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SPLIT SEMI-QUATERNIONS ALGEBRA IN SEMI-EUCLIDEAN 4-SPACE

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Abstract. The aim of this paper is to study the split semi-quaternions, H_{ss}, and to give some of their algebraic properties. We show that the set of unit split semi-quaternions is a subgroup of H_{ss}°. Furthermore, with the aid of De Moivre's formula, any powers of these quaternions can be obtained.
Keywords De Moivre's formula, Split semi-quaternion, Euler's formula.

1. Introduction

Quaternions were invented by Sir William Rowan Hamilton as an extension to the complex number in 1843. Hamilton's defining relation is most succinctly written as

$$i^2 = j^2 = k^2 = ijk = -1.$$

Quaternions have provided a successful and elegant means for the representation of three dimensional rotations, Lorentz transformations of special relativity, robotics, computer vision, problems of electrical engineering and so on. The Euler's and De-Moivre's formulas for the complex numbers are generalized for quaternions. Obtaining the roots of a quaternion was given by Niven[3] and Brand [1]. Brand proved De Moivre's theorem and used it to find n-th roots of a quaternion. These formulas are also investigated in the cases of split and semi-quaternions [2, 4]. A brief introduction of the split semi-quaternions is provided in [5]. In this paper, we investigate some algebraic properties of split semi-quaternions. Moreover, we obtain De-Moivre's and Euler's formulas for these quaternions in different cases. We use De-Moivre's formula to find $n-$th roots of a split semi-quaternion. Finally, we give some example for the purpose of more clarification.

2. Splitsemi-quaternions

Definition 2.1. A split semi-quaternion q is defined *as*

$$q = a_0 + a_1 i + a_2 j + a_3 k$$

where a_0, a_1, a_2 and a_3 are real numbers and i, j, k are quaternionic units with the properties that

$$\begin{aligned} i^2 &= 1, \quad j^2 = k^2 = 0 \\ ij &= k = -ji\ , \quad jk = 0 = kj \end{aligned}$$

and

$$ki = -j = -ik.$$

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The set of all split semi-quaternions are denoted by H_{ss}. A split semi-quaternion q is a sum of a scalar and a vector, called scalar part, $S_q = a_0$, and vector part $V_q = a_1 i + +a_2 j + a_3 k$. The set of split semi-quaternions $H_{ss} - \{[0,(0,0,0)]\}$ is written H_{ss}°.

Let $q, p \in H_{ss}$, where $q = S_q + V_q$ and $p = S_p + V_p$. The addition operator, $+$, is defined

$$\begin{aligned} q + p &= (S_q + S_p) + (V_p + V_q) \\ &= (a_0 + b_0) + (a_1 + b_1)i \ + (a_2 + b_2)j + (a_3 + b_3)k. \end{aligned}$$

This rule preserves the associativity and commutativity properties of addition. The product of scalar and a split semi-quaternion is defined in a straightforward manner. If c is a scaler and $q \in H_{ss}$,

$$cq \quad = \quad cS_q + cV_q = (ca_0)1 + (ca_1)i + (ca_2)j + (ca_3)k.$$

The multiplication rule for split semi-quaternions is defined as

$$qp = S_q S_p - < V_q, V_p > + S_q V_p + S_p V_q + V_q \times V_p,$$

where

$$< V_q, V_p >= -a_1 b_1, \ V_p \times V_q = 0i - (a_3 b_1 - a_1 b_3)j + (a_1 b_2 - a_2 b_1)k.$$

It could be written as

$$qp = \begin{bmatrix} a_0 & a_1 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Split semi-quaternion multiplication is not generally commutative. We state the following properties of quaternion multiplication:

Proposition 2.1. *Let $q, q\prime, p \in H_{ss}$ and $r \in \mathbb{R}$. Then*

$$\begin{aligned} (pq)q' &= p(qq') && (\textit{Quaternion multiplication is associative.}) \\ p(q+q') &= pq + pq' && (\textit{Quaternion multiplication distributes} \\ (q+q')p &= qp + q'p && \qquad \textit{across addition.}) \end{aligned}$$

Corollary 2.1. *H_{ss} with addition and multiplication has all the properties of a number field expect commutativity of the multiplication. It is therefore called the skew field of quaternions.*

3. Some properties of split semi-quaternions

Definition 3.1. Let $q \in H_{ss}$.Then $\overline{q}$ is called the conjugate of q is defined by

$$\overline{q} \quad = \quad a_0 - (a_1 i + a_2 j + a_3 k) = S_q - V_q.$$

It is clear the scalar and vector part of q is denoted by $S_q = \frac{q+\overline{q}}{2}$ and $V_q = \frac{q-\overline{q}}{2}$.

