



BEST BOUND FOR λ - PSEUDO STARLIKE FUNCTIONS

D. VAMSHEE KRISHNA AND D. SHALINI

ABSTRACT. In this paper, we obtain sharp upper bound to the second Hankel determinant for the functions belong to the class of λ - pseudo starlike functions, an interesting sub class of univalent functions defined in the open unit disc $E = \{z : |z| < 1\}$, using Toeplitz determinants.

1. INTRODUCTION

Let A denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture, i.e., for a univalent function its n^{th} Taylor coefficient is bounded by n (see [3]). The bounds for the coefficients of these functions give information about their geometric properties. In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [9] as follows, and has been extensively studied.

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} (a_1 = 1).$$

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In particular for $q = 2, n = 1$ and $q = 2, n = 2$, the Hankel determinant simplifies respectively to

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.$$

We refer to $H_2(2)$ as the second Hankel determinant. It is well known that for a univalent function of the form (1.1), the sharp inequality $H_2(1) = |a_3 - a_2^2| \leq 1$ holds true [4]. For a family \mathcal{T} of functions in S , the more general problem of finding sharp estimate for the functional $|a_3 - \mu a_2^2|$ ($\mu \in \mathbb{R}$ or $\mu \in \mathbb{C}$) is popularly known as the Fekete-Szegő problem for \mathcal{T} . Let \mathfrak{R} and S^* be the usual subclasses of S consisting of functions which are respectively, of bounded turning and starlike in E . That is, functions satisfying the conditions $\operatorname{Re} f'(z) > 0$ and $\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0$ respectively in E . In 1921 Nevanlinna obtained the criterion of starlikeness. The bounded turning functions were introduced by Alexander [1] in 1915 and a systematic study of their properties was conducted by MacGregor in 1962. Janteng et al. [6] obtained the sharp upper bound to the second Hankel determinant for these two classes and have shown that $|a_2a_4 - a_3^2| \leq \frac{4}{9}$ and $|a_2a_4 - a_3^2| \leq 1$ respectively.

In this paper, we consider λ - pseudo-starlike functions (see [2]) and obtain sharp upper bound to the functional $|a_2a_4 - a_3^2|$, for the functions belong to this class, defined as follows.

Definition 1.1. Let the function $f \in A$. Suppose $\lambda \geq 1$ is real. Then $f(z)$ belongs to the class \mathcal{L}_λ of λ - pseudo- starlike functions in the unit disc E if and only if

$$\operatorname{Re} \left\{ \frac{z (f'(z))^\lambda}{f(z)} \right\} > 0, \quad \forall z \in E. \quad (1.2)$$

- (1) Throughout this work, the powers are meant for principal values.
- (2) If $\lambda = 1$, we get S^* , which in this context are called as 1-pseudo-starlike functions.
- (3) For $\lambda = 2$, the functions in \mathcal{L}_λ are defined by

$$\operatorname{Re} \left\{ f'(z) \frac{zf'(z)}{f(z)} \right\} > 0, \quad \forall z \in E, \quad (1.3)$$

which is a product combination of geometric expressions for bounded turning and starlike functions. We shall need the following preliminary Lemmas required for proving our result, which has been used widely, are as follows:

2. PRELIMINARY RESULTS

Let \mathcal{P} denote the class of functions consisting of g , such that

$$g(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad (2.1)$$

which are analytic (regular) in the open unit disc E and satisfy $\text{Reg}(z) > 0$, for any $z \in E$. Here $g(z)$ is called a Carathéodory function [4].

Lemma 2.1. ([9, 11]) *If $g \in \mathcal{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2.2. ([5]) *The power series for g given in (2.1) converges in the open unit disc E to a function in \mathcal{P} if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, n = 1, 2, 3, \dots$$

and $c_{-k} = \bar{c}_k$, are all non-negative. They are strictly positive except for $g(z) = \sum_{k=1}^m \rho_k g_0(e^{it_k} z)$, where $\sum_{k=1}^m \rho_k = 1$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $g_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n = 0$ for $n \geq m$.

