z + h_1\int_S u^\xi - k_1\int_S u^{\xi+1}\end{aligned} \tag{2.1}$$

for all $t \in (0, T_{\max})$. Similarly,

$$\frac{1}{\xi}\cdot\frac{d}{dt}\int_S v^\xi = -(\xi-1)\int_S v^{\xi-2}|\nabla v|^2 + (\xi-1)\chi_2\int_S v^{\xi-1}\nabla v\cdot\nabla z + h_2\int_S v^\xi - k_2\int_S v^{\xi+1} \tag{2.2}$$

and

$$\frac{1}{\xi}\cdot\frac{d}{dt}\int_S w^\xi = -(\xi-1)\int_S w^{\xi-2}|\nabla w|^2 + (\xi-1)\chi_3\int_S w^{\xi-1}\nabla w\cdot\nabla z + h_3\int_S w^\xi - k_3\int_S w^{\xi+1} \tag{2.3}$$

for all $t \in (0, T_{\max})$. By adding (2.1)-(2.3),

$$\begin{aligned}\frac{1}{\xi}\cdot\frac{d}{dt}\left(\int_S u^\xi + \int_S v^\xi + \int_S w^\xi\right) &= -(\xi-1)\int_S u^{\xi-2}|\nabla u|^2 - (\xi-1)\int_S v^{\xi-2}|\nabla v|^2 - (\xi-1)\int_S w^{\xi-2}|\nabla w|^2 \\ &+ (\xi-1)\chi_1\int_S u^{\xi-1}\nabla u\cdot\nabla z + (\xi-1)\chi_2\int_S v^{\xi-1}\nabla v\cdot\nabla z + (\xi-1)\chi_3\int_S w^{\xi-1}\nabla w\cdot\nabla z \\ &+ h_1\int_S u^\xi + h_2\int_S v^\xi + h_3\int_S w^\xi - k_1\int_S u^{\xi+1} - k_2\int_S v^{\xi+1} - k_3\int_S w^{\xi+1}\end{aligned}$$

for all $t \in (0, T_{\max})$. By Lemma 2.2, there is a positive number $C > 0$ such that

$$h_1\int_S u^\xi \leq (\xi-1)\int_S u^{\xi-2}|\nabla u|^2 + \frac{C}{3}$$
$$h_2\int_S v^\xi \leq (\xi-1)\int_S v^{\xi-2}|\nabla v|^2 + \frac{C}{3}$$

and

$$h_3\int_S w^\xi \leq (\xi-1)\int_S w^{\xi-2}|\nabla w|^2 + \frac{C}{3}$$

for all $t \in (0, T_{\max})$. Moreover, by Lemma 2.3,

$$(\xi-1)\chi_1 \int_S u^{\xi-1}\nabla u \cdot \nabla z \leq (\xi-1)\chi_1 \left[\frac{b}{\xi} + \frac{c}{\xi+1} + \frac{d}{\xi+1}\right]\int_S u^{\xi+1} + \frac{c(\xi-1)\chi_1}{\xi(\xi+1)}\int_S v^{\xi+1} + \frac{d(\xi-1)\chi_1}{\xi(\xi+1)}\int_S w^{\xi+1}$$

$$(\xi-1)\chi_2 \int_S v^{\xi-1}\nabla v \cdot \nabla z \leq (\xi-1)\chi_2 \left[\frac{b}{\xi+1} + \frac{c}{\xi} + \frac{d}{\xi+1}\right]\int_S v^{\xi+1} + \frac{b(\xi-1)\chi_2}{\xi(\xi+1)}\int_S u^{\xi+1} + \frac{d(\xi-1)\chi_2}{\xi(\xi+1)}\int_S w^{\xi+1}$$

and

$$(\xi-1)\chi_3 \int_S w^{\xi-1}\nabla w \cdot \nabla z \leq \frac{b(\xi-1)\chi_3}{\xi(\xi+1)}\int_S u^{\xi+1} + \frac{c(\xi-1)\chi_3}{\xi(\xi+1)}\int_S v^{\xi+1} + (\xi-1)\chi_3 \left[\frac{b}{\xi+1} + \frac{c}{\xi+1} + \frac{d}{\xi}\right]\int_S w^{\xi+1}$$

which yields

$$\begin{aligned}(\xi-1)\left[\chi_1 \int_S u^{\xi-1}\nabla u \cdot \nabla z + \chi_2 \int_S v^{\xi-1}\nabla v \cdot \nabla z + \chi_3 \int_S w^{\xi-1}\nabla w \cdot \nabla z\right] &\leq (\xi-1)\left[\frac{b\chi_1}{\xi} + \frac{(c+d)\chi_1}{\xi+1} + \frac{b(\chi_2+\chi_3)}{\xi(\xi+1)}\right]\int_S u^{\xi+1} \\ &+ (\xi-1)\left[\frac{(b+d)\chi_2}{\xi+1} + \frac{c\chi_2}{\xi} + \frac{c(\chi_1+\chi_3)}{\xi(\xi+1)}\right]\int_S v^{\xi+1} \\ &+ (\xi-1)\left[\frac{(b+c)\chi_3}{\xi+1} + \frac{d(\chi_1+\chi_2)}{\xi(\xi+1)} + \frac{d\chi_3}{\xi}\right]\int_S w^{\xi+1}\end{aligned} \tag{2.4}$$