The above definition would lead to the following properties:

Proposition 3.1. *Let $q, p \in H_{ss}$. Then*

i) $\overline{\overline{q}} = q$ *ii)* $\overline{pq} = \overline{q}\,\overline{p}$ *iii)* $\overline{q+p} = \overline{q} + \overline{p}$ *iv)* $q\overline{q} = \overline{q}q$.

Definition 3.2. Let $q \in H_{ss}$ and let the mapping $\|.\| : H_{ss} \to \mathbb{R}$ be defined by $\|q\| = q\overline{q} = a_0^2 - a_1^2 \in \mathbb{R}$. This mapping is called the norm and $\|q\| \,(= N_q)$ is norm of q. If $\|q\| = a_0^2 - a_1^2 = 1$, then q is called a unit split semi-quaternion. We will use H_{ss}^1 to denote the set of unit split semi-quaternion.

A split semi-quaternion q for which $\|q\| = 0$ has the form $q = a_2 j + a_3 k$, $(a_0 = a_1 = 0)$ and it is a zero divisor, but not all zero divisors of this algebra have this form.

Definition 3.3. Let $q \in H_{ss}$ and $\|q\| \neq 0$. Then there exists $q^{-1} \in H_{ss}$ such that $qq^{-1} = q^{-1}\, q = I$. Furthermore q^{-1} is unique and it is given by

$$q^{-1} = \frac{\overline{q}}{\|q\|}.$$

Proposition 3.2. *Let $p, q \in H_{ss}$ and $\lambda \in \mathbb{R}$. The following three equations hold:*

i) $(qp)^{-1} = p^{-1}q^{-1}$, *ii)* $(\lambda q)^{-1} = \frac{1}{\lambda} q^{-1}$, *iii)* $\left\|q^{-1}\right\| = \frac{1}{\|q\|}$.

Proposition 3.3. *The set H_{ss}^1 of unit split semi-quaternions is a subgroup of the group H_{ss}°.*

Proof. Let $q, q' \in H_{ss}^1$. We have $\|qq'\| = 1$, *i.e.* $qq' \in H_{ss}^1$ and thus the first subgroup requirement is satisfied. Also, by proposition 3.2,

$$\|q\| = \|\overline{q}\| = \left\|q^{-1}\right\| = 1.$$

and thereby the second subgroup requirement $q^{-1} \in H_{ss}^1$.

4) To divide a split semi-quaternion p by the semi-quaternion $q(N_q \neq 0)$, one simply has to resolve the equation

$$xq = p \quad or \quad qy = p,$$

with the respective solutions

$$x \quad = \quad pq^{-1} = p\frac{\overline{q}}{N_q},$$
$$y \quad = \quad q^{-1}p = \frac{\overline{q}}{N_q}$$

and the relation $N_x = N_y = \frac{N_p}{N_q}$.

Definition 3.4. Let $q, p \in H_{ss}$, $q = S_q + V_q$ and $p = S_p + V_p$. The inner product is defined as

$$\begin{aligned} g(q,p) &= S_q S_p + < V_q, V_p > \\ &= S\left(q\overline{p}\right). \end{aligned}$$

Theorem 3.1. *The inner product has the properties;*
1) $g(pq_1, pq_2) = N_p \cdot g(q_1, q_2)$
2) $g(q_1 p, q_2 p) = N_p \cdot g(q_1, q_2)$
3) $g(pq_1, q_2) = g(q_1, \overline{p}q_2)$
4) $g(pq_1, q_2) = g(p, q_2\overline{q_1})$.