This necessary and sufficient condition found in [5] is due to Carathéodory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$ and $n = 3$ respectively, for some complex values y and ζ with $|y| \leq 1$ and $|\zeta| \leq 1$ respectively, we have

$$2c_2 = c_1^2 + y(4 - c_1^2) \quad (2.2)$$

and

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)y - c_1(4 - c_1^2)y^2 + 2(4 - c_1^2)(1 - |y|^2)\zeta. \quad (2.3)$$

In obtaining our result, we refer to the classical method devised by Libera and Zlotkiewicz [7], widely used by many authors.

3. MAIN RESULT

Theorem 3.1. *If $f(z) \in \mathcal{L}_\lambda$, ($\lambda \geq 1$ is real) then*

$$|a_2 a_4 - a_3^2| \leq \frac{4}{(3\lambda - 1)^2}$$

and the inequality is sharp.

Proof. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{L}_\lambda$, by virtue of Definition 1.1, there exists an analytic function $g \in \mathcal{P}$ in the open unit disc E with $g(0) = 1$ and $\text{Re}g(z) > 0$ such that

$$\frac{z(f'(z))^\lambda}{f(z)} = g(z) \Leftrightarrow z(f'(z))^\lambda = f(z)g(z). \tag{3.1}$$

Using the series representations for f and g in (3.1), we have

$$z \left\{ 1 + \sum_{n=2}^{\infty} na_n z^{n-1} \right\}^\lambda = z \left\{ 1 + \sum_{n=2}^{\infty} a_n z^{n-1} \right\} \left\{ 1 + \sum_{n=1}^{\infty} c_n z^n \right\}.$$

Applying the binomial expansion on the left hand side of the above expression subject to the condition $|\sum_{n=2}^{\infty} na_n z^{n-1}| < 1$, upon simplification, we obtain

$$\begin{aligned} & 1 + \{2a_2\lambda\}z + \{3\lambda a_3 + 2\lambda(\lambda - 1)a_2^2\}z^2 + \\ & \left\{ 4\lambda a_4 + 6\lambda(\lambda - 1)a_2 a_3 + \frac{4\lambda(\lambda - 1)(\lambda - 2)}{3}a_2^3 \right\}z^3 + \dots \\ & = 1 + (c_1 + a_2)z + (c_2 + c_1 a_2 + a_3)z^2 + (c_3 + c_2 a_2 + c_1 a_3 + a_4)z^3 + \dots \end{aligned} \tag{3.2}$$

Equating the coefficients of z , z^2 and z^3 respectively in (3.2), after simplifying, we get

$$\begin{aligned} a_2 &= \frac{c_1}{2\lambda - 1}; \quad a_3 = \frac{1}{(2\lambda - 1)^2(3\lambda - 1)} [(2\lambda - 1)^2 c_2 - (2\lambda^2 - 4\lambda + 1)c_1^2]; \\ a_4 &= \frac{1}{3(2\lambda - 1)^3(3\lambda - 1)(4\lambda - 1)} [3(2\lambda - 1)^3(3\lambda - 1)c_3 \\ & \quad - 3(2\lambda - 1)^2(6\lambda^2 - 11\lambda + 2)c_1 c_2 \\ & \quad + (24\lambda^4 - 80\lambda^3 + 84\lambda^2 - 28\lambda + 3)c_1^3]. \end{aligned} \tag{3.3}$$

Considering, the second Hankel functional $|a_2 a_4 - a_3^2|$ for the function \mathcal{L}_λ , substituting the values of a_2 , a_3 and a_4 from (3.3), it simplifies to give

$$\begin{aligned} |a_2 a_4 - a_3^2| &= \frac{1}{3(2\lambda - 1)^4(3\lambda - 1)^2(4\lambda - 1)} \times \\ & \left| 3(2\lambda - 1)^3(3\lambda - 1)^2 c_1 c_3 - 3\lambda(\lambda - 1)(2\lambda - 1)^3 c_1^2 c_2 \right. \\ & \left. - 3(2\lambda - 1)^4(4\lambda - 1)c_2^2 + (24\lambda^5 - 60\lambda^4 + 44\lambda^3 - 12\lambda^2 + \lambda)c_1^4 \right|. \end{aligned} \tag{3.4}$$

