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An A-Stable Uniformly Order Seven Block Hybrid Method for Solving Nonlinear Initial Value Problems

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Keywords

A-stable, Block Hybrid Method, Nonlinear Initial Value Problems, Numerical Methods, Stiffness, Accuracy, Computational Efficiency.

Abstract

This study presents the development of a new A-stable uniformly order seven block hybrid method for solving Nonlinear Initial Value Problems (NIVPs) in Ordinary Differential Equations (ODEs). Traditional numerical methods, including Euler's method and Runge-Kutta methods, often struggle with nonlinear problems due to stability and computational inefficiencies, especially when dealing with stiff equations. To address this limitation, the proposed method integrates the advantages of block hybrid techniques, ensuring A-stability and uniform order seven, which enhances both accuracy and computational efficiency. The formulation of the method involves applying a one-step linear multistep approach combined with interpolation and collocation techniques. Through extensive analysis, the method is shown to satisfy essential numerical properties such as consistency, zero-stability, and convergence. Numerical experiments demonstrate that the new method outperforms existing methods in terms of accuracy and computational cost, particularly for stiff nonlinear problems. The method's performance is validated by applying it to various test cases, yielding results consistent with previous studies and showing significant improvements in error reduction.

1. Introduction

Initial Value Problems (IVPs) for ordinary differential equations (ODEs) play a crucial role in modeling dynamic systems across scientific domains such as physics, biology, and engineering. Nonlinear initial value problems (NIVPs) are especially important because they often describe real-world phenomena with complex behaviors [1]. However, due to their nonlinear nature, these problems frequently lack analytical solutions, requiring the use of numerical methods for approximation. Classical numerical techniques like Euler's method, Runge-Kutta methods, and linear multistep methods have been widely used, but they often struggle with stiff or highly nonlinear problems due to limitations in stability and computational efficiency [2, 3].

Despite these advancements, many existing methods fall short of providing both A-stability and a uniformly high order in a block implementation. This gap in the literature and practice presents a significant challenge for solving nonlinear IVPs efficiently and accurately [4] To address this, the present study aims to develop a new block hybrid method that is not only A-stable but also maintains a uniform order of seven. Such a method promises improved stability, accuracy, and computational efficiency, offering a more robust and reliable tool for solving a wide range of nonlinear initial value problems [5].

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The nonlinear initial value problem (IVP) is given by the following first-order differential equation of the form

$$y'(u) = f(u,\ y), y(0) = u_0 \tag{1}$$

is considered in this study. This type of problem can be solved using various numerical methods, such as the Runge-Kutta method, Euler's method, Backward Euler or Backward Differentiation Formulas, Multistep methods, Advanced hybrid methods, especially when an analytical solution is difficult to obtain. The selection of a method depends on the problem's stiffness, required accuracy, and computational resources.

The study by [6] introduced a sixth-order optimized hybrid block Adams method designed to solve first-order linear and nonlinear ordinary differential equations (ODEs) more accurately and efficiently. The method incorporates Lagrange interpolation polynomials along with a three-point block approach, using an off-step point to simultaneously compute multiple solution values. The study demonstrated that their method satisfies crucial numerical properties such as convergence, consistency, zero-stability, and A-stability, making it especially suitable for stiff ODEs. The numerical results confirmed that the method outperformed existing techniques, minimizing errors and enhancing stability, particularly when reducing the step size. This improvement significantly reduced the computational steps required, offering a substantial improving in solving initial value problems.

Also, the study by [7] developed a one-sixth hybrid block method for solving general first-order initial value problems of ODEs. By employing collocation and interpolation based on Chebyshev polynomials, the authors constructed a continuous linear multistep method evaluated at off-grid points, resulting in a hybrid method that was zero-stable, consistent, and convergent. Their method demonstrated superior performance on standard problems, including the SIR model and highly stiff oscillatory problems. An implicit quarter-step first derivative block hybrid method for solving stiff ODEs, using interpolation and collocation techniques was studied by [8]. Their method, which exhibited A-stability, outperformed traditional methods, particularly for stiff equations. Similarly, [9] developed a continuous one-step hybrid block method using shifted Legendre polynomials, which was shown to be efficient, simple, and accurate for solving both linear and nonlinear ODEs. Finally [9] introduced a two-step second derivative hybrid block backward differentiation formula for stiff ODEs. Their method, based on interpolation and collocation, proved to be a stable and accurate solution for stiff problems, offering improvements over previous methods and promising better computational results in various scientific and engineering applications.