Proof. We will prove the identities (1) and (3).

$$\begin{aligned} g(pq_1, pq_2) &= S(pq_1\overline{pq}_2) = S(pq_1\overline{q}_2\ \overline{p}) \\ &= S(\overline{q}_2\ \overline{p}pq_1) = N_p\ S(\overline{q}_2 q_1) \\ &= N_p\ S(q_1\overline{q}_2) = N_p \cdot g(q_1, q_2), \end{aligned}$$

and

$$\begin{aligned} g(pq_1, q_2) &= S(pq_1\overline{q}_2) = S(q_1\overline{q}_2 p) \\ &= S(q_1\ \overline{\overline{p}q}_2) = g(q_1, \overline{p}q_2). \end{aligned}$$

Theorem 3.2. *The algebra H_{ss} is isomorphic to the subalgebra of the algebra $\mathbb{C}_2'$ consisting of the (2×2)-matrices*

$$\hat{A} = \begin{bmatrix} A & B \\ 0 & \overline{A} \end{bmatrix},$$

and to the subalgebra of the algebra $\mathbb{C}_2^{\circ}$ consisting of the (2×2)-matrices

$$\tilde{A} = \begin{bmatrix} A & B \\ 0 & A \end{bmatrix},$$

where $A, B \in \mathbb{C}$.

Proof. The proof can be found in [5].

4. De Moivre's formula for split semi-quaternions

In this section, we express De-Moivre's formula for split semi-quaternions. For this, we can consider two different cases:

Case 1: Let the norm of split semi-quaternion be positive.

Definition 4.1. Every nonzero split semi-quaternion $q = a_0 + a_1 i + a_2 j + a_3 k$ can be written in the polar form

$$q = r(\cosh\varphi + \overrightarrow{w}\sinh\varphi)$$

where $r = \sqrt{N_q}$ and

$$\cosh\varphi = \frac{|a_0|}{r}, \ \ \sinh\varphi = \frac{\sqrt{a_1^2}}{r} = \frac{|a_1|}{\sqrt{a_0^2 - a_1^2}}.$$

The unit vector $\overrightarrow{w}$ is given by

$$\overrightarrow{w} = \frac{1}{\sqrt{a_1^2}}(a_1 i + a_2 j + a_3 k), \ \ a_1 \neq 0.$$

Euler's formula for a unit split semi-quaternion holds. Since $\overrightarrow{w}\overrightarrow{w} = 1$, we have

$$\begin{aligned} e^{\overrightarrow{w}\varphi} &= 1 + \overrightarrow{w}\varphi + \frac{(\overrightarrow{w}\varphi)^2}{2!} + \frac{(\overrightarrow{w}\varphi)^3}{3!} + ... \\ &= (1 + \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} + ...) + \overrightarrow{w}(\varphi + \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} + ...) \\ &= \cosh\varphi + \overrightarrow{w}\sinh\varphi. \end{aligned}$$

Moreover, this can be shown by using the following method.

$$\begin{aligned} q &= \cosh\varphi + \overrightarrow{w}\sinh\varphi \Rightarrow dq = (\sinh\varphi + \overrightarrow{w}\cosh\varphi)d\varphi \\ dq &= \overrightarrow{w}(\cosh\varphi + \overrightarrow{w}\sinh\varphi)d\varphi = \overrightarrow{w}q d\varphi. \end{aligned}$$

thus, we get $\int \frac{dq}{q} = \int \overrightarrow{w} d\varphi \Rightarrow \ln q = \overrightarrow{w}\varphi \Rightarrow q = e^{\overrightarrow{w}\varphi} = \cosh\varphi + \overrightarrow{w}\sinh\varphi.$

Example 4.1. The polar form of split semi-quaternions $q_1 = 2+\sqrt{2}i - j + 2k, q_2 = 3+2i+j+k, q_3 = 4+i-j+2k$ are $q_1 = \sqrt{2}(\cosh\theta_1 + \overrightarrow{w}\sinh\theta_1)$ where $\theta_1 = \ln(\sqrt{2}+1)$, $q_2 = \sqrt{5}(\cosh\theta_2 + \overrightarrow{u}\sinh\theta_2)$ where $\theta_2 = \ln(\sqrt{5})$, and $q_3 = \sqrt{15}(\cosh\theta_3 + \overrightarrow{\varepsilon}\sinh\theta_3)$ where $\theta_3 = \ln(\frac{5}{\sqrt{15}})$, respectively.

