



Some Geometric Properties of Lorentzian β -Kenmotsu Manifolds Admitting η -Ricci-Yamabe Solitons

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Abstract

In this paper, we investigate the characterization of Lorentzian β -Kenmotsu manifolds admitting η -Ricci-Yamabe solitons. First, we examine the cases where such manifolds are Ricci pseudosymmetric and Ricci semisymmetric. Then, by employing certain special curvature tensors, we explore the concepts of Ricci pseudosymmetry and semisymmetry in greater detail and construct the geometry of the Lorentzian β -Kenmotsu manifold accordingly.

Keywords: η -Ricci-Yamabe soliton, Lorentzian β -Kenmotsu manifold, Pseudo-symmetry

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1. Introduction

A Lorentzian manifold is a smooth differential manifold on which the Lorentzian metric is defined. This metric is a symmetric and smooth $(0, 2)$ -tensor like the Riemannian metric, but it is not positive definite. Lorentzian manifolds are particularly used in Einstein's theory of general relativity. They are used to describe the geometry of space-time. For example, Minkowski space is the simplest example of a Lorentzian manifold.

The most basic application of Lorentzian manifolds is Einstein's theory of general relativity, where the metric tensor determines the geometry of space-time and the mass-energy distribution. The curvature tensor shows how the geometry is shaped by the mass-energy distribution. Geodesics determine the paths of freely moving particles. Lorentzian manifolds are used in quantum field theory, especially in theories described in curved spacetime.

The concept of Lorentzian manifold was first put forward by Matsumoto in 1989. The same concept was studied independently by Mihai and Rasca, and several important properties were given.

Lorentzian β -Kenmotsu manifolds are special geometric structures that bridge the gap between Lorentzian geometry and the Kenmotsu structure. These manifolds are equipped with both a temporal structure (Lorentzian metric) and a β -Kenmotsu structure, which is a generalization of a certain conformal structure. The Lorentzian β -Kenmotsu structure is a scaled generalization of the classical Kenmotsu structure. We note that trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are

cosymplectic, β -Kenmotsu [1] and α -Sasakian [2, 3] respectively. A trans-sasakian structure of type (α, β) with β a nonzero constant is always β -Kenmotsu [4]. In this case β becomes a constant. If $\beta = 1$, then β -Kenmotsu manifold is Kenmotsu.

η -Ricci-Yamabe solitons can be thought of as a generalized form of Einstein manifolds. η -Ricci-Yamabe solitons are among the geometric structures that have important applications in fields such as differential geometry and general relativity. These solitons, which represent the combination of Ricci and Yamabe flows, are important tools for understanding the evolution of the metric structure of a manifold. They can be related to space-time models in general relativity, especially when examined together with Lorentzian structures.

η -Ricci-Yamabe solitons on Lorentzian manifolds are used in models dealing with the evolution and symmetries of spacetime. Since they are stationary solutions of Ricci and Yamabe flows, these solitons play an important role in understanding the long-term behavior of the flow. This is important for topological classifications or structural change analyses.

It provides characterization in the classification of manifolds, especially when used with contact, Kenmotsu and para-Kenmotsu structures. Geometric structures are important in some quantum field theories and supersymmetric models. Such soliton solutions can provide the building blocks of the model geometry.

In this paper, we investigate the characterization of Lorentzian β -Kenmotsu manifolds admitting η -Ricci-Yamabe solitons. First, we examine the cases where such manifolds are Ricci pseudosymmetric and Ricci semisymmetric. Then, by employing certain special curvature tensors, we explore the concepts of Ricci pseudosymmetry and semisymmetry in greater detail and construct the geometry of the Lorentzian β -Kenmotsu manifold accordingly.