for all $t \in (0, T_{\max})$. It then follows that

$$\begin{aligned}\frac{1}{\xi}\cdot\frac{d}{dt}\left(\int_S u^\xi + \int_S v^\xi + \int_S w^\xi\right) &\leq \left\{(\xi-1)\left(\frac{b\chi_1}{\xi} + \frac{(c+d)\chi_1}{\xi+1} + \frac{b(\chi_2+\chi_3)}{\xi(\xi+1)}\right) - k_1\right\}\int_S u^{\xi+1} \\ &+ \left\{(\xi-1)\left(\frac{(b+d)\chi_2}{\xi+1} + \frac{c\chi_2}{\xi} + \frac{c(\chi_1+\chi_3)}{\xi(\xi+1)}\right) - k_2\right\}\int_S v^{\xi+1} \\ &+ \left\{(\xi-1)\left(\frac{(b+c)\chi_3}{\xi+1} + \frac{d(\chi_1+\chi_2)}{\xi(\xi+1)} + \frac{d\chi_3}{\xi}\right) - k_3\right\}\int_S v^{\xi+1} + C\end{aligned}$$

for all $t \in (0, T_{\max})$. Fix $\xi > 1$ sufficiently close to 1 such that $\xi := 1 + \varepsilon$, for $\varepsilon \ll 1$. By (1.3)-(1.5), for all k_1, k_2, and k_3,

$$\varepsilon \cdot \left[\frac{b\chi_1}{1+\varepsilon} + \frac{(c+d)\chi_1}{2+\varepsilon} + \frac{b(\chi_2+\chi_3)}{(1+\varepsilon)(2+\varepsilon)}\right] - k_1 < 0$$

$$\varepsilon \cdot \left[\frac{(b+d)\chi_2}{2+\varepsilon} + \frac{c\chi_2}{1+\varepsilon} + \frac{c(\chi_1+\chi_3)}{(1+\varepsilon)(2+\varepsilon)}\right] - k_2 < 0$$

and

$$\varepsilon \cdot \left[\frac{(b+c)\chi_3}{2+\varepsilon} + \frac{d(\chi_1+\chi_2)}{(1+\varepsilon)(2+\varepsilon)} + \frac{d\chi_3}{1+\varepsilon}\right] - k_3 < 0$$

Then, by Young's inequality with some elementary arrangements, there is a $k^* > 0$ such that

$$\frac{1}{\xi}\cdot\frac{d}{dt}\left(\int_S u^\xi + \int_S v^\xi + \int_S w^\xi\right) \leq -k^*\left(\int_S u^\xi + \int_S v^\xi + \int_S w^\xi\right) + C^*$$

for all $t \in (0, T_{\max})$. Let $y(t) := \int_S u^\xi + \int_S v^\xi + \int_S w^\xi$, which yields $y' \leq -\xi k^* y + \xi C^*$. Then, the ODE's comparison principle yields

$$\int_S u^\xi + \int_S v^\xi + \int_S w^\xi \leq C, \qquad \text{for all } t \in (0, T_{\max})$$

□

3. Main Results

This section provides the obtained primary results.

3.1. L^ρ-Bounds

This subsection establishes L^ρ-bounds of $u+v+w$.

Theorem 3.1 (L^ρ-boundedness)**.** Suppose that the initial functions u_0, v_0, and w_0 satisfy (1.2), and the assumptions in (1.3)-(1.5) are valid. Then, for any given $\frac{n}{2}<\rho<n$,

$$\int_S u^\rho + \int_S v^\rho + \int_S w^\rho \le C, \quad \text{for all } t\in(0,T_{\max})$$

Proof. Fix $\frac{n}{2}<\xi<n$. Then, the main assumptions in (1.3)-(1.5) guarantee that the following hold:

$$(\xi-1)\left(\frac{b\chi_1}{\xi}+\frac{(c+d)\chi_1}{\xi+1}+\frac{b(\chi_2+\chi_3)}{\xi(\xi+1)}\right)-k_1<0$$

$$(\xi-1)\left(\frac{(b+d)\chi_2}{\xi+1}+\frac{c\chi_2}{\xi}+\frac{c(\chi_1+\chi_3)}{\xi(\xi+1)}\right)-k_2<0$$

and

$$(\xi-1)\left(\frac{(b+c)\chi_3}{\xi+1}+\frac{d(\chi_1+\chi_2)}{\xi(\xi+1)}+\frac{d\chi_3}{\xi}\right)-k_3<0$$