Lemma 4.1. *Let $\overrightarrow{w}$ be a unit vector, then we have*

$$(\cosh\varphi + \overrightarrow{w}\sinh\varphi)(\cosh\psi + \overrightarrow{w}\sinh\psi) = \cosh(\varphi+\psi) + \overrightarrow{w}\sinh(\varphi+\psi).$$

Now, let's prove De Moivre's formula for a split semi-quaternion.

Theorem 4.1. *(De-Moivre's formula) Let $q = \cosh\varphi + \overrightarrow{w}\sinh\varphi$ be a unit split semi-quaternion. Then for every integer n;*

$$q^n = \cosh n\varphi + \overrightarrow{w}\sinh n\varphi.$$

Proof. We use induction on positive integers n. Assume that $q^n = \cosh n\varphi + \overrightarrow{w}\sinh n\varphi$ holds. Then

$$\begin{aligned} q^{n+1} &= (\cosh\varphi + \overrightarrow{w}\sinh\varphi)^n(\cosh\varphi + \overrightarrow{w}\sinh\varphi) \\ &= (\cosh n\varphi + \overrightarrow{w}\sinh n\varphi)(\cosh\varphi + \overrightarrow{w}\sinh\varphi) \\ &= \cosh(n\varphi+\varphi) + \overrightarrow{w}\ \sinh(n\varphi+\varphi) \\ &= \cosh(n+1)\varphi + \overrightarrow{w}\ \sinh(n+1)\varphi. \end{aligned}$$

The formula holds for all integer n, since

$$\begin{aligned} q^{-1} &= \cosh\varphi - \overrightarrow{w}\sinh\varphi, \\ q^{-n} &= \cosh(-n\varphi) + \overrightarrow{w}\sinh(-n\varphi) \\ &= \cosh n\varphi - \overrightarrow{w}\sinh n\varphi. \end{aligned}$$

Example 4.2. Let $q = 3 - 2i - j + 3k$ be a split semi-quaternion. Then, we can write it as $q = \sqrt{5}(\cosh\theta + \overrightarrow{w}\sinh\theta)$ where $\theta = \ln(\sqrt{5})$. Every power of this quaternion is found with the aid of Theorem 4.2 , for example, 10-th power of

$$q^{10} = 5^5[\cosh 10\theta + \overrightarrow{w}\sinh 10\theta].$$

where $\cosh 10\theta = \frac{5^5+5^{-5}}{2}$ and $\sinh 10\theta = \frac{5^5-5^{-5}}{2}$.

Theorem 4.2. *Let $q = \cosh\varphi + \overrightarrow{w}\sinh\varphi$ be a unit split semi-quaternion. The equation $x^n = q$ has only one root:*

$$x = \cosh\frac{\varphi}{n} + \overrightarrow{w}\sinh\frac{\varphi}{n}.$$

Proof. If $x^n = q$, q will have the same unit vector as $\overrightarrow{w}$. So, we assume that $x = \cosh\chi + \overrightarrow{w}\sinh\chi$ is a root of the equation $x^n = q$. From Theorem 4.2, we have

$$x^n = \cosh n\chi + \overrightarrow{w}\sinh n\chi,$$

Thus, $\chi = \frac{\varphi}{n}$. So, $x = \cosh\frac{\varphi}{n} + \overrightarrow{w}\sinh\frac{\varphi}{n}$ is a root of the equation $x^n = q$.

Example 4.3. Let $q = 2 + \sqrt{3}i - 2j + k = (\cosh\varphi + \overrightarrow{w}\sinh\varphi)$ be a split semi-quaternion. The equation $x^3 = q$ has one root and that is

$$x = (\cosh\frac{\ln(2+\sqrt{3})}{3} + \overrightarrow{w}\sinh\frac{\ln(2+\sqrt{3})}{3}).$$

Case 2: Let the norm of split semi-quaternion be negative.

