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İKİ NOKTADA SÜREKSİZLİK KOŞULLARINA SAHİP DİFÜZYON OPERATÖTÜ İÇİN İNTEGRAL GÖSTERİLİM

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Özet. Bu çalışmada iki noktada süreksizlik koşullarına sahip difüzyon operatörü ele alınmıştır. Verilen denklemin belirli başlangıç ve süreksizlik koşullarını sağlayan çözümleri için integral denklemler elde edilmiş ve süreksizlik koşullarına sahip difüzyon denklemini için oldukça kullanışlı olan integral gösterilim verilmiştir.

Anahtar kelimeler: İntegral denklemi, Sturm-Lioville

INTEGRAL REPRESENTATION FOR THE DIFFUSION OPERATOR WITH DISCONTINUITY CONDITIONS AT TWO POINTS

Abstract. In this paper, diffusion operator with discontinuity conditions at two point is considered. İntegral equations for solution which satisfy certain initial and jump conditions of given equation has been obtained and useful integral representation for diffusion operator with discontinuity conditions at two.

Keywords: Integral equation, Sturm-Liouville

1. GİRİŞ

Modern fonksiyonel analiz ve uygulamalarının ana dallarından biri olan spektral teori bazı uygulamalı bilimlerde önemli yere sahiptir. Özellikle fizikte ve matematiksel fizikte, diferansiyel operatörlerin spektral teorisi karşımıza çıkmaktadır.

Diferansiyel operatörlerin spektral teorisi düz spektral problemler ve ters spektral problemler olmak üzere ikiye ayrılır. Bir diferansiyel operatör verildiğinde operatörün spektrumunun ve öz fonksiyonlarının aranması ve verilen bir fonksiyonun bu operatörün öz fonksiyonlarına göre ayrışımının incelenmesine düz spektral problem denir. Spektral analizin ters problemleri ise; verilen belirli dizilere göre spektral karakteristikleri bu diziler olan operatörün inşasından ibarettir.

Fizikteki birçok problem ters problemlere indirgenmektedir. Örneğin; mekanikte verilen dalga boylarına göre homojen olmayan yayda yoğunluk dağılımının öğrenilmesi, parçacıkların enerji seviyelerine göre parçacıklar arasındaki etkileşim kuvvetlerinin belirlenmesi; kuantum fiziğinde saçılma verilerine göre alan potansiyellerinin bulunması; jeofizikte yer altı madenlerinin yer

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altındaki elementlerin dağılım karakteristiklerine göre belirlenmesi problemlerinin her biri birer ters problemdir.

İkinci mertebeden singüler operatörlerin spektral teorisine yeni bir yaklaşımı 1946 yılında Titchmarsh vermiştir. Doğru ekseninde tanımlı azalan (artan) potansiyelli

$$L=-\frac{d^2}{dx^2}+q(x)$$

Sturm-Liouville operatörleri için özdeğerlerin dağılımı formülü Titchmarsh tarafından bulunmuştur. Son yıllarda bu operatörlere bir boyutlu $q(x)$ potansiyelli Schrödinger denkleminde denir. Aynı zamanda bu çalışmada Schrödinger operatörü için özdeğerlerin dağılım formülü de verilmiştir.

Singüler diferansiyel operatörlerin incelenmesine ilişkin ve diferansiyel operatörlerin spektral teorisinde önemli bir yere sahip olan çalışmalar, 1949 yılında Levitan tarafından yapılmıştır. Levitan bu çalışmalarında spektral teoriyi esaslandırmak için önemli bir yöntem vermiştir.

2. İNTEGRAL DENKLEMİN OLUŞTURULMASI

Bu bölümde öncelikle ele alınan problem tanımlanmıştır. Daha sonra da sırasıyla problemin belirli başlangıç ve süreksizlik koşullarını sağlayan çözümleri için integral denklemler elde edilmiş, difüzyon denklemi için oldukça kullanışlı olan integral gösterilim verilmiştir.

L sınır değer problemini;

$$L:=\begin{cases}-y''+\left[2\lambda p(x)+q(x)\right]y=\lambda^2 y,\ x\in(0,\pi) & (2.1)\\ U(y)=y'(0)=0,\mathrm{V}(y)=y(\pi)=0 & (2.2)\\ y(a_1+0)=\alpha_1 y(a_1-0) & (2.3)\\ y'(a_1+0)=\beta_1 y'(a_1-0)+i\lambda\gamma_1 y(a_1-0) & (2.4)\\ y(a_2+0)=\alpha_2 y(a_2-0) & (2.5)\\ y'(a_2+0)=\beta_2 y'(a_2-0)+i\lambda\gamma_2 y(a_2-0) & (2.6)\end{cases}$$

şeklinde tanımlayalım. Burada λ spektral parametre, $q(x)\in L_2[0,\pi],\ p(x)\in W_2^1[0,\pi]$, $a_1,a_2\in(0,\pi)$, $a_1\le a_2$. Ayrıca $|\alpha_1-1|^2+\gamma_1{}^2\ne 0$, $|\alpha_2-1|^2+\gamma_2\ne 0$, $\left(\beta_i=\frac{1}{\alpha_i}(i=1,2)\right)$ reel sabitlerdir.

2.1 Çözümler için İntegral Denklem

(2.1)-(2.6) probleminin çözümleri denildiğinde aşağıdaki koşulları sağlayan fonksiyonlar kümesi olarak tanımlanacaktır.