Hence, by Lemma 2.4,

$$\int_S u^\xi + \int_S v^\xi + \int_S w^\xi \le C_\xi$$

for all $t\in(0,T_{\max})$. Moreover, by the Gagliardo-Nirenberg embedding theorem and Young's inequality, for all $\varepsilon>0$,

$$\begin{aligned}\int_S u^{\rho+1} &= \|u^{\frac{\rho}{2}}\|^{\frac{2(\rho+1)}{\rho}}_{L^{\frac{2(\rho+1)}{\rho}}(S)}\\
&\le C\|\nabla u^{\frac{\rho}{2}}\|^{\frac{2(\rho+1)\theta}{\rho}}_{L^2(S)}\|u^{\frac{\rho}{2}}\|^{\frac{2(\rho+1)(1-\theta)}{\rho}}_{L^{\frac{2\xi}{\rho}}(S)}+C\|u^{\frac{\rho}{2}}\|^{\frac{2(\rho+1)\theta}{\rho}}_{L^{\frac{2\xi}{\rho}}(S)}\\
&\le C\left(\frac{\rho^2}{4}\int_S u^{\rho-2}|\nabla u|^2\right)^{\frac{(\rho+1)\theta}{\rho}}(C_\xi)^{\frac{(\rho+1)(1-\theta)}{\xi}}+C(C_\xi)^{\frac{(\rho+1)\theta}{\xi}}\\
&\le \varepsilon\int_S u^{\rho-2}|\nabla u|^2 + C(\rho,\xi,\varepsilon,\theta,C_\xi,|S|) \quad \text{for all } t\in(0,T_{\max})\end{aligned} \tag{3.1}$$

where

$$\theta=\frac{\frac{\rho}{2\xi}-\frac{\rho}{2(\rho+1)}}{\frac{1}{n}+\frac{\rho}{2\xi}-\frac{1}{2}}=\frac{\rho n}{\rho+1}\cdot\frac{\rho+1-\xi}{2\xi+n(p-\xi)}\in(0,1) \quad \text{and} \quad \frac{(\rho+1)\theta}{\rho}<1$$

due to the fact that (1.3) implies

$$\chi_1<\frac{n}{n-2}\cdot\frac{k_1}{b_1} \quad \text{for all } \frac{n-2}{2}<\frac{n}{2}<\rho<n$$

Similarly,

$$\int_S v^{\rho+1}\le\varepsilon\int_S v^{\rho-2}|\nabla v|^2+C(\rho,\xi,\varepsilon,\theta,C_\xi,|S|) \tag{3.2}$$

and

$$\int_S w^{\rho+1}\le\varepsilon\int_S w^{\rho-2}|\nabla w|^2+C(\rho,\xi,\varepsilon,\theta,C_\xi,|S|) \tag{3.3}$$

for all $t\in(0,T_{\max})$. Besides, multiplying the first equality in (1.1) by $u^{\rho-1}$ with $\rho>1$, the second equality in (1.1) by $v^{\rho-1}$ with $\rho>1$, and the third equality in (1.1) by $w^{\rho-1}$ with $\rho>1$, integrating them over S, and adding these equations,

$$\frac{1}{\rho}\frac{d}{dt}\left(\int_S u^\rho + \int_S v^\rho + \int_S w^\rho\right) = -(\rho-1)\int_S u^{\rho-2}|\nabla u|^2 - (\rho-1)\int_S v^{\rho-2}|\nabla v|^2 - (\rho-1)\int_S w^{\rho-2}|\nabla w|^2$$
$$+ (\rho-1)\chi_1\int_S u^{\rho-1}\nabla u\cdot\nabla z + (\rho-1)\chi_2\int_S v^{\rho-1}\nabla v\cdot\nabla z + (\rho-1)\chi_3\int_S w^{\rho-1}\nabla w\cdot\nabla z \quad (3.4)$$
$$+ h_1\int_S u^\rho + h_2\int_S v^\rho + h_3\int_S w^\rho - k_1\int_S u^{\rho+1} - k_2\int_S v^{\rho+1} - k_3\int_S w^{\rho+1}$$

for all $t \in (0, T_{\max})$. In view of (2.4), (3.1), (3.2), and (3.3),

$$(\rho-1)\left[\chi_1\int_S u^{\rho-1}\nabla u\cdot\nabla z + \chi_2\int_S v^{\rho-1}\nabla v\cdot\nabla z + \chi_3\int_S w^{\rho-1}\nabla w\cdot\nabla z\right] \le (\xi-1)\left[\frac{b\chi_1}{\xi} + \frac{(c+d)\chi_1}{\xi+1} + \frac{b(\chi_2+\chi_3)}{\xi(\xi+1)}\right]\int_S u^{\rho+1}$$
$$+ (\xi-1)\left[\frac{(b+d)\chi_2}{\xi+1} + \frac{c\chi_2}{\xi} + \frac{c(\chi_1+\chi_3)}{\xi(\xi+1)}\right]\int_S v^{\rho+1}$$
$$+ (\xi-1)\left[\frac{(b+c)\chi_3}{\xi+1} + \frac{d(\chi_1+\chi_2)}{\xi(\xi+1)} + \frac{d\chi_3}{\xi}\right]\int_S w^{\rho+1} \quad (3.5)$$
$$\le (\rho-1)\int_S v^{\rho-2}|\nabla v|^2 + (\rho-1)\int_S v^{\rho-2}|\nabla v|^2$$
$$+ (\rho-1)\int_S w^{\rho-2}|\nabla w|^2 + C$$

for $t \in (0, T_{\max})$. Moreover, by Young's inequality,

$$h_1\int_S u^\rho \le \frac{k_1}{2}\int_S u^{\rho+1} + C(h_1, k_1, |S|) \quad (3.6)$$

$$h_2\int_S v^\rho \le \frac{k_2}{2}\int_S v^{\rho+1} + C(h_2, k_2, |S|) \quad (3.7)$$