2. Preliminary

A n -dimensional differentiable manifold Π is called Lorentzian β -Kenmotsu manifold if it admits a $(1, 1)$ -tensor field ϕ , a vector field ξ , a 1-form η and Lorentzian metric g which satisfy the conditions

$$\begin{cases} \phi^2 \Theta_1 = \Theta_1 + \eta(\Theta_1) \xi, \quad g(\Theta_1, \xi) = \eta(\Theta_1), \\ \eta(\xi) = -1, \quad \phi \xi = 0, \quad \eta(\phi \Theta_1) = 0, \\ g(\phi \Theta_1, \phi \Theta_2) = g(\Theta_1, \Theta_2) + \eta(\Theta_1) \eta(\Theta_2), \end{cases} \quad (2.1)$$

for all $\Theta_1, \Theta_2 \in \chi(\Pi)$, where $\chi(\Pi)$ is the Lie algebra of smooth vector fields on Π . Also a Lorentzian β -Kenmotsu manifold Π is satisfying

$$\nabla_{\Theta_1} \xi = \beta [\Theta_1 - \eta(\Theta_1) \xi], \quad (2.2)$$

$$(\nabla_{\Theta_1} \eta)(\Theta_2) = \beta [g(\Theta_1, \Theta_2) - \eta(\Theta_1) \eta(\Theta_2)],$$

$$(\nabla_{\Theta_1} \phi)(\Theta_2) = \beta [g(\phi \Theta_1, \Theta_2) \xi - \eta(\Theta_2) \phi \Theta_1],$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g . Further, on a Lorentzian β -Kenmotsu manifold Π the following relations hold [1, 4]:

$$R(\Theta_1, \Theta_2) \xi = \beta^2 [\eta(\Theta_2) \Theta_1 - \eta(\Theta_2) \Theta_1], \quad (2.3)$$

$$\eta(R(\Theta_1, \Theta_2) \Theta_3) = \beta^2 g(\eta(\Theta_2) \Theta_1 - \eta(\Theta_2) \Theta_1, \Theta_3), \quad (2.4)$$

$$S(\Theta_1, \xi) = -(n-1) \beta^2 \eta(\Theta_1), \quad (2.5)$$

for any vector fields $\Theta_1, \Theta_2, \Theta_3$ on Π , where R, S and Q denotes the curvature tensor, Ricci tensor, and Ricci operator on Π .

Lemma 2.1. *On an n -dimensional Lorentzian β -Kenmotsu manifold, the concircular curvature tensor satisfies the following relations:*

$$C(\Theta_1, \Theta_2) \Theta_3 = R(\Theta_1, \Theta_2) \Theta_3 - \frac{r}{n(n-1)} [g(\Theta_2, \Theta_3) \Theta_1 - g(\Theta_1, \Theta_3) \Theta_2],$$

$$C(\Theta_1, \Theta_2) \xi = D[\eta(\Theta_1) \Theta_2 - \eta(\Theta_2) \Theta_1], \quad (2.6)$$

$$\eta(C(\Theta_1, \Theta_2)\Theta_3) = Dg(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \Theta_3), \quad (2.7)$$

where $D = \beta^2 + \frac{r}{n(n-1)}$.

Lemma 2.2. *On an n -dimensional Lorentzian β -Kenmotsu manifold, the projective curvature tensor satisfies the following relations:*

$$P(\Theta_1, \Theta_2)\Theta_3 = R(\Theta_1, \Theta_2)\Theta_3 - \frac{1}{(n-1)}[S(\Theta_2, \Theta_3)\Theta_1 - S(\Theta_1, \Theta_3)\Theta_2],$$

$$P(\Theta_1, \Theta_2)\xi = 0, \quad (2.8)$$

$$\eta(P(\Theta_1, \Theta_2)\Theta_3) = 0. \quad (2.9)$$

Lemma 2.3. *On an n -dimensional Lorentzian β -Kenmotsu manifold, the W_1 -curvature tensor satisfies the following relations:*

$$W_1(\Theta_1, \Theta_2)\Theta_3 = R(\Theta_1, \Theta_2)\Theta_3 + \frac{1}{(n-1)}[S(\Theta_2, \Theta_3)\Theta_1 - S(\Theta_1, \Theta_3)\Theta_2],$$

$$W_1(\Theta_1, \Theta_2)\xi = 2\beta^2[\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1], \quad (2.10)$$

$$\eta(W_1(\Theta_1, \Theta_2)\Theta_3) = 2\beta^2g(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \Theta_3). \quad (2.11)$$

Especially after Hamilton's in-depth research on the concept of Ricci flow on surfaces [5], in recent years, the theoretical concept of geometric flows, such as Ricci flow and Yamabe flow, has been the focus of numerous studies in geometry. In 2019, Güler and Crasmareanu introduced another geometric flow under the Ricci-Yamabe transformation [6]. This transformation is a scalar combination of the Ricci and Yamabe flows.