2. Formulation of New Scheme

The new hybrid block method was derived using the one-step linear multistep method of the form

$$\alpha_{\frac{1}{2}} y_{n+\frac{1}{2}} = h\sum \beta_\tau f_{n+\tau},\ \tau = 0, \frac{1}{6}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{5}{6}, 1 \tag{2}$$

where $\alpha_{\frac{1}{2}}$ and β_τ are continuous coefficients of continuous scheme.

We consider an approximate solution of a series to Eq. 1 as a continuous function of the form

$$y(u) = \sum_{\tau=0}^{\rho+\nu-1} \vartheta_\tau u^\tau \tag{3}$$

Eq.3 are differentiated once to obtain

$$y'(u) = \sum_{\tau=0}^{\psi} \tau\vartheta_\tau u^{\tau-1} \tag{4}$$

where $\psi=\rho+\nu-1$, the point of interpolation and collocation. Using one-step linear multistep method in Eq.2, we obtain the reformulated matrix form as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{12} & \frac{1}{54} & \frac{5}{1296} & \frac{1}{1296} & \frac{7}{46656} \\ 0 & 1 & \frac{1}{2} & \frac{3}{16} & \frac{1}{16} & \frac{5}{256} & \frac{3}{512} & \frac{7}{4096} \\ 0 & 1 & 1 & \frac{3}{4} & \frac{1}{2} & \frac{5}{16} & \frac{3}{16} & \frac{7}{64} \\ 0 & 1 & \frac{3}{2} & \frac{27}{16} & \frac{27}{16} & \frac{405}{256} & \frac{512}{729} & \frac{5103}{4096} \\ 0 & 1 & \frac{5}{3} & \frac{25}{12} & \frac{125}{54} & \frac{3125}{1296} & \frac{3125}{1296} & \frac{109375}{46656} \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} \sigma_0 \\ \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \\ \sigma_7 \end{bmatrix} = \begin{bmatrix} y_{n+1} \\ f_n \\ f_{n+\frac{1}{6}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+\frac{5}{6}} \\ f_{n+1} \end{bmatrix} \tag{5}$$

In Eq.5, the method constructs a system of linear equations resulting from the application of the one-step linear multistep method combined with collocation and interpolation. The general matrix form of the system can be represented in Eq.5 is solved using the Gaussian elimination method, to obtain the continuous values of $\alpha_{\frac{1}{2}}$ and β_τ in Eq.2 as

$$\alpha_0(\upsilon)=1$$

$$\beta_0(\upsilon)=-\frac{31}{630}+\upsilon-\frac{233}{30}\upsilon^2+\frac{1358}{45}\upsilon^3-\frac{191}{3}\upsilon^4+\frac{1112}{15}\upsilon^5-\frac{224}{5}\upsilon^6+\frac{384}{35}\upsilon^7$$

$$\beta_{\frac{1}{6}}(\upsilon)=-\frac{243}{980}+\frac{729}{28}\upsilon^2-\frac{11583}{70}\upsilon^3-\frac{6075}{14}\upsilon^4-\frac{19926}{35}\upsilon^5+\frac{2592}{7}\upsilon^6-\frac{23328}{245}\upsilon^7$$

$$\beta_{\frac{1}{4}}(\upsilon)=-\frac{64}{2205}-\frac{160}{7}\upsilon^2+\frac{11072}{63}\upsilon^3-\frac{3552}{7}\upsilon^4+\frac{73984}{105}\upsilon^5-\frac{3328}{7}\upsilon^6+\frac{6144}{45}\upsilon^7$$

$$\beta_{\frac{1}{2}}(\upsilon)=-\frac{73}{210}+\frac{15}{2}\upsilon^2-\frac{203}{3}\upsilon^3+238\upsilon^4-\frac{1916}{5}\upsilon^5+288\upsilon^6-\frac{576}{7}\upsilon^7$$

$$\beta_{\frac{1}{2}}(\upsilon)=-\frac{64}{2205}-\frac{160}{21}\upsilon^2+\frac{4544}{63}\upsilon^3-\frac{5728}{21}\upsilon^4+\frac{50944}{105}\upsilon^5-\frac{2816}{7}\upsilon^6+\frac{6144}{49}\upsilon^7$$

$$\beta_{\frac{3}{4}}(\upsilon)=-\frac{243}{980}+\frac{729}{140}\upsilon^2-\frac{3483}{70}\upsilon^3+\frac{2673}{14}\upsilon^4-\frac{2430}{7}\upsilon^5+\frac{10368}{35}\upsilon^6-\frac{23328}{245}\upsilon^7$$

$$\beta_1(\upsilon)=-\frac{31}{630}-\frac{1}{2}\upsilon^2+\frac{218}{45}\upsilon^3-19\upsilon^4+\frac{536}{15}\upsilon^5-32\upsilon^6+\frac{384}{35}\upsilon^7$$