and

$$h_3\int_S w^\rho \le \frac{k_3}{2}\int_S w^{\rho+1} + C(h_3, k_3, |S|) \quad (3.8)$$

for all $t \in (0, T_{\max})$. Collecting (3.4)-(3.8),

$$\frac{1}{\rho}\frac{d}{dt}\left(\int_S u^\rho + \int_S v^\rho + \int_S w^\rho\right) \le -\min\left\{\frac{k_1}{2}, \frac{k_2}{2}, \frac{k_3}{2}\right\}\left(\int_S u^\rho + \int_S u^\rho + \int_S w^\rho\right) + C^*$$

which implies by the ODE's comparison principle that

$$\int_S u^\rho + \int_S v^\rho + \int_S w^\rho \le \max\left\{\int_S (u_0^\rho + v_0^\rho + w_0^\rho),\ \frac{4C^*}{\min\{k_1, k_2, k_3\}}\right\}$$

for all $t \in (0, T_{\max})$. The proof is thus over. □

3.2. Global Existence and Boundedness

This subsection presents the subsequent observation related to the global existence and boundedness of solutions to (1.1).

Theorem 3.2 (Global existence and boundedness)**.** Assume that the initial functions u_0, v_0, and w_0 satisfy (1.2), and (1.3) and (1.4) hold. Then, the solution (u, v, w, z) is global, i.e.,

$$T_{\max}(u_0, v_0, w_0) = \infty$$

Moreover, there is a $K_\infty > 0$ such that

$$\|u+v+w\|_{L^\infty(S)} \le K_\infty, \qquad \text{for all } t>0$$

Proof. It is well known that if $\rho > \frac{n}{2}$, then L^ρ-boundedness of solutions in time implies the L^∞-boundedness in time of solutions. Thus, by Theorem 3.1 and similar operations in the proof of Theorem 2.5 in [27], $T_{\max}(u_0,v_0,w_0)=\infty$ and

$$\sup \|u(t,\cdot)+v(t,\cdot)+w(t,\cdot)\|_{L^\infty(S)} < \infty, \qquad \text{for all } t>0$$

□

3.3. Combined Mass Persistence

This section analyzes the combined mass persistence of solutions to (1.1). It first present the following key estimate.

Lemma 3.3. Let β_0, β_1, and β_2 be positive, $\theta_1>1$, $\theta_2>1$, $t_0\in\mathbb{R}$, and $y\in C^1([t_0,\infty))$ be nonnegative and satisfy the following inequality, for all $t>0$:

$$y'(t) \ge \beta_0 y(t) - \beta_1 y^{\theta_1}(t) - \beta_2 y^{\theta_2}(t)$$

Then,

$$y(t) \ge \min\left\{ y(t_0), \left(\frac{\beta_0}{2\beta_1}\right)^{\frac{1}{\theta_1-1}}, \left(\frac{\beta_0}{2\beta_2}\right)^{\frac{1}{\theta_2-1}} \right\}$$

The proof follows from the argument of Lemma 2.5 in [42]

Afterward, we provide an estimate from below for $u(t,x)+v(t,x)+w(t,x)$.

Lemma 3.4. Assume that $\delta\in(0,1)$. Then, there is a $\sigma>0$ such that

$$\int_S \left(u^\delta(t,x)+v^\delta(t,x)+w^\delta(t,x)\right)dx \ge \sigma, \quad \text{for all } t>0$$

Proof. Let $\delta\in(0,1)$. Then, multiplying the first equality in (1.1) by $u^{\delta-1}$ with , the second equality in (1.1) by $v^{\delta-1}$ with $\delta\in(0,1)$, and the third equality in (1.1) by $w^{\delta-1}$ with $\delta\in(0,1)$, integrating them over S, and adding these equations, for all $t>0$,

$$\begin{aligned}
\frac{1}{\delta}\cdot\frac{d}{dt}\left(\int_S u^\delta+\int_S v^\delta+\int_S w^\delta\right) = {}& (1-\delta)\int_S u^{\delta-2}|\nabla u|^2 + (1-\delta)\int_S v^{\delta-2}|\nabla v|^2 + (1-\delta)\int_S w^{\delta-2}|\nabla w|^2 \\
& - (1-\delta)\chi_1\int_S u^{\delta-1}\nabla u\cdot\nabla z - (1-\delta)\chi_2\int_S v^{\delta-1}\nabla v\cdot\nabla z \\
& - (1-\delta)\chi_3\int_S w^{\delta-1}\nabla w\cdot\nabla z + h_1\int_S u^\delta + h_2\int_S v^\delta + h_3\int_S w^\delta \\
& - k_1\int_S u^{\delta+1} - k_2\int_S v^{\delta+1} - k_3\int_S w^{\delta+1}
\end{aligned} \tag{3.9}$$