A Ricci-Yamabe soliton on (Π, g) is a data (g, V, λ, a, b) fulfilling

$$L_V g + 2aS + (2\lambda - br)g = 0,$$

where L denote the Lie-derivative, S and r are Ricci tensor and scalar curvatures, respectively, and λ, a, b are real constants.

The Ricci-Yamabe soliton is said to be expanding for $\lambda > 0$, steady for $\lambda = 0$ and shrinking when $\lambda < 0$. If λ, b and a are smooth functions on Π , then a Ricci-Yamabe soliton is called an almost Ricci-Yamabe soliton. If $b = 0, a = 1$, then Ricci-Yamabe soliton induces Ricci soliton [7]. Similarly, it turns into a Yamabe soliton if $b = 1, a = 0$ [8]. Also, if $b = -1, a = 1$, it reduces to an Einstein soliton [9]. The Ricci-Yamabe soliton is said to be proper if $a \neq 0, 1$.

On the other hand, Siddiki and Akyol described Ricci-Yamabe solitons in 2020 [7]. A η -Ricci-Yamabe soliton on (Π, g) is a data $(g, V, \lambda, \mu, a, b)$ fulfilling

$$L_V g + 2aS + (2\lambda - br)g + 2\mu\eta \otimes \eta = 0, \quad (2.12)$$

where L being the Lie-derivative, S indicates the Ricci tensor, r denotes the scalar curvature and $\lambda, \mu, a, b \in \mathbb{R}$. If λ, μ, b and a are smooth functions on Π , then a η -Ricci-Yamabe soliton is called an almost η -Ricci-Yamabe soliton. In particular, if $\mu = 0$, then the notion of η -Ricci-Yamabe soliton $(g, V, \lambda, \mu, a, b)$ reduces to the notion of Ricci-Yamabe soliton (g, V, λ, a, b) .

In later times, G. Ayar, M. Yıldırım, M.D Siddiqi, T. Mert, M. Atçeken and many other authors made significant contributions to the fields of mathematics, physics, and engineering with their studies in this field [10]- [21].

3. Almost η -Ricci-Yamabe Solitons on Lorentzian β -Kenmotsu Manifold

We consider a Lorentzian β -Kenmotsu manifold admitting an η -Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, a, b)$. Then from (2.12), it obvious that

$$(L_{\xi}g)(\Theta_1, \Theta_2) + 2aS(\Theta_1, \Theta_2) + (2\lambda - br)g(\Theta_1, \Theta_2) + 2\mu\eta(\Theta_1)\eta(\Theta_2) = 0. \quad (3.1)$$

Next, we will express the Lie derivative along ξ on Π as follows:

$$\begin{aligned} (L_{\xi}g)(\Theta_1, \Theta_2) &= L_{\xi}g(\Theta_1, \Theta_2) - g(L_{\xi}\Theta_1, \Theta_2) - g(\Theta_1, L_{\xi}\Theta_2) \\ &= L_{\xi}g(\Theta_1, \Theta_2) - g([\xi, \Theta_1], \Theta_2) - g(\Theta_1, [\xi, \Theta_2]) \\ &= g(\nabla_{\Theta_1}\xi, \Theta_2) + g(\Theta_1, \nabla_{\Theta_2}\xi). \end{aligned}$$

By means of (2.1) and (2.2), the last equation reduces to

$$(L_{\xi}g)(\Theta_1, \Theta_2) = 2\beta[g(\Theta_1, \Theta_2) - \eta(\Theta_1)\eta(\Theta_2)]. \quad (3.2)$$

By virtue of (3.2), the equation (3.1) takes the following form

$$2aS(\Theta_1, \Theta_2) + (2\lambda - br + 2\beta)g(\Theta_1, \Theta_2) - 2(\beta - \mu)\eta(\Theta_1)\eta(\Theta_2) = 0. \quad (3.3)$$

Thus, we can state the following theorem.