Evaluate Eq.5 at non-interpolating pints $t=0,\frac{1}{6},\frac{1}{4},\frac{3}{4},\frac{5}{6},1$ to gives

$$y_n=y_{n+\frac{1}{2}}-\frac{271}{5040}hf_n-\frac{1539}{7840}hf_{n+\frac{1}{6}}-\frac{242}{2205}hf_{n+\frac{1}{4}}-\frac{73}{420}hf_{n+\frac{1}{2}}+\frac{178}{2205}hf_{n+\frac{3}{4}}-\frac{81}{1568}hf_{n+\frac{5}{6}}+\frac{23}{5040}hf_{n+1} \tag{6}$$

$$\left.\begin{aligned}
y_{n+\frac{1}{6}}&=y_{n+\frac{1}{2}}-\frac{31}{8505}hf_n+\frac{53}{1764}hf_{n+\frac{1}{6}}-\frac{43168}{178605}hf_{n+\frac{1}{4}}-\frac{1178}{8505}hf_{n+\frac{1}{2}}+\frac{928}{19845}hf_{n+\frac{3}{4}}-\frac{253}{8820}hf_{n+\frac{5}{6}}+\frac{61}{25515}hf_{n+1}\\
y_{n+\frac{1}{4}}&=y_{n+\frac{1}{2}}-\frac{341}{80640}hf_n+\frac{8667}{125440}hf_{n+\frac{1}{6}}-\frac{6871}{35280}hf_{n+\frac{1}{4}}-\frac{1903}{13440}hf_{n+\frac{1}{2}}+\frac{1739}{35280}hf_{n+\frac{3}{4}}-\frac{3807}{125440}hf_{n+\frac{5}{6}}+\frac{41}{16128}hf_{n+1}\\
y_{n+\frac{3}{4}}&=y_{n+\frac{1}{2}}-\frac{41}{16128}hf_n+\frac{3807}{125440}hf_{n+\frac{1}{6}}-\frac{1739}{35280}hf_{n+\frac{1}{4}}+\frac{1903}{13440}hf_{n+\frac{1}{2}}+\frac{6871}{35280}hf_{n+\frac{3}{4}}-\frac{8667}{125440}hf_{n+\frac{5}{6}}+\frac{341}{80640}hf_{n+1}\\
y_{n+\frac{5}{6}}&=y_{n+\frac{1}{2}}-\frac{61}{25515}hf_n+\frac{253}{8820}hf_{n+\frac{1}{6}}-\frac{928}{19845}hf_{n+\frac{1}{4}}+\frac{1178}{8505}hf_{n+\frac{1}{2}}+\frac{43168}{178605}hf_{n+\frac{3}{4}}-\frac{53}{1764}hf_{n+\frac{5}{6}}+\frac{31}{8505}hf_{n+1}\\
y_{n+1}&=y_{n+\frac{1}{2}}-\frac{23}{5040}hf_n+\frac{81}{1568}hf_{n+\frac{1}{6}}-\frac{178}{2205}hf_{n+\frac{1}{4}}+\frac{73}{420}hf_{n+\frac{1}{2}}+\frac{242}{2205}hf_{n+\frac{3}{4}}+\frac{1539}{7840}hf_{n+\frac{5}{6}}+\frac{271}{5040}hf_{n+1}
\end{aligned}\right\} \tag{7}$$

Making the $y_{n+\frac{1}{2}}$ subject Eq.6 to obtained

$$y_{n+\frac{1}{2}} = y_n + \frac{271}{5040}hf_n + \frac{1539}{7840}hf_{n+\frac{1}{6}} + \frac{242}{2205}hf_{n+\frac{1}{4}} + \frac{73}{420}hf_{n+\frac{1}{2}} - \frac{178}{2205}hf_{n+\frac{3}{4}} + \frac{81}{1568}hf_{n+\frac{5}{6}} - \frac{23}{5040}hf_{n+1} \quad (8)$$