(2.1) denkleminin $y(0)=1, y'(0)=0$ başlangıç ve (2.3)-(2.6) süreksizlik koşullarını sağlayan $y(x,\lambda)$ çözümü;

$i)$ $y(x,\lambda)$, $y'(x,\lambda)$ fonksiyonları $(0,a_1),(a_1,a_2)$ ve (a_2,π) aralıklarının her birinde mutlak sürekli,

$ii)$ $l(y)\in L_2(0,\pi)$,

$iii)$ $y(x,\lambda)$ fonksiyonu $(0,a_1),(a_1,a_2)$ ve (a_2,π) aralıklarında (2.1) denklemini ve $x=a_1$ noktasında (2.3)-(2.4) ; $x=a_2$ noktasında (2.5)-(2.6) süreksizlik koşullarını sağlamaktadır.

Bu tanım altında (2.1) denkleminin

$$\varphi(0,\lambda)=1, \varphi'(0,\lambda)=0 \qquad (2.1.1)$$

şeklindeki başlangıç ve (2.3)-(2.6) süreksizlik koşullarını sağlayan çözümü $\varphi(x,\lambda)$ olsun. Bu durumda aşağıdaki lemma geçerlidir.

Lemma 2.1.1 $Q(t)=2\lambda p(t)+q(t)$ olmak üzere (2.1) denkleminin (2.2) başlangıç ve (2.3)-(2.6) süreksizlik koşullarını sağlayan çözümleri aşağıdaki integral denklemlerini sağlar.

$0\le x<a_1$ için ;

$$y(x,\lambda)=e^{i\lambda x}+\frac{1}{\lambda}\int_0^x \sin\lambda(x-t)Q(t)y(\mathrm{t},\lambda)dt \qquad (2.1.2)$$

$a_1<x<a_2$ için ;

$$y(x,\lambda)=\frac{1}{2}(\alpha_1+\beta_1)e^{i\lambda x}+\frac{1}{2}(\alpha_1-\beta_1)e^{i\lambda(2a_1-x)}+\frac{\gamma_1}{2}e^{i\lambda x}-\frac{\gamma_1}{2}e^{i\lambda(2a_1-x)}$$
$$+\frac{1}{2}(\alpha_1+\beta_1)\int_0^{a_1}\frac{\sin\lambda(x-t)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt-\frac{1}{2}(\alpha_1-\beta_1)\int_0^{a_1}\frac{\sin\lambda(x+t-2a_1)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt$$
$$-i\frac{\gamma_1}{2}\int_0^{a_1}\frac{\cos\lambda(x-t)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt+i\frac{\gamma_1}{2}\int_0^{a_1}\frac{\cos\lambda(x+t-2a_1)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt$$
$$+\int_{a_1}^{x}\frac{\sin\lambda(x-t)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt \qquad (2.1.3)$$

$a_2<x\le\pi$ için,

$$y(x,\lambda)=\alpha_1^+\alpha_2^+e^{i\lambda x}+\alpha_1^-\alpha_2^-e^{i\lambda(2a_1-2a_2+x)}+\alpha_1^+\alpha_2^-e^{i\lambda(2a_2-x)}+\alpha_1^-\alpha_2^+e^{i\lambda(2a_1-x)}+\frac{\gamma_1\alpha_2^+}{2}e^{i\lambda x}$$

$$-\frac{\gamma_1\alpha_2^-}{2}e^{i\lambda(2a_1-2a_2+x)}+\frac{\gamma_1\alpha_2^-}{2}e^{i\lambda(2a_2-x)}-\frac{\gamma_1\alpha_2^+}{2}e^{i\lambda(2a_1-x)}-\frac{\gamma_1\gamma_2}{4}e^{i\lambda x}-\frac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_1-2a_2+x)}$$

$$-\frac{\gamma_2\alpha_1^+}{2}e^{i\lambda(2a_2-x)}-\frac{\gamma_2\alpha_1^-}{2}e^{i\lambda(2a_1-x)}+\frac{\gamma_2\alpha_1^+}{2}e^{i\lambda x}+\frac{\gamma_2\alpha_1^-}{2}e^{i\lambda(2a_1-2a_2+x)}-\frac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_2-x)}+\frac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_1-x)}$$

$$+\left(\alpha_1^+\alpha_2^++\frac{\gamma_1\gamma_2}{4}\right)\int_0^{a_1}\frac{\sin\lambda(x-t)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt$$

$$+\left(-\alpha_1^+\alpha_2^-+\frac{\gamma_1\gamma_2}{4}\right)\int_0^{a_1}\frac{\sin\lambda(x+t-2a_2)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt$$

$$+\left(-\alpha_1^-\alpha_2^+-\frac{\gamma_1\gamma_2}{4}\right)\int_0^{a_1}\frac{\sin\lambda(x+t-2a_1)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt-\alpha_2^-\int_{a_1}^{a_2}\frac{\sin\lambda(x+t-2a_2)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt$$

$$+\left(\alpha_1^-\alpha_2^--\frac{\gamma_1\gamma_2}{4}\right)\int_0^{a_1}\frac{\sin\lambda(2a_1-2a_2+x-t)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt+\alpha_2^+\int_{a_1}^{a_2}\frac{\sin\lambda(x-t)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt$$

$$-\frac{i}{2}\left(\gamma_1\alpha_2^+-\gamma_2\alpha_1^+\right)\int_0^{a_1}\frac{\cos\lambda(x-t)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt$$

$$+\frac{i}{2}\left(-\gamma_1\alpha_2^--\gamma_2\alpha_1^-\right)\int_0^{a_1}\frac{\cos\lambda(x+t-2a_2)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt$$

$$+\frac{i}{2}\left(\gamma_1\alpha_2^++\gamma_2\alpha_2^-\right)\int_0^{a_1}\frac{\cos\lambda(x+t-2a_1)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt+\frac{i\gamma_2}{2}\int_{a_1}^{a_2}\frac{\cos\lambda(x+t-2a_2)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt$$

$$+\frac{i}{2}\left(\gamma_1\alpha_2^-+\gamma_2\alpha_2^-\right)\int_0^{a_1}\frac{\cos\lambda(2a_1-2a_2+x-t)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt-\frac{i\gamma_2}{2}\int_{a_1}^{a_2}\frac{\cos\lambda(x-t)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt$$

$$+\int_{a_2}^{x}\frac{\sin\lambda(x-t)}{\lambda}Q(t)y(\mathrm{t},\lambda)dt \qquad (2.1.4)$$

İspat.