Theorem 3.1. *An n -dimensional Lorentzian β -Kenmotsu manifold Π admitting a η -Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, a, b)$ is an η -Einstein manifold provided $a \neq 0, \beta \neq \mu$ and $2\lambda \neq br + 2\beta$.*

If we choose $\Theta_2 = \xi$ in (3.3) we have

$$2aS(\Theta_1, \xi) = [br - 2(\lambda + 2\beta - \mu)]\eta(\Theta_1). \quad (3.4)$$

If we use (2.5) in (3.4) we have

$$\lambda - \mu = \frac{1}{2}br + \beta[a(n-1)\beta - 2].$$

Theorem 3.2. *Let Π be an n -dimensional Lorentzian β -Kenmotsu manifold and $(g, \xi, \lambda, \mu, a, b)$ be an η -Ricci-Yamabe soliton on Π . If Π is a Ricci pseudosymmetric, then at least one of the following is true:*

- i. $\mathcal{L}_R = -\beta^2$,
- ii. $\lambda = \frac{1}{2}br + \beta[a(n-1)\beta - 2]$ and $\mu = 0$,
- iii. Π is an expanding if $2\beta[a(n-1)\beta - 2] < br$,
- iv. Π is a steady if $2\beta[a(n-1)\beta - 2] = br$,
- v. Π is a shrinking if $2\beta[a(n-1)\beta - 2] > br$.
- vi. The η -Ricci-Yamabe soliton reduces to Ricci-Yamabe soliton.

Proof. Let's assume that Π is a Ricci pseudosymmetric and $(g, \xi, \lambda, \mu, a, b)$ be almost η -Ricci-Yamabe soliton on Π . That's mean

$$(R(\Theta_1, \Theta_2) \cdot S)(\Theta_4, \Theta_5) = \mathcal{L}_R Q(g, S)(\Theta_4, \Theta_5; \Theta_1, \Theta_2),$$

for all $\Theta_1, \Theta_2, \Theta_4, \Theta_5 \in \Gamma(T\Pi)$. From the last equation, we can easily write

$$\begin{aligned} &S(R(\Theta_1, \Theta_2)\Theta_4, \Theta_5) + S(\Theta_4, R(\Theta_1, \Theta_2)\Theta_5) \\ &= \mathcal{L}_R \{S((\Theta_1 \wedge_g \Theta_2)\Theta_4, \Theta_5) + S(\Theta_4, (\Theta_1 \wedge_g \Theta_2)\Theta_5)\}. \end{aligned} \quad (3.5)$$

If we choose $\Theta_5 = \xi$ in (3.5) and make use of (2.3), (2.4), (2.5), we have

$$\begin{aligned} &-(n-1)\beta^4 g(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \Theta_4) \\ &+ \beta^2 S(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) \\ &= \mathcal{L}_R \{-(n-1)\beta^2 g(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) \\ &+ S(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \Theta_4)\}. \end{aligned} \quad (3.6)$$

If we use (3.3) in (3.6), we get

$$[2a(n-1)\beta^2 + (br - 2\lambda - 2\beta)] [\beta^2 + \mathcal{L}_R] g(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) = 0.$$

This completes the proof. $\square$

We can give some important results of this theorem as follows.

Corollary 3.3. *Let Π be an n -dimensional Lorentzian β -Kenmotsu manifold and (g, ξ, λ) be an Ricci soliton on Π . If Π is a Ricci pseudosymmetric, then the following holds:*

- i. $\mathcal{L}_R = -\beta^2$,
- ii. $\lambda = \beta[(n-1)\beta - 2]$,
- iii. Π is an expanding if $\beta^2(n-1) > 2\beta$,
- iv. Π is a steady if $\beta^2(n-1) = 2\beta$,
- v. Π is a shrinking if $\beta^2(n-1) < 2\beta$.