The above expression is equivalent to

$$|a_2 a_4 - a_3^2| = \frac{|d_1 c_1 c_3 + d_2 c_1^2 c_2 + d_3 c_2^2 + d_4 c_1^4|}{3(2\lambda - 1)^4(3\lambda - 1)^2(4\lambda - 1)}, \tag{3.5}$$

where

$$\begin{aligned} d_1 &= 3(2\lambda - 1)^3(3\lambda - 1)^2 \\ d_2 &= -3\lambda(\lambda - 1)(2\lambda - 1)^3 \\ d_3 &= -3(2\lambda - 1)^4(4\lambda - 1) \\ d_4 &= (24\lambda^5 - 60\lambda^4 + 44\lambda^3 - 12\lambda^2 + \lambda). \end{aligned} \quad (3.6)$$

Substituting the values of c_2 and c_3 given in (2.2) and (2.3) respectively from Lemma 2.2 on the right-hand side of (3.5), we have

$$\begin{aligned} &|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \\ &= \left| d_1c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4 - c_1^2)y - c_1(4 - c_1^2)y^2 + 2(4 - c_1^2)(1 - |y|^2)\zeta\} + \right. \\ &\quad \left. d_2c_1^2 \times \frac{1}{2}\{c_1^2 + y(4 - c_1^2)\} + d_3 \times \frac{1}{4}\{c_1^2 + y(4 - c_1^2)\}^2 + d_4c_1^4 \right|. \end{aligned}$$

Using the triangle inequality and the fact that $|\zeta| < 1$, upon simplification, we obtain

$$\begin{aligned} 4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| &\leq |(d_1 + 2d_2 + d_3 + 4d_4)c_1^4 + 2d_1c_1(4 - c_1^2) + \\ &2(d_1 + d_2 + d_3)c_1^2(4 - c_1^2)|y| + \{(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3\}(4 - c_1^2)|y|^2. \end{aligned} \quad (3.7)$$

From (3.6), we can now write

$$\begin{aligned} d_1 + 2d_2 + d_3 + 4d_4 &= 72\lambda^5 - 156\lambda^4 + 86\lambda^3 - 9\lambda^2 - 2\lambda; \\ d_1 + d_2 + d_3 &= 3(2\lambda - 1)^3(3\lambda - 1)^2; \quad d_1 + d_2 + d_3 = 3\lambda(2\lambda - 1)^3 \end{aligned} \quad (3.8)$$

$$\begin{aligned} \text{and } \{(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3\} \\ = 3(2\lambda - 1)^3 [\lambda^2c_1^2 + 2(3\lambda - 1)^2c_1 + 4(2\lambda - 1)(4\lambda - 1)]. \end{aligned} \quad (3.9)$$

$$\begin{aligned} \text{Consider } &[\lambda^2c_1^2 + 2(3\lambda - 1)^2c_1 + 4(2\lambda - 1)(4\lambda - 1)] \\ &= \lambda^2 \left[c_1 + \left\{ \left(\frac{3\lambda - 1}{\lambda} \right)^2 - \sqrt{\frac{49\lambda^4 - 84\lambda^3 + 50\lambda^2 - 4\lambda + 1}{\lambda^4}} \right\} \right] \times \\ &\quad \left[c_1 + \left\{ \left(\frac{3\lambda - 1}{\lambda} \right)^2 + \sqrt{\frac{49\lambda^4 - 84\lambda^3 + 50\lambda^2 - 4\lambda + 1}{\lambda^4}} \right\} \right]. \end{aligned} \quad (3.10)$$