Substitute Eq.8 into Eq. 7 to obtained the new schemes as

$$\begin{pmatrix} y_{n+\frac{1}{2}} \\ y_{n+\frac{1}{6}} \\ y_{n+\frac{1}{4}} \\ y_{n+\frac{3}{4}} \\ y_{n+\frac{5}{6}} \\ y_{n+1} \end{pmatrix} - y_n = \begin{pmatrix} \frac{271}{5040} & \frac{1539}{7840} & \frac{242}{2205} & \frac{73}{420} & -\frac{178}{2205} & \frac{81}{1568} & -\frac{23}{5040} \\ \frac{6821}{136080} & \frac{15971}{70560} & -\frac{23566}{178605} & \frac{1201}{34020} & -\frac{674}{19845} & \frac{1621}{70560} & -\frac{887}{408240} \\ \frac{799}{16128} & \frac{33291}{125440} & -\frac{2999}{35280} & \frac{433}{13440} & -\frac{1109}{35280} & \frac{2673}{125440} & -\frac{163}{40824} \\ \frac{459}{8960} & \frac{28431}{125440} & -\frac{237}{3920} & \frac{1413}{4480} & \frac{447}{3920} & \frac{2187}{12440} & -\frac{3}{8960} \\ \frac{4195}{81648} & \frac{3175}{14112} & \frac{250}{3969} & \frac{21}{6804} & \frac{5750}{35721} & \frac{305}{14112} & -\frac{25}{27216} \\ \frac{31}{630} & \frac{243}{980} & \frac{64}{2205} & \frac{73}{210} & \frac{64}{2205} & \frac{243}{980} & \frac{31}{630} \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{1}{6}} \\ f_{n+\frac{1}{4}} \\ f_{n+\frac{3}{4}} \\ f_{n+\frac{5}{6}} \\ f_{n+1} \end{pmatrix} \quad (9)$$

3. Analysis of the Uniformly Order Seven Computational Method

The subsequent analysis of the newly developed scheme encompasses key aspects including its order, error constant, consistency, zero stability, stability region, and convergence properties.

3.1. Orders and Error Constants

Examine the linear operator corresponding to the Eq.2 associated with uniform order seven block hybrid method (9), we compute the new method as

$$\begin{bmatrix} \sum_{j=0}^{\infty}\frac{\left(\frac{1}{2}\right)^j}{j!} - y_n - \sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^j\left[-\frac{1539}{7840}\left(\frac{1}{2}\right) - \frac{242}{2205}\left(\frac{1}{6}\right) - \frac{73}{420}\left(\frac{1}{4}\right) + \frac{178}{2205}\left(\frac{3}{4}\right) - \frac{81}{1568}\left(\frac{5}{6}\right) + \frac{23}{5040}(1)\right] \\ \sum_{j=0}^{\infty}\frac{\left(\frac{1}{6}\right)^j}{j!} - y_n - \sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^j\left[-\frac{15971}{70560}\left(\frac{1}{2}\right) + \frac{23566}{178605}\left(\frac{1}{6}\right) - \frac{1201}{34020}\left(\frac{1}{4}\right) + \frac{674}{19845}\left(\frac{3}{4}\right) + -\frac{1621}{70560}\left(\frac{5}{6}\right) + \frac{887}{408240}(1)\right] \\ \sum_{j=0}^{\infty}\frac{\left(\frac{1}{4}\right)^j}{j!} - y_n - \sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^j\left[-\frac{33291}{125440}\left(\frac{1}{2}\right) + \frac{2999}{35280}\left(\frac{1}{6}\right) - \frac{433}{13440}\left(\frac{1}{4}\right) + \frac{1109}{35280}\left(\frac{3}{4}\right) - \frac{2673}{125440}\left(\frac{5}{6}\right) + \frac{163}{40824}(1)\right] \\ \sum_{j=0}^{\infty}\frac{\left(\frac{3}{4}\right)^j}{j!} - y_n - \sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^j\left[-\frac{28431}{125440}\left(\frac{1}{2}\right) + \frac{237}{3920}\left(\frac{1}{6}\right) - \frac{1413}{4480}\left(\frac{1}{4}\right) - \frac{447}{3920}\left(\frac{3}{4}\right) - \frac{2187}{12440}\left(\frac{5}{6}\right) + \frac{3}{8960}(1)\right] \\ \sum_{j=0}^{\infty}\frac{\left(\frac{5}{6}\right)^j}{j!} - y_n - \sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^j\left[-\frac{3175}{14112}\left(\frac{1}{2}\right) - \frac{250}{3969}\left(\frac{1}{6}\right) - \frac{21}{6804}\left(\frac{1}{4}\right) - \frac{5750}{35721}\left(\frac{3}{4}\right) - \frac{305}{14112}\left(\frac{5}{6}\right) + \frac{25}{27216}(1)\right] \\ \sum_{j=0}^{\infty}\frac{(1)^j}{j!} - y_n - \sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^j\left[-\frac{243}{980}\left(\frac{1}{2}\right) - \frac{64}{2205}\left(\frac{1}{6}\right) - \frac{73}{210}\left(\frac{1}{4}\right) - \frac{64}{2205}\left(\frac{3}{4}\right) - \frac{243}{980}\left(\frac{5}{6}\right) - \frac{31}{630}(1)\right] \end{bmatrix} \quad (10)$$