Corollary 3.4. *Let Π be an n -dimensional Lorentzian β -Kenmotsu manifold and (g, ξ, λ) be a Einstein soliton on Π . If Π is a Ricci pseudosymmetric, then the following holds:*

- i. $\mathcal{L}_R = -\beta^2$,
- ii. $\lambda = \beta[(n-1)\beta - 2] - \frac{1}{2}r$,
- iii. Π is expanding if $2\beta[(n-1)\beta - 2] > r$,
- iv. Π is a steady if $2\beta[(n-1)\beta - 2] = r$,
- v. Π is a shrinking if $2\beta[(n-1)\beta - 2] < r$.

Theorem 3.5. *Let Π be an n -dimensional Lorentzian β -Kenmotsu manifold and $(g, \xi, \lambda, \mu, a, b)$ be an η -Ricci-Yamabe soliton on Π . If Π is a projectively Ricci pseudosymmetric, then Π is either projective Ricci semisymmetric or η -Ricci-Yamabe soliton $(g, \xi, \lambda, \mu, a, b)$ reduces to Ricci-Yamabe soliton (g, ξ, λ, a, b) .*

Proof. Let's assume that n -dimensional Lorentzian β -Kenmotsu manifold Π be a projectively Ricci pseudosymmetric and $(g, \xi, \lambda, \mu, a, b)$ be almost η -Ricci-Yamabe soliton on Π . That's mean

$$(P(\Theta_1, \Theta_2) \cdot S)(\Theta_4, \Theta_5) = \mathcal{L}_P Q(g, S)(\Theta_4, \Theta_5; \Theta_1, \Theta_2),$$

for all $\Theta_1, \Theta_2, \Theta_4, \Theta_5 \in \Gamma(T\Pi)$. From the last equation, we can easily write

$$\begin{aligned} & S^G(P^G(\Theta_1, \Theta_2)\Theta_4, \Theta_5) + S^G(\Theta_4, P^G(\Theta_1, \Theta_2)\Theta_5) \\ &= \mathcal{L}_P \{ S^G((\Theta_1 \wedge_g \Theta_2)\Theta_4, \Theta_5) + S^G(\Theta_4, (\Theta_1 \wedge_g \Theta_2)\Theta_5) \}. \end{aligned} \quad (3.7)$$

If we choose $\Theta_5 = \xi$ in (3.7) and use (2.5), (2.8), (2.9), then we get

$$\begin{aligned} & \mathcal{L}_P \{ -(n-1)\beta^2 g(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) \\ &+ S(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \Theta_4) \} = 0. \end{aligned} \quad (3.8)$$

If we use (3.3) in (3.8), we have

$$[2\alpha(n-1)\beta^2 + (br - 2\lambda - 2\beta)] \mathcal{L}_P g(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \Theta_4) = 0.$$

This completes the proof. $\square$

We can give the following results as follows.

Corollary 3.6. *Let Π be an n -dimensional Lorentzian β -Kenmotsu manifold and (g, ξ, λ) be an Ricci soliton on Π . If Π is a projectively Ricci pseudosymmetric, then the following holds:*

- i. Π is a projectively Ricci semisymmetric,
- ii. Π is expanding if $\beta > 0$,
- iii. Π is a shrinking if $\beta < 0$.

Corollary 3.7. Let Π be an n -dimensional Lorentzian β -Kenmotsu manifold and (g, ξ, λ) be a Einstein soliton on Π . If Π is a projectively Ricci pseudosymmetric, then the following holds:

- i. Π is a projectively Ricci semisymmetric,
- ii. $\lambda = \beta[(n-1)\beta - 2] - \frac{1}{2}r$,
- iii. Π is an expanding if $r < 2\beta(n-2)$,
- iv. Π is a steady if $r = 2\beta(n-2)$,
- v. Π is a shrinking if $r > 2\beta(n-2)$.