Öncelikle $(0,a_1)$ aralığında (2.1) denkleminin çözümü için sağladığı integral denklemi yazalım. Bunun için öncelikle $p(x)\equiv 0, q(x)\equiv 0$ durumunu ele alalım. Bu durumda

$$-y''=\lambda^2 y$$

denkleminin genel çözümü

$$y=c_1e^{i\lambda x}+c_2e^{-i\lambda x}$$

şeklinde olduğundan sabitlerin değişimi yöntemi ile $(0,a_1)$ aralığında (2.1) denkleminin $\varphi(0,\lambda)=1,\varphi'(0,\lambda)=0$ başlangıç koşullarını sağlayan çözümü için

$$\varphi(x,\lambda)=e^{i\lambda x}+\frac{1}{\lambda}\int_0^x\sin\lambda(x-t)Q(t)\varphi(\mathrm{t},\lambda)dt$$

integral denklemi elde edilmiş olur.

Diğer taraftan (2.1.3) eşitliğini ispatlamak için $y(x,\lambda)$ fonksiyonunu (2.3) ve (2.4) koşullarını sağlayacak şekilde (a_1,a_2) aralığına devam ettirmek gerekir. Bu aralıkta

$$\varphi(x,\lambda)=A(\lambda)e^{i\lambda x}+B(\lambda)e^{-i\lambda x}+\frac{1}{\lambda}\int_{a_1}^{x}\sin\lambda(x-t)Q(t)\varphi(\mathrm{t},\lambda)\,dt \tag{2.1.5}$$

şeklinde aransın. Bu şekilde alınan $\varphi(x,\lambda)$ fonksiyonu (2.1) denkleminin çözümüdür. $A(\lambda)$ ve $B(\lambda)$ katsayılarını bulmak için (2.3) ve (2.4) süreksizlik koşullarını uygulayalım. O halde $A(\lambda)$ ve $B(\lambda)$ katsayıları için

$$A(\lambda)e^{i\lambda a_1}+B(\lambda)e^{-i\lambda a_1}=\alpha_1 e^{i\lambda a_1}+\alpha_1\int_0^{a_1}\frac{\sin\lambda(a_1-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt \tag{2.1.6}$$

$$\begin{aligned}A(\lambda)e^{i\lambda a_1}-B(\lambda)e^{-i\lambda a_1}&=\beta_1 e^{i\lambda a_1}+\frac{\beta_1}{i\lambda}\int_0^{a_1}\frac{\sin\lambda(a_1-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt\\&+\gamma_1 e^{i\lambda a_1}+\gamma_1\int_0^{a_1}\frac{\sin\lambda(a_1-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt\end{aligned} \tag{2.1.7}$$

lineer cebirsel denklemler sistemi alınır. Buradan da

$$\begin{aligned}A(\lambda)&=\frac{\alpha_1+\beta_1}{2}+\frac{\alpha_1}{2}e^{-i\lambda a_1}\int_0^{a_1}\frac{\sin\lambda(a_1-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt\\&+\frac{\beta_1}{2i}e^{-i\lambda a_1}\int_0^{a_1}\frac{\cos\lambda(a_1-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt+\frac{\gamma_1}{2}+\frac{\gamma_1}{2}e^{-i\lambda a_1}\int_0^{a_1}\frac{\sin\lambda(a_1-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt\end{aligned}$$

$$\begin{aligned}B(\lambda)&=\frac{\alpha_1-\beta_1}{2}e^{2i\lambda a_1}+\frac{\alpha_1}{2}e^{i\lambda a_1}\int_0^{a_1}\frac{\sin\lambda(a_1-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt\\&-\frac{\beta_1}{2i}e^{i\lambda a_1}\int_0^{a_1}\frac{\cos\lambda(a_1-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt-\frac{\gamma_1}{2}e^{2i\lambda a_1}-\frac{\gamma_1}{2}e^{i\lambda a_1}\int_0^{a_1}\frac{\sin\lambda(a_1-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt\end{aligned}$$

olur. $A(\lambda)$ ve $B(\lambda)$ fonksiyonlarının bu ifadeleri (2.1.5) denkleminde yerine yazılırsa $a_1<x<a_2$ için ;

$$\varphi(x,\lambda)=\frac{\alpha_1+\beta_1}{2}e^{i\lambda x}+\frac{\alpha_1}{2}e^{i\lambda(x-a_1)}\int_0^{a_1}\frac{\sin\lambda(a_1-t)}{\lambda}Q(t)\varphi(t,\lambda)\,dt$$

$$+\frac{\beta_1}{2i}e^{i\lambda(x-a_1)}\int_0^{a_1}\frac{\cos\lambda(a_1-t)}{\lambda}Q(t)\varphi(t,\lambda)\,dt+\frac{\gamma_1}{2}e^{i\lambda x}+\frac{\gamma_1}{2}e^{i\lambda(x-a_1)}\int_0^{a_1}\frac{\sin\lambda(a_1-t)}{\lambda}Q(t)\varphi(t,\lambda)\,dt$$

$$+\frac{\alpha_1-\beta_1}{2}e^{i\lambda(2a_1-x)}+\frac{\alpha_1}{2}e^{-i\lambda(x-a_1)}\int_0^{a_1}\frac{\sin\lambda(a_1-t)}{\lambda}Q(t)\varphi(t,\lambda)\,dt$$