From Lemma 2.3, the third equality in (1.1), and integration by parts over S,

$$(1-\delta)\chi_1\int_S u^{\delta-1}\nabla u\cdot\nabla z \le (1-\delta)\chi_1\left(\frac{b}{\delta}+\frac{c+d}{\delta+1}\right)\int_S u^{\delta+1} + \frac{(1-\delta)c\chi_1}{\delta(\delta+1)}\int_S v^{\delta+1} + \frac{(1-\delta)d\chi_1}{\delta(\delta+1)}\int_S w^{\delta+1}$$

$$(1-\delta)\chi_2\int_S v^{\delta-1}\nabla v\cdot\nabla z \le \frac{(1-\delta)b\chi_2}{\delta(\delta+1)}\int_S u^{\delta+1} + (1-\delta)\chi_2\left(\frac{b+d}{\delta+1}+\frac{c}{\delta}\right)\int_S v^{\delta+1} + \frac{(1-\delta)b\chi_2}{\delta(\delta+1)}\int_S w^{\delta+1}$$

and

$$(1-\delta)\chi_3\int_S w^{\delta-1}\nabla w\cdot\nabla z \le \frac{b(\delta-1)\chi_3}{\delta(\delta+1)}\int_S u^{\delta+1} + \frac{c(\delta-1)\chi_3}{\delta(\delta+1)}\int_S v^{\delta+1} + (\delta-1)\chi_3\left(\frac{b}{\delta+1}+\frac{c}{\delta+1}+\frac{d}{\delta}\right)\int_S w^{\delta+1}$$

which entail for all $t > 0$ that

$$(1-\delta)\left[\chi_1\int_S u^{\delta-1}\nabla u\cdot\nabla z+\chi_2\int_S v^{\delta-1}\nabla v\cdot\nabla z+\chi_3\int_S w^{\delta-1}\nabla w\cdot\nabla z\right]\le C_1\int_S u^{\delta+1}+C_2\int_S v^{\delta+1}+C_3\int_S w^{\delta+1} \quad (3.10)$$

where

$$C_1=(1-\delta)\chi_1\left(\frac{b}{\delta}+\frac{c+d}{\delta+1}\right)+\frac{(1-\delta)b\chi_2}{\delta(\delta+1)}\frac{b(\delta-1)\chi_3}{\delta(\delta+1)}$$

$$C_2=\frac{(1-\delta)c\chi_1}{\delta(\delta+1)}+(1-\delta)\chi_2\left(\frac{b+d}{\delta+1}+\frac{c}{\delta}\right)+\frac{c(\delta-1)\chi_3}{\delta(\delta+1)}$$

and

$$C_3=\frac{(1-\delta)d\chi_1}{\delta(\delta+1)}+\frac{(1-\delta)b\chi_2}{\delta(\delta+1)}+(\delta-1)\chi_3\left(\frac{b}{\delta+1}+\frac{c}{\delta+1}+\frac{d}{\delta}\right)$$

Define $\zeta > 0$ such that

$$0<\frac{\xi(n-2\delta)}{n(\xi-\delta)}<\zeta<1<\xi$$

where $\xi > 1$ is as in Lemma 2.4. By Hölder's inequality, for all $t > t_0 > 0$,

$$\int_S u^{\delta+1}=\int_S u^{\zeta}\cdot u^{\delta+1-\zeta}\le\Big(\int_S u^{\xi}\Big)^{\frac{\zeta}{\xi}}\Big(\int_S u^{\frac{\xi(\delta+1-\zeta)}{\xi-\zeta}}\Big)^{\frac{\xi-\zeta}{\xi}}\le(C_\xi)^{\frac{\zeta}{\xi}}\Big(\int_S u^{\frac{\xi(\delta+1-\zeta)}{\xi-\zeta}}\Big)^{\frac{\xi-\zeta}{\xi}} \quad (3.11)$$

Employing the Gagliardo–Nirenberg Theorem and Young's inequality yields that

$$\begin{aligned}\Big(\int_S u^{\frac{\xi(\delta+1-\zeta)}{\xi-\zeta}}\Big)^{\frac{\xi-\zeta}{\xi}}&=\|u^{\frac{\delta}{2}}\|^{\frac{2(\delta+1-\zeta)}{\delta}}_{L^{\frac{2q(\delta+1-\zeta)}{\delta(\xi-\zeta)}}(S)}\\&\le C\|\nabla u^{\frac{\delta}{2}}\|^{\frac{2(\delta+1-\zeta)\theta}{\delta}}_{L^2(S)}\|u^{\frac{\delta}{2}}\|^{\frac{2(\delta+1-\zeta)(1-\theta)}{\delta}}_{L^2(S)}+C\|u^{\frac{\delta}{2}}\|^{\frac{2(\delta+1-\zeta)}{\delta}}_{L^2(S)}\\&\le C\Big(\int_S u^{\delta-2}|\nabla u|^2\Big)^{\frac{(\delta+1-\zeta)\theta}{\delta}}\Big(\int_S u^{\delta}\Big)^{\frac{(\delta+1-\zeta)(1-\theta)}{\delta}}+C\Big(\int_S u^{\delta}\Big)^{\frac{\delta+1-\zeta}{r}}\\&\le(1-\delta)C_1^{-1}(C_\xi)^{-\frac{\zeta}{\xi}}\int_S u^{\delta-2}|\nabla u|^2+\tilde{C}\Big(\int_S u^{\delta}\Big)^{\frac{(\delta+1-\zeta)(1-\theta)}{\delta-\theta(\delta+1-\zeta)}}+C\Big(\int_S u^{\delta}\Big)^{\frac{\delta+1-\zeta}{\delta}}\end{aligned}$$