Since $c_1 \in [0, 2]$, noting that $(c_1 + a)(c_1 + b) \geq (c_1 - a)(c_1 - b)$, where $a, b \geq 0$ on the right-hand side of (3.10), after simplifying, we get

$$\begin{aligned} &[\lambda^2c_1^2 + 2(3\lambda - 1)^2c_1 + 4(2\lambda - 1)(4\lambda - 1)] \\ &\geq [\lambda^2c_1^2 - 2(3\lambda - 1)^2c_1 + 4(2\lambda - 1)(4\lambda - 1)]. \end{aligned} \quad (3.11)$$

From the relations (3.9) and (3.11), we can write

$$\begin{aligned} & \{(d_1 + d_2)c_1^2 + 2d_1c_1 - 4d_2\} \\ & \leq 3(2\lambda - 1)^3 [\lambda^2c_1^2 - 2(3\lambda - 1)^2c_1 + 4(2\lambda - 1)(4\lambda - 1)]. \end{aligned} \quad (3.12)$$

Substituting the calculated values from (3.8) and (3.12) on the right-hand side of (3.7), we have

$$\begin{aligned} 4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| & \leq |(72\lambda^5 - 156\lambda^4 + 86\lambda^3 - 9\lambda^2 - 2\lambda)c_1^4 + \\ & 6(2\lambda - 1)^3(3\lambda - 1)^2c_1(4 - c_1^2) + 6\lambda(2\lambda - 1)^3c_1^2(4 - c_1^2)|y| \\ & + 3(2\lambda - 1)^3 [\lambda^2c_1^2 - 2(3\lambda - 1)^2c_1 + 4(2\lambda - 1)(4\lambda - 1)] (4 - c_1^2)|y|^2. \end{aligned}$$

Choosing $c_1 = c \in [0, 2]$, applying the triangle inequality and replacing $|y|$ by μ on the right-hand side of the above inequality, will give

$$\begin{aligned} 4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| & \leq [(-72\lambda^5 + 156\lambda^4 - 86\lambda^3 + 9\lambda^2 + 2\lambda)c^4 + \\ & 6(2\lambda - 1)^3(3\lambda - 1)^2c(4 - c^2) + 6\lambda(2\lambda - 1)^3c^2(4 - c^2)\mu \\ & + 3(2\lambda - 1)^3 \{\lambda^2c^2 - 2(3\lambda - 1)^2c + 4(2\lambda - 1)(4\lambda - 1)\} (4 - c^2)\mu^2] \\ & = F(c, \mu), \text{ for } 0 \leq \mu = |y| \leq 1, \end{aligned} \quad (3.13)$$

where

$$\begin{aligned} F(c, \mu) & = [(-72\lambda^5 + 156\lambda^4 - 86\lambda^3 + 9\lambda^2 + 2\lambda)c^4 + \\ & 6(2\lambda - 1)^3(3\lambda - 1)^2c(4 - c^2) + 6\lambda(2\lambda - 1)^3c^2(4 - c^2)\mu \\ & + 3(2\lambda - 1)^3 \{\lambda^2c^2 - 2(3\lambda - 1)^2c + 4(2\lambda - 1)(4\lambda - 1)\} (4 - c^2)\mu^2]. \end{aligned} \quad (3.14)$$

We next maximize the function $F(c, \mu)$ on the closed region $[0, 2] \times [0, 1]$. Let us suppose that $F(c, \mu)$ have maximum value at any point in the interior of the closed region $[0, 2] \times [0, 1]$. Differentiating $F(c, \mu)$ in (3.14) partially with respect to μ , we get

$$\begin{aligned} \frac{\partial F}{\partial \mu} & = 6(2\lambda - 1)^3[\lambda c^2 + \{\lambda^2c^2 - 2(3\lambda - 1)^2c + 4(2\lambda - 1)(4\lambda - 1)\} \mu] \\ & \quad \times (4 - c^2). \end{aligned} \quad (3.15)$$