$$-\frac{\beta_1}{2i}e^{-i\lambda(x-a_1)}\int_0^{a_1}\frac{\cos\lambda(a_1-t)}{\lambda}Q(t)\varphi(t,\lambda)\,dt-\frac{\gamma_1}{2}e^{i\lambda(2a_1-x)}$$

$$-\frac{\gamma_1}{2}e^{-i\lambda(x-a_1)}\int_0^{a_1}\frac{\sin\lambda(a_1-t)}{\lambda}Q(t)\varphi(t,\lambda)\,dt+\frac{1}{\lambda}\int_{a_1}^{x}\sin\lambda(x-t)Q(t)\varphi(t,\lambda)\,dt$$

alınır. Buradan

$$\varphi(x,\lambda)=\frac{\alpha_1+\beta_1}{2}e^{i\lambda x}+\frac{\alpha_1-\beta_1}{2}e^{i\lambda(2a_1-x)}+\frac{\gamma_1}{2}e^{i\lambda x}-\frac{\gamma_1}{2}e^{i\lambda(2a_1-x)}$$

$$+\alpha_1\cos(x-a_1)\int_0^{a_1}\frac{\sin\lambda(a_1-t)}{\lambda}Q(t)\varphi(t,\lambda)\,dt+\beta_1\sin(x-a_1)\int_0^{a_1}\frac{\cos\lambda(a_1-t)}{\lambda}Q(t)\varphi(t,\lambda)\,dt$$

$$+i\gamma_1\sin(x-a_1)\int_0^{a_1}\frac{\sin\lambda(a_1-t)}{\lambda}Q(t)\varphi(t,\lambda)\,dt+\frac{1}{\lambda}\int_{a_1}^{x}\sin\lambda(x-t)Q(t)\varphi(t,\lambda)\,dt$$

integral denklemleri elde edilir. Burada gerekli ters dönüşüm formülleri uygulanırsa,

$$\varphi(x,\lambda)=\frac{1}{2}(\alpha_1+\beta_1)e^{i\lambda x}+\frac{1}{2}(\alpha_1-\beta_1)e^{i\lambda(2a_1-x)}+\frac{\gamma_1}{2}e^{i\lambda x}-\frac{\gamma_1}{2}e^{i\lambda(2a_1-x)}$$

$$+\frac{1}{2}(\alpha_1+\beta_1)\int_0^{a_1}\frac{\sin\lambda(x-t)}{\lambda}Q(t)\varphi(t,\lambda)\,dt-\frac{1}{2}(\alpha_1-\beta_1)\int_0^{a_1}\frac{\sin\lambda(x+t-2a_1)}{\lambda}Q(t)\varphi(t,\lambda)\,dt$$

$$-i\frac{\gamma_1}{2}\int_0^{a_1}\frac{\cos\lambda(x-t)}{\lambda}Q(t)\varphi(t,\lambda)\,dt+i\frac{\gamma_1}{2}\int_0^{a_1}\frac{\cos\lambda(x+t-2a_1)}{\lambda}Q(t)\varphi(t,\lambda)\,dt$$

$$+\int_{a_1}^{x}\frac{\sin\lambda(x-t)}{\lambda}Q(t)\varphi(t,\lambda)\,dt$$

(2.1.3) eşitliği elde edilir.

Böylece (2.1) denkleminin verilen başlangıç ve (2.3-(2.4) süreksizlik koşullarını sağlayan her bir çözümü (2.1.3) integral denklemini sağlar.

Diğer taraftan (2.1.4) eşitliğini ispatlamak için $y(x,\lambda)$ fonksiyonunu (2.5) ve (2.6) koşullarını sağlayacak şekilde (a_2,π) aralığına devam ettirmek gerekir. Bu aralıkta

$$\varphi(x,\lambda)=A_1(\lambda)e^{i\lambda x}+B_1(\lambda)e^{i\lambda(2a_1-x)}+\frac{1}{\lambda}\int_{a_2}^{x}\sin\lambda(x-t)Q(t)y(\mathrm{t},\lambda)dt \qquad (2.1.8)$$

şeklinde aransın. Bu şekilde alınan $y(x,\lambda)$ fonksiyonu (2.1) denkleminin çözümüdür. $A_1(\lambda)$ ve $B_1(\lambda)$ katsayılarını bulmak için (2.5) ve (2.6) süreksizlik koşullarını uygulayalım. O halde $A_1(\lambda)$ ve $B_1(\lambda)$ katsayıları için

$$A_1(\lambda)e^{i\lambda a_2}+B_1(\lambda)e^{i\lambda(2a_1-a_2)}=\frac{1}{2}\alpha_2(\alpha_1+\beta_1)e^{i\lambda a_2}+\frac{1}{2}\alpha_2(\alpha_1-\beta_1)e^{i\lambda(2a_1-a_2)}$$
$$+\alpha_2\frac{\gamma_1}{2}e^{i\lambda a_2}-\alpha_2\frac{\gamma_1}{2}e^{i\lambda(2a_1-a_2)}+\frac{1}{2}\alpha_2(\alpha_1+\beta_1)\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$-\frac{1}{2}\alpha_2(\alpha_1-\beta_1)\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$-i\alpha_2\frac{\gamma_1}{2}\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt+i\alpha_2\frac{\gamma_1}{2}\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$+\alpha_2\int_{a_1}^{a_2}\frac{\sin\lambda(x-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt \qquad (2.1.9)$$