By expanding Eq.10 using Taylor series and collecting the similar terms of $h^{(d)}$ such that $d = 0, 1, 2, \cdots$ to get

$$L[u(t_n):h] = c_0\, y(u_n) + c_1 hy'(u_n) + c_2 h^2 y''(u_n) + \cdots + c_d h^d y^{(d)}(u_n) + \cdots \quad (11)$$

where c_d are vectors [2]. From Eq.11, it follows that

$c_0 = c_1 = c_2 = \cdots = c_d = 0$, but $c_{d+1} \neq 0$

The uniform order seven block hybrid method (9) is defined to be of order *d*, with its error constant given by c_{d+1} By applying Taylor series expansion to the uniform order seven block hybrid method in Eq.9 and organizing like terms, the method is confirmed to be of order seven, with the corresponding error constants obtained as follows: $\left(1.1510(-06), 4.9197(-09), 8.9705(-09), 2.8962(-06), 5.2297(-09), 1.9259(-06)\right)^T$

3.2. Consistency

A uniform order seven block hybrid method (9) is regarded as consistent if it achieves a certain order of convergence [3]. The new method satisfies this requirement, as all calculated orders exceed one.

3.3. Zero Stability

Zero-stability of the uniform order seven block hybrid method in Eq.9 is established if all roots of its first characteristic polynomial lie on or within the unit circle in the complex plane, with any root on the circle being simple [8]. To obtain the normalized block form, Eq.9 is multiplied by the inverse of the associated coefficient matrix. The resulting first characteristic polynomial of the newly derived hybrid scheme is given as:

$$\delta(\varpi) = \det\left[\varpi A^0 - A^1\right] \tag{12}$$

where

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence,

$$\delta(\varpi) = \det\left(\varpi \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\right) = \det\begin{bmatrix} \varpi & 0 & 0 & 0 & 0 & -1 \\ 0 & \varpi & 0 & 0 & 0 & -1 \\ 0 & 0 & \varpi & 0 & 0 & -1 \\ 0 & 0 & 0 & \varpi & 0 & -1 \\ 0 & 0 & 0 & 0 & \varpi & -1 \\ 0 & 0 & 0 & 0 & 0 & \varpi - 1 \end{bmatrix}$$

This implies, $\delta(\varpi) = \varpi^8 - \varpi^7$. Given that $\delta(\varpi) = 0$, so $\varpi^8 - \varpi^7 = 0$. As a result, the roots of the first characteristic polynomial are $\varpi_1 = \varpi_2 = \varpi_3 = \varpi_4 = \varpi_5 = \varpi_6 = \varpi_7 = 0$, $\Omega_8 = 1$ and $\delta(\varpi)$ fulfills the condition $|\varpi| \leq 1$ and $|\varpi| = 1$ is simple, thus uniform order seven block hybrid method (2.10) is zero stable.

3.4. Convergence

The convergence of the uniform order seven block hybrid method outlined in Eq.9 is evaluated through its fundamental properties consistency and zero-stability as guided by Dahlquist's fundamental theorem for linear multistep methods [8]. According to this theorem, a multistep method is convergent if and only if it is both consistent and zero-stable. Since the method in Eq.9 fulfills these conditions, it is therefore deemed convergent.

3.5. Region of Absolute Stability

The boundary locus method is a graphical tool used to analyze the stability properties of numerical methods applied to solve ordinary differential equations (ODEs), especially when dealing with the linear test equation $y' = \lambda y$. The method involves determining the stability function $R(z)$ of the numerical scheme, where $z = \lambda \Delta t$ and examining the region in the complex plane where $|R(z)| = 1$. This region is known as the region of absolute stability and the boundary of this region where $|R(z)| \leq 1$ is called the boundary locus [10]. By using the boundary locus, one can assess whether a given step size and eigenvalue combination will result in stable numerical behavior.