For $0 < \mu < 1$, for any fixed c with $0 < c < 2$ and $\lambda \geq 1$, from (3.15), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c, \mu)$ is an increasing function of μ and hence it cannot have maximum value at any point in the interior of the closed region $[0, 2] \times [0, 1]$. The maximum value of $F(c, \mu)$ occurs on the boundary i.e., when $\mu = 1$. Therefore, for fixed $c \in [0, 2]$, we have

$$\max_{0 \leq \mu \leq 1} F(c, \mu) = F(c, 1) = G(c). \quad (3.16)$$

In view of (3.16), replacing μ by 1 in (3.14), which simplifies to

$$G(c) = F(c, 1) = -2\lambda(48\lambda^4 - 72\lambda^3 + 25\lambda^2 + 12\lambda - 4)c^4 - 12(\lambda - 1)(2\lambda - 1)^3(7\lambda - 1)c^2 + 48(2\lambda - 1)^4(4\lambda - 1), \quad (3.17)$$

$$G'(c) = -8\lambda(48\lambda^4 - 72\lambda^3 + 25\lambda^2 + 12\lambda - 4)c^3 - 24(\lambda - 1)(2\lambda - 1)^3(7\lambda - 1)c. \quad (3.18)$$

From the expression (3.18), we observe that $G'(c) \leq 0$, for every $c \in [0, 2]$ and for fixed λ with $\lambda \geq 1$. Therefore, $G(c)$ is a decreasing function of c in the interval $[0, 2]$, whose maximum value occurs at $c = 0$ only. For $c = 0$ in (3.17), the maximum value of $G(c)$ is

$$G_{max} = G(0) = 48(2\lambda - 1)^4(4\lambda - 1). \quad (3.19)$$

From the expressions (3.13) and (3.19), we have

$$|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq 12(2\lambda - 1)^4(4\lambda - 1). \quad (3.20)$$

Simplifying the relations (3.5) and (3.20), we obtain

$$|a_2a_4 - a_3^2| \leq \frac{4}{(3\lambda - 1)^2}. \quad (3.21)$$

Choosing $c_1 = c = 0$ and selecting $y = 1$ in (2.2) and (2.3), we find that $c_2 = 2$ and $c_3 = 0$. Substituting the values $c_1 = c_3 = 0$ and $c_2 = 2$ in (3.3) and then the obtained values in (3.20) along with the values in (3.6), we observe that equality is attained, which shows that our result is sharp. For the values $c_1 = c_3 = 0$ and $c_2 = 2$, from (2.1), we derive that

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \dots = 1 + 2z^2 + 2z^4 + \dots = \frac{1 + z^2}{1 - z^2}. \quad (3.22)$$

Therefore, the extremal function in this case is

$$z \frac{(f'(z))^\lambda}{f(z)} = 1 + 2z^2 + 2z^4 + \dots = \frac{1 + z^2}{1 - z^2}. \quad (3.23)$$

This completes the proof of our Theorem. \square

Remark 3.2. Choosing $\lambda = 1$, we have the class of starlike functions, for which from (3.21), we get $|a_2a_4 - a_3^2| \leq 1$, coincides with that of Janteng et al. [6].

Remark 3.3. Selecting $\lambda = 2$, we have $\operatorname{Re} \left\{ f'(z) \frac{zf'(z)}{f(z)} \right\} > 0$, which is a product combination of bounded turning and starlike functions, from (3.21), we obtain $|a_2a_4 - a_3^2| \leq \frac{4}{25}$.

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Current address: D. Vamshee Krishna (Corresponding author): Department of Mathematics, GIT, GITAM University Visakhapatnam- 530 045, A.P., India

E-mail address: vamsheekrishna1972@gmail.com

ORCID Address: <http://orcid.org/0000-0002-3334-9079>

Current address: D. Shalini: Department of Mathematics, Sri Venkateswara College of Engineering and Technology, Affiliated to JNTUK, Etcherla-532 410, A. P., India.

E-mail address: shaliniraj1005@gmail.com

ORCID Address: <http://orcid.org/0000-0003-4059-8900>