ve

$$A_1(\lambda)e^{i\lambda a_2}-B_1(\lambda)e^{i\lambda(2a_1-a_2)}=\frac{1}{2}\beta_2(\alpha_1+\beta_1)e^{i\lambda a_2}-\frac{1}{2}\beta_2(\alpha_1-\beta_1)e^{i\lambda(2a_1-a_2)}$$
$$+\beta_2\frac{\gamma_1}{2}e^{i\lambda a_2}+\beta_2\frac{\gamma_1}{2}e^{i\lambda(2a_1-a_2)}+\frac{1}{2i}\beta_2(\alpha_1+\beta_1)\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$-\frac{1}{2i}\beta_2(\alpha_1-\beta_1)\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$+\beta_2\frac{\gamma_1}{2}\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt-\beta_2\frac{\gamma_1}{2}\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$+\frac{\beta_2}{i}\int_{a_1}^{a_2}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt+\frac{1}{2}\gamma_2(\alpha_1+\beta_1)e^{i\lambda a_2}+\frac{1}{2}\gamma_2(\alpha_1-\beta_1)e^{i\lambda(2a_1-a_2)}$$
$$+\frac{\gamma_1\gamma_2}{2}e^{i\lambda a_2}-\frac{\gamma_1\gamma_2}{2}e^{i\lambda(2a_1-a_2)}+\frac{1}{2}\gamma_2(\alpha_1+\beta_1)\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$-\frac{1}{2}\gamma_2(\alpha_1-\beta_1)\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$-i\frac{\gamma_1\gamma_2}{2}\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt+i\frac{\gamma_1\gamma_2}{2}\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$+\gamma_2\int_{a_1}^{a_2}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt \qquad (2.1.10)$$

lineer cebirsel denklemler sistemi alınır. (2.1.9) ve (2.1.10) taraf tarafa toplanırsa,

$$A_1(\lambda)=\frac{1}{4}(\alpha_1+\beta_1)(\alpha_2+\beta_2)+\frac{1}{4}(\alpha_1-\beta_1)(\alpha_2-\beta_2)e^{2i\lambda(a_1-a_2)}+\frac{1}{4}\gamma_1(\alpha_2+\beta_2)$$

$$-\frac{1}{4}\gamma_1(\alpha_2-\beta_2)e^{2i\lambda(a_1-a_2)}+\frac{1}{4}\alpha_2(\alpha_1+\beta_1)e^{-i\lambda a_2}\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-\frac{1}{4}\alpha_2(\alpha_1-\beta_1)e^{-i\lambda a_2}\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-i\frac{\alpha_2\gamma_1}{4}e^{-i\lambda a_2}\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+i\frac{\alpha_2\gamma_1}{4}e^{-i\lambda a_2}\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+\frac{1}{2}\alpha_2 e^{-i\lambda a_2}\int_{a_1}^{a_2}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+\frac{1}{4i}\beta_2(\alpha_1+\beta_1)e^{-i\lambda a_2}\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-\frac{1}{4i}\beta_2(\alpha_1-\beta_1)e^{-i\lambda a_2}\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+\frac{\beta_2\gamma_1}{4}e^{-i\lambda a_2}\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-\frac{\beta_2\gamma_1}{4}e^{-i\lambda a_2}\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt+$$

$$+\frac{1}{2i}\beta_2 e^{-i\lambda a_2}\int_{a_1}^{a_2}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+\frac{1}{4}(\alpha_1+\beta_1)\gamma_2+\frac{1}{4}(\alpha_1-\beta_1)\gamma_2 e^{2i\lambda(a_1-a_2)}+\frac{\gamma_1\gamma_2}{4}-\frac{\gamma_1\gamma_2}{4}e^{2i\lambda(a_1-a_2)}$$

$$+\frac{1}{4}(\alpha_1+\beta_1)\gamma_2 e^{-i\lambda a_2}\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-\frac{1}{4}(\alpha_1-\beta_1)\gamma_2 e^{-i\lambda a_2}\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-i\frac{\gamma_1\gamma_2}{4}e^{-i\lambda a_2}\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+i\frac{\gamma_1\gamma_2}{4}e^{-i\lambda a_2}\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+\frac{\gamma_2}{2}e^{-i\lambda a_2}\int_{a_1}^{a_2}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$B_1(\lambda)=\frac{1}{4}(\alpha_1+\beta_1)(\alpha_2-\beta_2)e^{2i\lambda(a_1-a_2)}+\frac{1}{4}(\alpha_1-\beta_1)(\alpha_2+\beta_2)+\frac{1}{4}\gamma_1(\alpha_2-\beta_2)e^{2i\lambda(a_1-a_2)}$$

$$-\frac{1}{4}\gamma_1(\alpha_2+\beta_2)+\frac{1}{4}\alpha_2(\alpha_1+\beta_1)e^{-i\lambda(2a_1-a_2)}\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-\frac{1}{4}\alpha_2(\alpha_1-\beta_1)e^{-i\lambda(2a_1-a_2)}\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-i\frac{\alpha_2\gamma_1}{4}e^{-i\lambda(2a_1-a_2)}\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+i\frac{\alpha_2\gamma_1}{4}e^{-i\lambda(2a_1-a_2)}\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+\frac{1}{2}\alpha_2e^{-i\lambda(2a_1-a_2)}\int_{a_1}^{a_2}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+\frac{1}{4i}\beta_2(\alpha_1+\beta_1)e^{-i\lambda(2a_1-a_2)}\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-\frac{1}{4i}\beta_2(\alpha_1-\beta_1)e^{-i\lambda(2a_1-a_2)}\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-\frac{\beta_2\gamma_1}{4}e^{-i\lambda(2a_1-a_2)}\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+\frac{\beta_2\gamma_1}{4}e^{-i\lambda(2a_1-a_2)}\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt+$$