where

$$\theta=\frac{\frac{1}{2}-\frac{\delta(\xi-\zeta)}{2\xi(\delta+1-\zeta)}}{\frac{1}{n}+\frac{1}{2}-\frac{1}{2}}=\frac{n}{2\xi}\cdot\frac{\xi-\zeta\xi+\zeta\delta}{\delta+1-\zeta}\in(0,1)$$

$$\frac{(\delta+1-\zeta)\theta}{\delta}=\frac{n(\xi-\xi\zeta+\delta\zeta)}{2\xi\delta}\in(0,1)$$

$$\frac{(\delta+1-\zeta)(1-\theta)}{\delta-\theta(\delta+1-\zeta)}=1+\frac{1-\zeta}{\delta-\theta(\delta+1-\zeta)}>1$$

and

$$\frac{\delta+1-\zeta}{\delta}>1$$

It then follows that for all $t > t_0$,

$$C_1\int_S u^{r+1}\le(1-\delta)\int_S u^{\delta-2}|\nabla u|^2+\beta_1\Big(\int_S u^{\delta}\Big)^{\theta_1}+\beta_2\Big(\int_S u^{\delta}\Big)^{\theta_2} \quad (3.12)$$

where $\beta_1, \beta_2 > 0$ are certain positive constants and $\theta_1, \theta_2 > 1$. Similarly, for all $t > t_0$,

$$C_2\int_S v^{\delta+1}\le(1-\delta)\int_S v^{\delta-2}|\nabla v|^2+\beta_3\Big(\int_S v^{\delta}\Big)^{\theta_1}+\beta_4\Big(\int_S v^{\delta}\Big)^{\theta_2} \quad (3.13)$$

and

$$C_3\int_S w^{\delta+1}\le(1-\delta)\int_S w^{\delta-2}|\nabla w|^2+\beta_5\Big(\int_S w^{\delta}\Big)^{\theta_1}+\beta_6\Big(\int_S w^{\delta}\Big)^{\theta_2} \quad (3.14)$$

where $\beta_3, \beta_4, \beta_5, \beta_6 > 0$ are certain positive constants. Hence, from (3.10)-(3.14),

$$\begin{aligned}
(1-\delta)\left[\chi_1 \int_S u^{\delta-1}\nabla u\cdot\nabla z + \chi_2 \int_S v^{\delta-1}\nabla v\cdot\nabla z + \chi_3 \int_S w^{\delta-1}\nabla w\cdot\nabla z\right] &\le (1-\delta)\left[\int_S u^{\delta-2}|\nabla u|^2 + \int_S v^{\delta-2}|\nabla v|^2 + \int_S w^{\delta-2}|\nabla w|^2\right] \\
&\quad + \beta_1\Big(\int_S u^\delta\Big)^{\theta_1} + \beta_3\Big(\int_S v^\delta\Big)^{\theta_1} + \beta_5\Big(\int_S w^\delta\Big)^{\theta_1} \\
&\quad + \beta_2\Big(\int_S u^\delta\Big)^{\theta_2} + \beta_4\Big(\int_S v^\delta\Big)^{\theta_2} + \beta_6\Big(\int_S w^\delta\Big)^{\theta_2} \\
&\le (1-\delta)\left[\int_S u^{\delta-2}|\nabla u|^2 + \int_S v^{\delta-2}|\nabla v|^2 + \int_S w^{\delta-2}|\nabla w|^2\right] \\
&\quad + \beta_7\Big(\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\Big)^{\theta_1} \\
&\quad + \beta_8\Big(\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\Big)^{\theta_2}
\end{aligned}$$

for some $\beta_7, \beta_8 > 0$. Together with (3.9), this yields that

$$\begin{aligned}
\frac{1}{\delta}\cdot\frac{d}{dt}\left(\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\right) \ge \min\{h_1,h_2,h_3\}\left(\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\right) - \beta_5\Big(\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\Big)^{\theta_1} \\
- \beta_6\Big(\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\Big)^{\theta_2}
\end{aligned}$$

for all $t > t_0$. Consequently, by Lemma 3.3, there is a $\sigma > 0$ such that for all $t > t_0$,

$$\int_S u^\delta + \int_S v^\delta + \int_S w^\delta \ge \sigma$$

□

Theorem 3.5 (Combined mass persistence)**.** Suppose that initial functions u_0, v_0, and w_0 satisfy (1.2), and the main assumptions in (1.3) and (1.4) are valid. Then, there is a $\sigma_* > 0$ such that

$$\int_S (u+v+w) \ge \sigma_*, \qquad \text{for all } t > 0$$

Proof. By Hölder inequality, for all $\delta \in (0,1)$ and for all $t > 0$,

$$\int_S (u+v+w) \ge |S|^{\frac{\delta-1}{\delta}}\left(\int_S (u+v+w)^\delta\right)^{\frac{1}{\delta}} \ge |S|^{\frac{\delta-1}{\delta}}\left\{\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\right\}^{\frac{1}{\delta}}$$

Afterward, by Lemma 3.4, for all $\delta \in (0,1)$, there is a $\sigma > 0$ such that for all $t > 0$,

$$\int_S u^\delta + \int_S v^\delta + \int_S w^\delta \ge \sigma$$

Therefore, for all $t > 0$,

$$\int_S (u+v+w) \ge |S|^{\frac{\delta-1}{\delta}}\left\{\int_S u^\delta + \int_S v^\delta + \int_S w^\delta\right\}^{\frac{1}{\delta}} \ge |S|^{\frac{\delta-1}{\delta}}\sigma^{\frac{1}{\delta}}$$

□

4. Conclusion

In this section, we analyze the obtained findings, outline open problems related to the system in (1.1), and suggest potential directions for future research. To begin with, we remark that the system in (1.1) represents a mixed-type Keller-Segel chemotaxis model, incorporating three mobile species and a single chemical stimulus. This model combines standard sensitivities with competitive dynamics defined by weak logistic sources. Furthermore, it is significant to point out that this is the inaugural study documented related to the system in (1.1) in the related literature.