The boundary locus method was used on uniform order seven block hybrid method (9), we obtain the stability polynomial as

$$\bar{h}(w) = \left(-\frac{1}{11612160}w^6 - \frac{163}{1463132160\ 0}w^7\right)h^{14} + \left(-\frac{810757}{8778792960\ 0}w^6 + \frac{113}{34836480}w^7\right)h^{12}$$
$$+\left(-\frac{4473503}{3762339840\ 0}w^6 - \frac{61}{829440}w^7\right)h^{10} + \left(-\frac{1235057}{522547200}w^6 + \frac{481}{414720}w^7\right)h^8 + \left(-\frac{128809}{6967296}w^6 - \frac{67}{5184}w^7\right)h^6$$
$$+\left(-\frac{817}{5040}w^6 + \frac{113}{1152}w^7\right)h^4 + \left(-\frac{17}{30}w^6 - \frac{11}{24}w^7\right)h^2 - 2w^6 + w^7 \qquad (13)$$

Using the stability polynomial (13), the region of absolute stability of (9) is shown in Figure 1 as

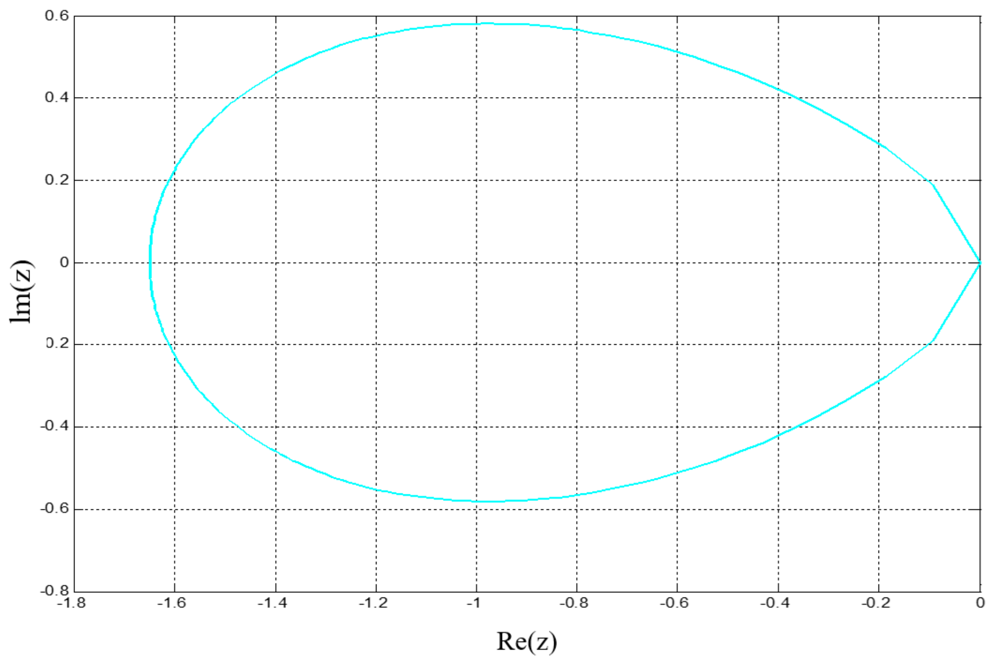

Figure 1. An A-stable region of Absolute Stability

4. Numerical Results and Discussion

The uniform order seven block hybrid method (9) in the form of Eq. 1 was applied to solve three nonlinear first-order initial value problems of ordinary differential equations. The associated error is expressed as follows:

$$E(y_n) = |y(u) - y(u_n)| \qquad (14)$$

In Eq.14, $y(u)$ is the exact solution for the problem considered and $y(u_n)$ is the approximate.

System 4.1:

A nonlinear first-order initial value problem of an ordinary differential equation has been explored in the works of [9, 10].

$$y'(u) = -10(y-1)^2,\ y(0) = 2,\ h = 0.1 \tag{15}$$

whose exact solution is:

$$y(u) = 1 + \frac{1}{1+10u} \tag{16}$$

Table 1. The numerical results obtained for system 4.1 align with the findings reported by [9, 10]

u	Exact Solution	Computed Solution	Errors in new method	Errors [9]	Errors in [10]
0.1	1.9090909090909090909	1.9090909090909104507590	1.3599e-13	1.75330e-10	1.55825e-06
0.2	1.8333333333333333333	1.8333333333333350273780	1.6941e-13	2.32000e-10	2.39975e-06
0.3	1.7692307692307692308	1.76923076923093767140	1.6844-13	2.41150e-10	2.83045e-06
0.4	1.7142857142857142857	1.71428571428587074170	1.5646e-13	2.31400e-10	3.02094e-06
0.5	1.6666666666666666667	1.666666666666680847250	1.4181e-13	2.14840e-10	3.06956e-06
0.6	1.6250000000000000000	1.62500000000012747620	1.2748e-13	1.96600e-10	3.03457e-06
0.7	1.5882352941176470588	1.58823529411776150520	1.1445e-13	1.78870e-10	2.95115e-06
0.8	1.5555555555555555556	1.55555555555556565848900	1.0293e-13	1.62500e-10	2.84088e-06
0.9	1.5263157894736842105	1.52631578947377708310	9.2873e-13	1.47730e-10	2.71713e-06
1.0	1.5000000000000000000	1.50000000000008410710	8.4107e-13	1.34570e-10	2.58816e-06