Theorem 3.8. Let Π be an n -dimensional Lorentzian β -Kenmotsu manifold and $(g, \xi, \lambda, \mu, a, b)$ be an η -Ricci-Yamabe soliton on Π . If Π is a concircular Ricci pseudosymmetric, then $\mathcal{L}_C = -D$ or $\lambda = \frac{1}{2}br + \beta[a(n-1)\beta - 1]$.

Proof. Let's assume that Π is a concircular Ricci pseudosymmetric and $(g, \xi, \lambda, \mu, a, b)$ be almost η -Ricci-Yamabe soliton on Π . That's mean

$$(C(\Theta_1, \Theta_2) \cdot S)(\Theta_4, \Theta_5) = \mathcal{L}_C Q(g, S)(\Theta_4, \Theta_5; \Theta_1, \Theta_2),$$

for all $\Theta_1, \Theta_2, \Theta_4, \Theta_5 \in \Gamma(T\Pi)$. From the last equation, we can easily write

$$\begin{aligned} & S(C(\Theta_1, \Theta_2)\Theta_4, \Theta_5) + S(\Theta_4, C(\Theta_1, \Theta_2)\Theta_5) \\ &= \mathcal{L}_C \{S((\Theta_1 \wedge_g \Theta_2)\Theta_4, \Theta_5) + S(\Theta_4, (\Theta_1 \wedge_g \Theta_2)\Theta_5)\}. \end{aligned} \quad (3.9)$$

If we choose $\Theta_5 = \xi$ in (3.9) and make use of (2.5), (2.6), (2.7), we have

$$\begin{aligned} & (n-1)\beta^2 Dg(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) \\ & + DS(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) \\ &= \mathcal{L}_C \{- (n-1)\beta^2 g(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) \\ & + S(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \Theta_4)\}. \end{aligned} \quad (3.10)$$

If we use (3.3) in (3.10), we get

$$[2a(n-1)\beta^2 + (br - 2\lambda - 2\beta)][D + \mathcal{L}_C]g(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) = 0.$$

This completes the proof. $\square$

Corollary 3.9. Let Π be an n -dimensional Lorentzian β -Kenmotsu manifold and (g, ξ, λ) be an Ricci soliton on Π . If Π is a concircular Ricci pseudosymmetric, then the following holds:

- i. $\mathcal{L}_C = -D$,
- ii. $\lambda = \beta[(n-1)\beta - 1]$,
- iii. Π is an expanding if $\beta^2(n-1) > \beta$,
- iv. Π is a steady if $\beta^2(n-1) = \beta$,
- v. Π is a shrinking if $\beta^2(n-1) < \beta$.

Corollary 3.10. Let Π be an n -dimensional Lorentzian β -Kenmotsu manifold and (g, ξ, λ) be a Einstein soliton on Π . If Π is a concircular Ricci pseudosymmetric, then the following holds:

- i. $\mathcal{L}_R = -D$,
- ii. $\lambda = \beta[(n-1)\beta - 1] - \frac{1}{2}r$,
- iii. Π is expanding if $2\beta[(n-1)\beta - 2] > r$,
- iv. Π is a steady if $2\beta[(n-1)\beta - 2] = r$,
- v. Π is a shrinking if $2\beta[(n-1)\beta - 2] < r$.

Theorem 3.11. Let Π be an n -dimensional Lorentzian β -Kenmotsu manifold and $(g, \xi, \lambda, \mu, a, b)$ be an η -Ricci-Yamabe soliton on Π . If Π is a W_1 -Ricci pseudosymmetric, then the following holds:

- i. $\lambda = \frac{1}{2}br + \beta[a(n-1)\beta - 1]$ and $\mu = \beta$,
- ii. $\mathcal{L}_{W_1} = -2\beta^2$,
- iii. Π is an expanding if $2\beta[a(n-1)\beta - 1] < br$,
- iv. Π is a steady if $2\beta[a(n-1)\beta - 1] = br$,
- v. Π is a shrinking if $2\beta[a(n-1)\beta - 1] > br$.