Definition 4.2. Every nonzero split semi-quaternion $q = a_0 + a_1 i + a_2 j + a_3 k$ can be written in the polar form

$$q = r(\sinh\psi + \overrightarrow{u}\cosh\psi)$$

where $r = \sqrt{|N_q|}$ and

$$\sinh\psi = \frac{|a_0|}{r}, \ \cosh\psi = \frac{\sqrt{a_1^2}}{r} = \frac{|a_1|}{\sqrt{|a_0^2 - a_1^2|}}.$$

The unit vector $\overrightarrow{u}$ is given by

$$\overrightarrow{u} = \frac{1}{\sqrt{a_1^2}}(a_1 i + a_2 j + a_3 k), \ a_1 \neq 0.$$

Example 4.4. The polar form of the split semi-quaternions $q_1 = 2 + 3i - j + 2k, q_2 = 1 + \sqrt{2}i + 2j + k$ are $q_1 = \sqrt{5}(\sinh\theta_1 + \overrightarrow{u}\cosh\theta_1)$ where $\theta_1 = \ln\sqrt{5}$, $q_2 = \sinh\theta_2 + \overrightarrow{u}\cosh\theta_2$ where $\theta_2 = \ln(1+\sqrt{2})$, respectively.

Theorem 4.3. *(De-Moivre's formula) Let $q = \sinh\varphi + \overrightarrow{u}\cosh\varphi$ be a unit split semi-quaternion. Then for every integer n;*

$$q^n = \sinh n\varphi + \overrightarrow{u}\cosh n\varphi.$$

Example 4.5. Let $q = \sqrt{2} + 2i - j + 3k = \sqrt{2}(\sinh\theta + \overrightarrow{u}\cosh\theta)$ be a split semi-quaternion. Every power of this spit semi-quaternion is found by the aid of Theorem 4.4 , for example, 40-th power is

$$q^{40} = 2^{20}[\sinh 40\theta + \overrightarrow{u}\cosh 40\theta],$$

where $\sinh 40\theta = \frac{(1+\sqrt{2})^{40}-(1+\sqrt{2})^{-40}}{2}$ and $\cosh 40\theta = \frac{(1+\sqrt{2})^{40}+(1+\sqrt{2})^{-40}}{2}$.

Theorem 4.4. *Let $q = \sinh\varphi + \overrightarrow{u}\cosh\varphi$ be a unit split semi-quaternion. The equation $x^n = q$ has only one root:*

$$x = \sinh\frac{\varphi}{n} + \overrightarrow{u}\cosh\frac{\varphi}{n}.$$

Proof. If $x^n = q$, q will have the same unit vector as $\overrightarrow{u}$. So, we assume that $x = \sinh\theta + \overrightarrow{u}\cosh\theta$ is a root of the equation $x^n = q$. From Theorem 4.4, we have

$$x^n = \cosh n\chi + \overrightarrow{u}\sinh n\chi,$$

Thus, $\theta = \frac{\varphi}{n}$. So, $x = \sinh\frac{\varphi}{n} + \overrightarrow{u}\cosh\frac{\varphi}{n}$ is a root of the equation $x^n = q$.

Example 4.6. Let $q = 2 + 3i - 2j + k = (\sinh\varphi + \overrightarrow{u}\cosh\varphi)$ be a split semi-quaternion. The equation $x^4 = q$ has one root and that is

$$x = (\sinh\frac{\ln(\sqrt{5})}{4} + \overrightarrow{u}\cosh\frac{\ln(\sqrt{5})}{4}).$$

5. Conclusion

In this paper, we give some of algebraic properties of the split semi-quaternions and investigate the Euler's and De Moivre's formulas for these quaternions in different cases. We use it to find $n-$th roots of a split semi-quaternion.

References

[1] Brand L.. *The roots of a Quaternion*, American Mathematical Monthly 49-8(1942) 519-520.
[2] Mortezaasl H., Jafari M., *A study on Semi-quaternions Algebra in Semi-Euclidean 4-space*, Mathematical Science and Applications E-Notes, Vol. 1/2,(2013) 20-27.
[3] Niven I. *The roots of a quaternion*, American Mathematical Monthly 49-6(1942) 386-388.
[4] Ozdemir M., *The roots of a split quaternion*, Appl. Math. Lett. 22(2009) 258-263.
[5] Rosenfeld B., *Geometry of Lie groups*, Kluwer Academic Publishers, Netherlands,1997.