System 4.2:

Previous study by [11] examined a nonlinear first-order ordinary differential equation, which is presented as follows:

$$y'(u) = 4 - 4y + y^2 = 0,\ y(0) = 1,\ h = 0.1 \tag{17}$$

whose exact solution is:

$$y(u) = \frac{2u-1}{u-1} \tag{18}$$

Table 2. The numerical results obtained for system 4.2 align with the findings reported by [11]

u	Exact Solution	Computed Solution	Errors in new method	Errors in [11]
0.1	0.98989898989898989899	0.98989898989898989899	0.0000e00	1.11400e-04
0.2	0.97959183673469387755	0.97959183673469387759	4.0000e-20	1.85700e-03
0.3	0.96907216494845360825	0.96907216494845360833	8.0000e-20	2.53600e-04
0.4	0.95833333333333333333	0.95833333333333333345	1.2000e-19	7.14800e-05
0.5	0.94736842105263157895	0.94736842105263157911	1.6000e-19	9.72000e-06
0.6	0.93617021276595744681	0.93617021276595744701	2.0000e-19	2.06400e-06
0.7	0.92473118279569892473	0.92473118279569892498	2.5000e-19	2.80400e-07
0.8	0.91304347826086956522	0.91304347826086956550	2.8000e-19	5.29200e-08
0.9	0.90109890109890109890	0.90109890109890109923	3.3000e-19	7.16300e-09
1.0	0.88888888888888888889	0.88888888888888888926	3.7000e-19	1.09300e-09

System 4.3:

The nonlinear first-order ordinary differential equation was explored in the study by [9] and is formulated as follows:

$$y'(u) = -y^2,\ y(0) = 1,\ h = 0.01 \tag{19}$$

whose exact solution is:

$$y(u) = \frac{1}{1+u} \tag{20}$$

Table 3. The numerical results obtained for system 4.3 align with the findings reported by [9]

u	Exact Solution	Computed Solution	Errors in new method	Errors in [9]
0.01	0.99009900990099009899	0.99009900990099009902	3.0000e-20	1.00000e-15
0.02	0.98039215686274509804	0.98039215686274509806	2.0000e-20	1.00000e-15
0.03	0.97087378640776699029	0.97087378640776699032	3.0000e-20	1.00000e-15
0.04	0.96153846153846153846	0.96153846153846153850	4.0000e-20	1.00000e-15
0.05	0.95238095238095238095	0.95238095238095238100	5.0000e-20	1.00000e-15
0.06	0.94339622641509433962	0.94339622641509433968	6.0000e-20	1.00000e-15
0.07	0.93457943925233644860	0.93457943925233644866	6.0000e-20	1.00000e-15
0.08	0.92592592592592592593	0.92592592592592592600	7.0000e-20	1.00000e-15
0.09	0.91743119266055045872	0.91743119266055045879	7.0000e-20	1.00000e-15
0.10	0.90909090909090909091	0.90909090909090909099	8.0000e-20	1.00000e-15

The application of the uniform order seven block hybrid method (9) to solve three nonlinear first-order initial value problems of ordinary differential equations demonstrated its high accuracy. In System 4.1, which aligns with the works of [9, 10] the computed solutions were very close to the exact solutions. The errors associated with the new method were extremely small, indicating a strong performance of the method in approximating the solution with minimal deviation.

In System 4.2, based on the work of [11], the hybrid method produced highly accurate results, with errors remaining very small throughout the calculations. The new method outperformed the existing approach from [11], suggesting that the uniform order seven block hybrid method provides an improved solution to the problem. This reinforces the method’s effectiveness in handling nonlinear first-order initial value problems with high precision.

Similarly, in System 4.3, the hybrid method's results aligned well with the findings of [9] and the computed errors were smaller compared to those reported in [9]. This demonstrates that the new method offers better accuracy and reliability, further validating its suitability for solving such nonlinear equations. The results confirm the method's potential to deliver more precise solutions compared to previous methods.

The numerical results obtained for all three systems confirm the uniform order seven block hybrid method’s robustness and accuracy. The method consistently produced errors of significantly smaller magnitude compared to earlier studies, highlighting its effectiveness in solving nonlinear first-order initial value problems. The results suggest that this method can be a valuable tool for solving a wide range of nonlinear differential equations, offering a higher degree of precision and stability in numerical approximations.