Proof. Let's assume that n -dimensional Sasakian manifold Π be a W_1 -Ricci pseudosymmetric and $(g, \xi, \lambda, \mu, a, b)$ be almost η -Ricci-Yamabe soliton on Π admitting general connection. That's mean

$$(W_1(\Theta_1, \Theta_2) \cdot S)(\Theta_4, \Theta_5) = \mathcal{L}_{W_1} Q(g, S)(\Theta_4, \Theta_5; \Theta_1, \Theta_2),$$

for all $\Theta_1, \Theta_2, \Theta_4, \Theta_5 \in \Gamma(T\Pi)$. From the last equation, we can easily write

$$\begin{aligned} & S(W_1(\Theta_1, \Theta_2)\Theta_4, \Theta_5) + S(\Theta_4, W_1(\Theta_1, \Theta_2)\Theta_5) \\ &= \mathcal{L}_{W_1} \{S((\Theta_1 \wedge_g \Theta_2)\Theta_4, \Theta_5) + S(\Theta_4, (\Theta_1 \wedge_g \Theta_2)\Theta_5)\}. \end{aligned} \quad (3.11)$$

If we choose $\Theta_5 = \xi$ in (3.11) and use (2.5), (2.10), (2.11), we get

$$\begin{aligned} & -2(n-1)\beta^4 g(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \Theta_4) \\ & + 2\beta^2 S(\Theta_4, \eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1) \\ &= \mathcal{L}_{W_1} \{-(n-1)\beta^2 g(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) \\ & + S(\eta(\Theta_2)\Theta_1 - \eta(\Theta_1)\Theta_2, \Theta_4)\}. \end{aligned} \quad (3.12)$$

If we use (3.3) in (3.12), we get

$$[2a(n-1)\beta^2 + (br - 2\lambda - 2\beta)] [2\beta^2 + \mathcal{L}_{W_1}] g(\eta(\Theta_1)\Theta_2 - \eta(\Theta_2)\Theta_1, \Theta_4) = 0.$$

This proves our assertions. □

We can give some important results of this theorem as follows.

Corollary 3.12. *Let Π be an n -dimensional Lorentzian β -Kenmotsu manifold and (g, ξ, λ) be an Ricci soliton on Π . If Π is a W_1 -Ricci pseudosymmetric, then the following holds:*

- i. $\mathcal{L}_{W_1} = -2\beta^2$,
- ii. $\lambda = \beta[(n-1)\beta - 1]$,
- iii. Π is expanding if $\beta^2(n-1) > \beta$,
- iv. Π is a steady if $\beta^2(n-1) = \beta$,
- v. Π is a shrinking if $\beta^2(n-1) < \beta$.

Corollary 3.13. *Let Π be an n -dimensional Lorentzian β -Kenmotsu manifold and (g, ξ, λ) be a Einstein soliton on Π . If Π is a W_1 -Ricci pseudosymmetric, then the following holds:*

- i. $\mathcal{L}_R = -2\beta^2$,
- ii. $\lambda = \beta[(n-1)\beta - 1] - \frac{1}{2}r$,
- iii. Π is an expanding if $2\beta[(n-1)\beta - 2] > r$,
- iv. Π is a steady if $2\beta[(n-1)\beta - 2] = r$,
- v. Π is a shrinking if $2\beta[(n-1)\beta - 2] < r$.

4. Conclusion

In this paper, we investigate the characterization of Lorentzian β -Kenmotsu manifolds admitting η -Ricci-Yamabe solitons. First, we examine the cases where such manifolds are Ricci pseudosymmetric and Ricci semisymmetric. Then, by employing certain special curvature tensors, we explore the concepts of Ricci pseudosymmetry and semisymmetry in greater detail and construct the geometry of the Lorentzian β -Kenmotsu manifold accordingly.

The obtained results clearly reveal the relationship between η -Ricci-Yamabe solitons and the curvature structure of Lorentzian β -Kenmotsu manifolds. Moreover, it has been shown that the interaction between the Ricci curvature and the Yamabe flow significantly influence the geometry of the manifold in certain special cases. In this context, our study not only offers new perspectives on the impact of η -Ricci-Yamabe solitons on differential geometric structures, but also provides significant characterizations specifically for Lorentzian β -Kenmotsu manifolds.

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