Afterward, we discuss the results obtained in Theorems 3.1, 3.2, and 3.5. To begin with, we note that compared to the Lotka-Volterra kinetics, which involves interactions among multiple species such as competition, predation, or mutualism, the current logistic source for any cell represented by u, v, w in the system described in (1.1) does not interact with other species. This situation presents both advantages and disadvantages for the system in (1.1). The benefit of the existing logistic source is its ability to prevent species extinction, refers to persistence. In contrast, the limitations are connected to the continuous evolution of the cell population over time, avoiding infinite growth or collapse within a finite period, which relates to global existence and boundedness, under more stronger assumptions regarding the parameters, particularly k_1, k_2, and k_3. The Lotka-Volterra kinetics offers more beneficial conditions for achieving outcomes associated with global existence and boundedness; however, it can also cause the extinction of one or two species as time progresses. While the current logistic kinetics requires more rigorous conditions on the parameters to secure the stated results on existence, it will ensure the strict positivity of species at any moment they exist. Therefore, the assumptions herein in establishing the main results presented in Theorem 3.5 compared to the previous works are considerably more stronger than those in earlier studies [12, 35] in terms of the persistency of species. On the other hand, these current results indicate the upper bounds for global existence, boundedness, and persistence in the (1.1) if the Lotka-Volterra kinetics is integrated into the system. We highlight that the global existence, boundedness and combined mass persistence of the current system has been established for the first time in this paper. Hence, future works associated with the system in (1.1) can focus on the following topics:

i. If (1.3) and (1.4) are not valid, then the global existence, boundedness, and mass persistence of the solution to the system in (1.1) is still open. The next phases of this research could involve an analysis of the asymptotic stability, co-existence, extinction, and bifurcation analysis of solutions, along with their numerical simulations.

ii. Another future works related to system in (1.1) may involve integrating Lotka-Volterra kinetics into the system to investigate its dynamics, followed by comparing the results from each model.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

References

[1] E. F. Keller, L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, Journal of Theoretical Biology 26 (1970) 399–415.

[2] E. F. Keller, L. A. Segel, *Traveling bans of chemotactic bacteria: A theoretical analysis*, Journal of Theoretical Biology 30 (1971) 377–380.

[3] N. Bellomo, A. Bellouquid, Y. Tao, M. Winkler, *Toward a mathematical theory of Keller–Segel models of pattern formation in biological tissues*, Mathematical Models and Methods in Applied Sciences 25 (09) (2015) 1663–1763.

[4] T. Hillen, K. Painter, *A user's guide to PDE models for chemotaxis*, Journal of Mathematical Biology 58 (2009) 183–217.

[5] D. Horstmann, *From 1970 until present: The Keller–Segel model in chemotaxis*, Jahresber Deutsch. Math.-Verein. 106 (2004) 51–69.

[6] M. A. Herrero, E. Medina, J. J. L. Velzquez, *Singularity patterns in a chemotaxis model*, Mathematische Annalen 306 (1996) 583–623.

[7] M. A. Herrero, J. J. L. Velzquez, *Finite-time aggregation into a single point in a reaction-diffusion system*, Nonlinearity 10 (6) (1997) 1739–1754.

[8] T. Nagai, *Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains*, Journal of Inequalities and Applications 6 (2001) 37–55.

[9] T. Nagai, T. Senba, *Global existence and blow-up of radial solutions to a parabolic-elliptic system of chemotaxis*, Advances in Mathematical Sciences and Applications 8 (1998) 145–156.

[10] J. I. Tello, M. Winkler, *A chemotaxis system with logistic source*, Common Partial Differential Equations 32 (2007) 849–877.

[11] B. Hu, Y. Tao, *Boundedness in a parabolic–elliptic chemotaxis-growth system under a critical parameter condition*, Applied Mathematics Letters 64 (2017) 1–7.

[12] Y. Tao, M. Winkler, *Persistence of mass in a chemotaxis system with logistic source*, Journal of Differential Equations 259 (2015) 6142–6161.

[13] H. I. Kurt, *Global boundedness and mass persistence of solutions to a chemotaxis-competition system with logistic source*, Süleyman Demirel University Journal of Natural and Applied Sciences 29 (1) (2025) 167–175.

[14] M. A. J. Chaplain, J. I. Tello, *On the stability of homogeneous steady states of a chemotaxis system with logistic growth term*, Applied Mathematics Letters 57 (2016) 1–6.

[15] T. B. Issa, W. Shen, *Dynamics in chemotaxis models of parabolic-elliptic type on bounded domain with time and space dependent logistic sources*, SIAM Journal on Applied Dynamical Systems 16 (2) (2017) 926–973.