$$-\frac{1}{2i}\beta_2e^{-i\lambda(2a_1-a_2)}\int_{a_1}^{a_2}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-\frac{1}{4}(\alpha_1+\beta_1)\gamma_2e^{2i\lambda(a_2-a_1)}-\frac{1}{4}(\alpha_1-\beta_1)\gamma_2-\frac{\gamma_1\gamma_2}{4}-\frac{\gamma_1\gamma_2}{4}e^{2i\lambda(a_2-a_1)}$$

$$-\frac{1}{4}(\alpha_1+\beta_1)\gamma_2e^{-i\lambda(2a_1-a_2)}\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+\frac{1}{4}(\alpha_1-\beta_1)\gamma_2e^{-i\lambda(2a_1-a_2)}\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+i\frac{\gamma_1\gamma_2}{4}e^{-i\lambda(2a_1-a_2)}\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-i\frac{\gamma_1\gamma_2}{4}e^{-i\lambda(2a_1-a_2)}\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-\frac{\gamma_2}{2}e^{-i\lambda(2a_1-a_2)}\int_{a_1}^{a_2}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

elde edilir. $A_1(\lambda)$ ve $B_1(\lambda)$ fonksiyonlarının bu ifadeleri (2.1.8) denkleminde yerine yazılırsa $a_2 < x \le \pi$ için,

$$\varphi(x,\lambda)=\frac{1}{4}(\alpha_1+\beta_1)(\alpha_2+\beta_2)e^{i\lambda x}+\frac{1}{4}(\alpha_1-\beta_1)(\alpha_2-\beta_2)e^{i\lambda(2a_1-2a_2+x)}+\frac{1}{4}\gamma_1(\alpha_2+\beta_2)e^{i\lambda x}$$

$$-\frac{1}{4}\gamma_1(\alpha_2-\beta_2)e^{i\lambda(2a_1-2a_2+x)}+\frac{1}{4}\alpha_2(\alpha_1+\beta_1)e^{i\lambda(x-a_2)}\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$-\frac{1}{4}\alpha_2(\alpha_1-\beta_1)e^{i\lambda(x-a_2)}\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$-i\frac{\alpha_2\gamma_1}{4}e^{i\lambda(x-a_2)}\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$+i\frac{\alpha_2\gamma_1}{4}e^{i\lambda(x-a_2)}\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$+\frac{1}{2}\alpha_2e^{i\lambda(x-a_2)}\int_{a_1}^{a_2}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$+\frac{1}{4i}\beta_2(\alpha_1+\beta_1)e^{i\lambda(x-a_2)}\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$-\frac{1}{4i}\beta_2(\alpha_1-\beta_1)e^{i\lambda(x-a_2)}\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$+\frac{\beta_2\gamma_1}{4}e^{i\lambda(x-a_2)}\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$-\frac{\beta_2\gamma_1}{4}e^{i\lambda(x-a_2)}\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt+$$

$$+\frac{1}{2i}\beta_2e^{i\lambda(x-a_2)}\int_{a_1}^{a_2}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$+\frac{1}{4}(\alpha_1+\beta_1)\gamma_2e^{i\lambda x}+\frac{1}{4}(\alpha_1-\beta_1)\gamma_2e^{i\lambda(2a_1-2a_2+x)}+\frac{\gamma_1\gamma_2}{4}e^{i\lambda x}-\frac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_1-2a_2+x)}$$

$$+\frac{1}{4}(\alpha_1+\beta_1)\gamma_2e^{i\lambda(x-a_2)}\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$-\frac{1}{4}(\alpha_1-\beta_1)\gamma_2e^{i\lambda(x-a_2)}\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$-i\frac{\gamma_1\gamma_2}{4}e^{i\lambda(x-a_2)}\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$+i\frac{\gamma_1\gamma_2}{4}e^{i\lambda(x-a_2)}\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$+\frac{\gamma_2}{2}e^{i\lambda(x-a_2)}\int_{a_1}^{a_2}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt+\frac{1}{4}(\alpha_1+\beta_1)(\alpha_2-\beta_2)e^{i\lambda(2a_2-x)}$$

$$+\frac{1}{4}(\alpha_1-\beta_1)(\alpha_2+\beta_2)e^{i\lambda(2a_1-x)}+\frac{1}{4}\gamma_1(\alpha_2-\beta_2)e^{i\lambda(2a_2-x)}$$

$$-\frac{1}{4}\gamma_1(\alpha_2+\beta_2)e^{i\lambda(2a_1-x)}+\frac{1}{4}\alpha_2(\alpha_1+\beta_1)e^{i\lambda(a_2-x)}\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-\frac{1}{4}\alpha_2(\alpha_1-\beta_1)e^{i\lambda(a_2-x)}\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-i\frac{\alpha_2\gamma_1}{4}e^{i\lambda(a_2-x)}\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+i\frac{\alpha_2\gamma_1}{4}e^{i\lambda(a_2-x)}\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+\frac{1}{2}\alpha_2e^{i\lambda(a_2-x)}\int_{a_1}^{a_2}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+\frac{1}{4i}\beta_2(\alpha_1+\beta_1)e^{i\lambda(a_2-x)}\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-\frac{1}{4i}\beta_2(\alpha_1-\beta_1)e^{i\lambda(a_2-x)}\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-\frac{\beta_2\gamma_1}{4}e^{i\lambda(a_2-x)}\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+\frac{\beta_2\gamma_1}{4}e^{i\lambda(a_2-x)}\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt+$$

$$-\frac{1}{2i}\beta_2e^{i\lambda(a_2-x)}\int_{a_1}^{a_2}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-\frac{1}{4}(\alpha_1+\beta_1)\gamma_2e^{i\lambda(2a_2-x)}-\frac{1}{4}(\alpha_1-\beta_1)\gamma_2e^{i\lambda(2a_1-x)}-\frac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_2-x)}-\frac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_1-x)}$$