5. Conclusion

This study introduces a novel A-stable uniformly order seven block hybrid method designed for solving nonlinear initial value problems (IVPs) in ordinary differential equations (ODEs). The method builds on previous advancements in hybrid block techniques, offering improvements in stability, accuracy, and computational efficiency. The method's development follows a series of formulations and mathematical analyses, ensuring it meets the necessary criteria for high performance, including consistency, zero stability, and convergence. With the application of the new method, numerical tests were conducted on nonlinear IVPs, yielding promising results that demonstrate its superior performance in terms of error minimization when compared to existing methods. The method also exhibits A-stability, making it a robust tool for solving stiff and nonlinear problems that are common in scientific applications.

The newly developed uniformly order seven block hybrid method provides an effective and reliable solution for nonlinear initial value problems in ODEs, particularly in cases where existing methods struggle with stability and accuracy. The method's A-stability, high order, and convergence properties position it as a significant advancement in computational mathematics. Numerical results show that the method significantly reduces error when compared to previous techniques, demonstrating its potential to outperform traditional methods in real-world applications. The study offers a promising approach to solving nonlinear ODEs, with potential applications across various fields, including physics, biology, and engineering, where such problems are prevalent.

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Declaration of Competing Interest

No conflict of interest was declared by the authors.

Authorship Contribution Statement

Bello Kareem Akanbi: Conceptualization, Methodology, Supervision, Reviewing, and Editing
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Ayinde Muhammed Abdullahi: Mathematical Analysis, Validation, and Visualization
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References

[1] J.F. Baiyeri, O.A. Esan, S.O. Salawu and I.S. Oke, "Numerical solutions of stiff initial value problems using modified extended backward differentiation formula," IOSR-JM, vol. 11, no. 1, pp. 37-40, Feb. 2015.

[2] J. Sunday, A.O. Adesanya and M.R. Odekunle, "A self-starting four-step fifth-order block integrator for stiff and oscillatory differential equations," J. Math. Comput. Sci., vol. 4, no.1, pp. 73-84, Jan. 2014.

[3] D.G. Yakubu and S. Markus, "Second derivative of high-order accuracy methods for the numerical integration of stiff initial value problems," Afrika Matematika, vol. 27, no 5, pp.963-977, Jan. 2016, doi: 10.1007/s13370-015-0389-5.

[4] E.A. Areo and M.O. Omojola, "A new one-twelfth step continuous block method for the solution of modeled problems of ordinary differential equations," Am. J. Comput. Math., vol. 5, no. 4, pp. 447-457, Dec. 2015, doi:10.4236/ajcm.2015.54039.

[5] Y. Skwame, J. Sunday and M.R. Odekunle, "A continuous block method integrator for the solution of stiff and oscillatory differential equations," IOSR-JM, vol. 8, no. 3, pp. 75-80, May. 2013.

[6] H. Soomro, N. Zainuddin, H. Daud and J. Sunday, "Optimized hybrid block Adams method for solving first order ordinary differential equations," Computers, Materials & Continua, vol. 72, no. 2, pp. 2948–2950, Mar. 2022, doi:10.32604/cmc.2022.025933.

[7] M.A. Rufai, M.K. Duromola and A.A. Ganiyu, "Derivation of one-sixth hybrid block method for solving general first order ordinary differential equations," IOSR-JM, vol. 12, no. 5, pp. 20–27, Sept.-Oct. 2016, doi:10.9790/5728-1205022027.

[8] P. Tumba, J. Sabo, A.A. Okeke, D.I. Yakoko, "An accurate implicit quarter step first derivative block hybrid method (AIQSFDBHM) for solving ordinary differential equations," Asian Res. J. Math., vol. 13, no.3, pp. 1-13, Apr. 2019, doi:10.9734/ARJOM/2019/v13i330106.

[9] N. Kamoh, G.M. Kumleng and J. Sunday, " Continuous one-step linear multi-step hybrid block method for the solution of first order linear and nonlinear initial value problems of ordinary differential equations," in Recent Developments in the Solution of Nonlinear Differential Equations, vol. 1, IntechOpen, 2021, ch.5, pp. 1-12.

[10] B.S. Kashkari and M.I. Syam, "Optimization of one-step block method with three hybrid points for solving first order ordinary differential equations," Results Phys., vol. 12, pp. 592–596, Dec. 2018, doi:10.1016/j.rinp.2018.12.015.

[11] I.O. Isah, A.S. Salawu, K.S. Olayemi and L.O. Enest , "An efficient 4-step block method for first order initial value problems via shifted Chebyshev polynomial," TJEST, vol. 1, no. 2, pp. 25-36, Mar. 2021, doi:10.47524/tjst.v1i2.5.