[16] H. I. Kurt, *Boundedness in a chemotaxis system with weak singular sensitivity and logistic kinetics in any dimensional setting*, Journal of Differential Equations 416 (2) (2025) 1429–1461.

[17] H. I. Kurt, *Improvement of criteria for global boundedness in a minimal parabolic-elliptic chemotaxis system with singular sensitivity*, Applied Mathematics Letters 167 (2025) 109570.

[18] H. I. Kurt, W. Shen, *Finite-time blow-up prevention by logistic source in chemotaxis models with singular sensitivity in any dimensional setting*, SIAM Journal on Mathematical Analysis 53 (1) (2021) 973–1003.

[19] H. I. Kurt, W. Shen, *Chemotaxis models with singular sensitivity and logistic source: Boundedness, persistence, absorbing set, and entire solutions*, Nonlinear Analysis: Real World Applications 69 (2023) 103762.

[20] H.I. Kurt, W. Shen, S. Xue, *Stability, bifurcation and spikes of stationary solutions in a chemotaxis system with ingular sensitivity and logistic source*, Mathematical Models and Methods in Applied Sciences 34 (9) (2024) 1649–1700.

[21] J. Lankeit, *Chemotaxis can prevent thresholds on population density*, Discrete and Continuous Dynamical Systems - Series B 20 (2015) 1499–1527.

[22] J. Lankeit, *Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source*, International Journal of Differential Equations 258 (2015) 1158–1191.

[23] M. Le, H. I. Kurt, *Global boundedness in a chemotaxis-growth system with weak singular sensitivity in any dimensional setting*, Nonlinear Analysis: Real World Applications 86 (2025) 104392.

[24] J. I. Tello, *Mathematical analysis and stability of a chemotaxis problem with a logistic growth term*, Mathematical Methods in the Applied Sciences 27 (2004) 1865–1880.

[25] G. Viglialoro, *Very weak global solutions to a parabolic-parabolic chemotaxis-system with logistic source*, Journal of Mathematical Analysis and Applications 439 (1) (2016) 197–212.

[26] M. Winkler, *Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source*, Communications in Partial Differential Equations 35 (8) (2010) 1516–1537.

[27] J. I. Tello, M. Winkler, *Stabilization in a two-species chemotaxis system with a logistic source*, Nonlinearity 25 (2012) 1413–1425.

[28] T. Black, J. Lankeit, M. Mizukami, *On the weakly competitive case in a two-species chemotaxis model*, IMA Journal of Applied Mathematics 81 (2016) 860–876.

[29] C. Stinner, J. I. Tello, M. Winkler, *Competitive exclusion in a two-species chemotaxis model*, Journal of Mathematical Biology 68 (2014) 1607–1626.

[30] M. Mizukami, *Boundedness and stabilization in a two-species chemotaxis-competition system of parabolic-parabolic-elliptic type*, Mathematical Methods in the Applied Sciences 41 (2018) 234–249.

[31] K. Lin, C. Mu, H. Zhong, *A new approach toward stabilization in a two-species chemotaxis model with logistic source*, Computers and Mathematics with Applications 75 (2018) 837–849.

[32] L. Wang, *Improvement of conditions for boundedness in a two-species chemotaxis competition system of parabolic-parabolic-elliptic type*, Journal of Mathematical Analysis and Applications 484 (2020) 123705.

[33] X. Bai, M. Winkler, *Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics*, The Indiana University Mathematics Journal 65 (2016) 553–583.

[34] T. B. Issa, R.B. Salako, *Asymptotic dynamics in a two-species chemotaxis model with non-local terms*, Discrete and Continuous Dynamical Systems - B 22 (10) (2017) 3839–3874.

[35] T. B. Issa, W. Shen, *Persistence, coexistence and extinction in two species chemotaxis models on bounded heterogeneous environments*, Journal of Dynamics and Differential Equations 31 (2019) 1839–1871.

[36] H. I. Kurt, W. Shen, *Two-species chemotaxis-competition system with singular sensitivity: Global existence, boundedness, and persistence*, Journal of Differential Equations 355 (2023) 248–295.

[37] H. I. Kurt, W. Shen, *Stabilization in two-species chemotaxis systems with singular sensitivity and Lotka-Volterra competitive kinetics*, Discrete and Continuous Dynamical Systems 44 (4) (2024) 882–904.

[38] K. Lin, C. Mu, *Convergence of global and bounded solutions of a two-species chemotaxis model with a logistic source*, Discrete and Continuous Dynamical Systems - Series B 22 (2017) 2233–2260.

[39] T. Xiang, *How strong a logistic damping can prevent blow-up for the minimal Keller-Segel chemotaxis system?*, Journal of Mathematical Analysis and Applications 459 (2018) 1172–1200.

[40] L. Xie, *On a fully parabolic chemotaxis system with nonlinear signal secretion*, Nonlinear Analysis: Real World Applications 49 (2019) 24–44.

[41] M. Winkler, *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model*, Journal of Differential Equations 248 (2010) 2889–2905.

[42] Y. Tao, M. Winkler, *Large time behavior in a multidimensional chemotaxis-haptotaxis model with slow signal diffusion*, SIAM Journal on Mathematical Analysis 47 (6) (2015) 4229–4250.