$$-\frac{1}{4}(\alpha_1+\beta_1)\gamma_2e^{i\lambda(a_2-x)}\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+\frac{1}{4}(\alpha_1-\beta_1)\gamma_2e^{i\lambda(a_2-x)}\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+i\frac{\gamma_1\gamma_2}{4}e^{i\lambda(a_2-x)}\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-i\frac{\gamma_1\gamma_2}{4}e^{i\lambda(a_2-x)}\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$-\frac{\gamma_2}{2}e^{i\lambda(a_2-x)}\int_{a_1}^{a_2}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt+\int_{a_2}^{x}\frac{\sin\lambda(x-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

eşitliği alınır. Buradan

$$\varphi(x,\lambda)=\frac{1}{4}(\alpha_1+\beta_1)(\alpha_2+\beta_2)e^{i\lambda x}+\frac{1}{4}(\alpha_1-\beta_1)(\alpha_2-\beta_2)e^{i\lambda(2a_1-2a_2+x)}+\frac{1}{4}\gamma_1(\alpha_2+\beta_2)e^{i\lambda x}$$

$$-\frac{1}{4}\gamma_1(\alpha_2-\beta_2)e^{i\lambda(2a_1-2a_2+x)}+\frac{1}{4}(\alpha_1+\beta_1)\gamma_2e^{i\lambda x}+\frac{1}{4}(\alpha_1-\beta_1)\gamma_2e^{i\lambda(2a_1-2a_2+x)}$$

$$+\frac{\gamma_1\gamma_2}{4}e^{i\lambda x}-\frac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_1-2a_2+x)}+\frac{1}{4}(\alpha_1+\beta_1)(\alpha_2-\beta_2)e^{i\lambda(2a_2-x)}$$

$$+\frac{1}{4}(\alpha_1-\beta_1)(\alpha_2+\beta_2)e^{i\lambda(2a_1-x)}+\frac{1}{4}\gamma_1(\alpha_2-\beta_2)e^{i\lambda(2a_2-x)}$$

$$-\frac{1}{4}\gamma_1(\alpha_2+\beta_2)e^{i\lambda(2a_1-x)}-\frac{1}{4}(\alpha_1+\beta_1)\gamma_2e^{i\lambda(2a_2-x)}-\frac{1}{4}(\alpha_1-\beta_1)\gamma_2e^{i\lambda(2a_1-x)}-\frac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_2-x)}$$

$$-\frac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_1-x)}+\frac{1}{2}\alpha_2(\alpha_1+\beta_1)\cos\lambda(x-a_2)\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$-\frac{1}{2}\alpha_2(\alpha_1-\beta_1)\cos\lambda(x-a_2)\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$-i\frac{\alpha_2\gamma_1}{2}\cos\lambda(x-a_2)\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$+i\frac{\alpha_2\gamma_1}{2}\cos\lambda(x-a_2)\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$+\alpha_2\cos\lambda(x-a_2)\int_{a_1}^{a_2}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$+\frac{1}{2}\beta_2(\alpha_1+\beta_1)\sin\lambda(x-a_2)\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$-\frac{1}{2}\beta_2(\alpha_1-\beta_1)\sin\lambda(x-a_2)\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$+i\frac{\beta_2\gamma_1}{2}\sin\lambda(x-a_2)\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$-i\frac{\beta_2\gamma_1}{2}\sin\lambda(x-a_2)\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt+$$

$$+\beta_2\sin\lambda(x-a_2)\int_{a_1}^{a_2}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$+\frac{i}{2}(\alpha_1+\beta_1)\gamma_2\sin\lambda(x-a_2)\int_0^{a_1}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$-\frac{i}{2}(\alpha_1-\beta_1)\gamma_2\sin\lambda(x-a_2)\int_0^{a_1}\frac{\sin\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$+\frac{\gamma_1\gamma_2}{2}\sin\lambda(x-a_2)\int_0^{a_1}\frac{\cos\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$-\frac{\gamma_1\gamma_2}{2}\sin\lambda(x-a_2)\int_0^{a_1}\frac{\cos\lambda(a_2+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$+i\gamma_2\sin\lambda(x-a_2)\int_{a_1}^{a_2}\frac{\sin\lambda(a_2-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt+\int_{a_2}^{x}\frac{\sin\lambda(x-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

İfadesi elde edilir. Burada gerekli ters dönüşüm formülleri uygulanırsa,

$$\varphi(x,\lambda)=\alpha_1^+\alpha_2^+e^{i\lambda x}+\alpha_1^-\alpha_2^-e^{i\lambda(2a_1-2a_2+x)}+\alpha_1^+\alpha_2^-e^{i\lambda(2a_2-x)}+\alpha_1^-\alpha_2^+e^{i\lambda(2a_1-x)}+\frac{\gamma_1\alpha_2^+}{2}e^{i\lambda x}$$
$$-\frac{\gamma_1\alpha_2^-}{2}e^{i\lambda(2a_1-2a_2+x)}+\frac{\gamma_1\alpha_2^-}{2}e^{i\lambda(2a_2-x)}-\frac{\gamma_1\alpha_2^+}{2}e^{i\lambda(2a_1-x)}-\frac{\gamma_1\gamma_2}{4}e^{i\lambda x}-\frac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_1-2a_2+x)}$$
$$-\frac{\gamma_2\alpha_1^+}{2}e^{i\lambda(2a_2-x)}-\frac{\gamma_2\alpha_1^-}{2}e^{i\lambda(2a_1-x)}+\frac{\gamma_2\alpha_1^+}{2}e^{i\lambda x}+\frac{\gamma_2\alpha_1^-}{2}e^{i\lambda(2a_1-2a_2+x)}-\frac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_2-x)}$$
$$+\frac{\gamma_1\gamma_2}{4}e^{i\lambda(2a_1-x)}+\left(\alpha_1^+\alpha_2^++\frac{\gamma_1\gamma_2}{4}\right)\int_0^{a_1}\frac{\sin\lambda(x-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$+\left(-\alpha_1^+\alpha_2^-+\frac{\gamma_1\gamma_2}{4}\right)\int_0^{a_1}\frac{\sin\lambda(x+t-2a_2)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$+\left(-\alpha_1^-\alpha_2^+-\frac{\gamma_1\gamma_2}{4}\right)\int_0^{a_1}\frac{\sin\lambda(x+t-2a_1)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$-\alpha_2^-\int_{a_1}^{a_2}\frac{\sin\lambda(x+t-2a_2)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$+\left(\alpha_1^-\alpha_2^--\frac{\gamma_1\gamma_2}{4}\right)\int_0^{a_1}\frac{\sin\lambda(2a_1-2a_2+x-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$+\alpha_2^+\int_{a_1}^{a_2}\frac{\sin\lambda(x-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt-\frac{i}{2}\left(\gamma_1\alpha_2^+-\gamma_2\alpha_1^+\right)\int_0^{a_1}\frac{\cos\lambda(x-t)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$
$$+\frac{i}{2}\left(-\gamma_1\alpha_2^--\gamma_2\alpha_1^-\right)\int_0^{a_1}\frac{\cos\lambda(x+t-2a_2)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)dt$$

$$+\frac{i}{2}\left(\gamma_1\alpha_2^+ + \gamma_2\alpha_2^-\right)\int_0^{a_1}\frac{\cos\lambda\left(x+t-2a_1\right)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$+\frac{i\gamma_2}{2}\int_{a_1}^{a_2}\frac{\cos\lambda\left(x+t-2a_2\right)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$+\frac{i}{2}\left(\gamma_1\alpha_2^- + \gamma_2\alpha_2^-\right)\int_0^{a_1}\frac{\cos\lambda\left(2a_1-2a_2+x-t\right)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

$$-\frac{i\gamma_2}{2}\int_{a_1}^{a_2}\frac{\cos\lambda\left(x-t\right)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt+\int_{a_2}^{x}\frac{\sin\lambda\left(x-t\right)}{\lambda}Q(t)\varphi(\mathrm{t},\lambda)\,dt$$

(2.1.4) eşitliği elde edilir.

Böylece (2.1) denkleminin verilen başlangıç ve (2.5-(2.6) süreksizlik koşullarını sağlayan her bir çözümü (2.1.4) integral denklemini sağlar. Gerekli hesaplamalar yapıldığında görülebilmektedir ki; $\varphi(x,\lambda)$ fonksiyonu (2.1) denklemini, verilen başlangıç koşullarını ve süreksizlik koşullarını sağlamaktadır. Bu halde lemmanın ispatı tamamlanmış olur.

Teorem 2.1.2. $p(x)\in W_2^1(0,\pi)$ ve $q(x)\in L_2(0,\pi)$ olmak üzere (2.1) denkleminin $(2.1.1)$ başlangıç ve (2.3)-(2.6) süreksizlik koşullarını sağlayan $y_\upsilon(x,\lambda)$ çözümü için

$$y_\upsilon(x,\lambda)=y_{0\upsilon}(x,\lambda)+\int_{-x}^{x}K_\upsilon(x,\lambda)e^{i\lambda t}dt \quad \left(\upsilon=\overline{1,3}\right)$$

integral gösterilimi vardır. Burada

$$y_{0\upsilon}(x,\lambda)=\begin{cases}R_0(x)e^{i\lambda x} & ;0\le x<a_1\\ R_1(x)e^{i\lambda x}+R_2(x)e^{i\lambda(2a_1-x)} & ;a_1<x<a_2\\ R_3(x)e^{i\lambda(2a_1-x)}+R_2(x)e^{i\lambda(2a_1-2a_2+x)} & ;a_2<x\le\pi\end{cases}$$

$$R_0(x)=e^{-i\int_0^x p(x)dx},\ R_1(x)=R_0(a_1)\left(\alpha_1^+ +\frac{\gamma_1}{2}\right)e^{-i\int_{a_1}^x p(t)dt},\ R_2(x)=R_0(a_1)\left(\alpha_1^- -\frac{\gamma_1}{2}\right)e^{i\int_{a_1}^x p(t)dt}$$

$$R_3(x)=\left(-\frac{\gamma_2}{2}\left(\alpha_1^+ +\frac{\gamma_1}{2}\right)R_0(a_1)-\alpha_2^- R_1(a_2)\right)e^{i\int_{a_2}^x p(t)dt},$$

$$R_4(x)=\left(\frac{\gamma_2}{2}\left(\alpha_1^- +\frac{\gamma_1}{2}\right)R_0(a_1)+\alpha_2^- R_2(a_2)\right)e^{-i\int_{a_2}^x p(t)dt}$$

Ve $\sigma(x)=\int_0^x\left(2\left|p(t)\right|+(x-t)\left|q(t)\right|\right)dt$ olmak üzere $K_\upsilon(x,t)$ fonksiyonları

$\int_{-x}^{x} \left| K_{\upsilon}\left(x,\lambda\right) \right| dt \leq e^{c_{\upsilon}\sigma(x)} - 1$ eşitsizliğini sağlamaktadır. Burada , $c_1 = 1$,

$c_2 = \left(\alpha_1^+ + \left| \alpha_1^- \right| + \frac{\gamma_1}{2} + 2 \right)$, $c_3 = \left[\left(\left| \alpha_2^- \right| + \gamma_2 \right) \left(\alpha_1 + 1 \right) + \alpha_2^+ + 1 \right]$ şeklindedir.

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