HİTİT JOURNAL OF SCIENCE

e-ISSN: 3061-9629
Volume: 2 • Number: 2
July 2025

On the Eigenvalue Problems with Integrable Potential and Boundary Conditions Rationally Dependent on the Eigenparameter

Elif Başkaya

Department of Mathematics, Karadeniz Technical University, 61080, Trabzon, Türkiye.

Corresponding Author

Elif BAŞKAYA

E-mail: elifbekar@ktu.edu.tr Phone: +90 462 377 2569

RORID: ror.org/03z8fyr40

Article Information

Article Type: Research Article

Doi: -

Received: 31.05.2025

Accepted: 17.06.2025

Published: 31.07.2025

Cite As

Başkaya E. On the eigenvalue problems with integrable potential and boundary conditions rationally dependent on the eigenparameter. Hitit journal of Science. 2025;2(2):37-41.

Peer Review: Evaluated by independent reviewers working in at least two different institutions appointed by the field editor.

Ethical Statement: Not available.

Plagiarism Checks: Yes - iThenticate

Conflict of Interest: In this work, asymptotic expansions of the eigenvalues of an eigenvalue problem are calculated with better error terms than previous works.

CRedit Author Statement

Elif Başkaya: Design of the study, Conceptualization, Writing-original draft, Editing, Supervision.

On the Eigenvalue Problems with Integrable Potential and Boundary Conditions Rationally Dependent on the Eigenparameter

Elif Başkaya
Department of Mathematics, Karadeniz Technical University, 61080, Trabzon, Türkiye.

Abstract
We present the asymptotic estimates of the eigenvalues for an eigenvalue problem that the problem has also the eigenparameter in the second boundary condition, rationally. The potential of the problem is integrable.

Keywords: Asymptotic Estimates, Boundary Condition Rationally Dependent on the Eigenvalue, Eigenvalue Problem, Integrable Potential, Riccati Equation

INTRODUCTION
In this paper, we consider the following eigenvalue problem:

$$y''(t)+[\lambda-q(t)]y(t)=0, t\in[0,1] \tag{1}$$

$$y(0)\cos\alpha-y'(0)\sin\alpha=0, \alpha\in[0,\pi) \tag{2}$$

$$\frac{y'(1)}{y(1)}=s\left(\lambda\right)=\frac{h(\lambda)}{g(\lambda)} \tag{3}$$

where λ is a real parameter; the potential q is a real-valued L_1 function on the interval , also has a mean value zero, i.e. $\int_0^1 q\left(t\right)dt=0$; g and h are polynomials with real coefficients and no common zeros. When $\alpha=0$, the boundary condition Equation (2) is taken as $y(0)=0$ and when $s(\lambda)=\infty$ the boundary condition Equation (3) is accepted as $y(1)=0$. In addition, if $M=\deg(g)\geq\deg(h)$, let $h(\lambda)=A_M\lambda^M+\cdots+A_0$ where $A_M\in\mathbb{R}$ (it may be zero) and assume that g is monic, and if $\deg(g)<\deg(h)=M$, let $g(\lambda)=A_{M-1}\lambda^{M-1}+\cdots+A_0$ where $A_{M-1}\in\mathbb{R}$ (it may be zero) and assume that h is monic.

The above eigenvalue equation (1) are common to many areas of application. For example, Hooke' s law describes a mass on a spring as

$$F=-Kx$$

where K is the spring constant. Also, for the potential energy $V(x)$ of the spring, we have

$$V\left(x\right)=\frac{1}{2}Kx^2$$

and in classical mechanics we can write the force as

$$F=-\frac{\partial V}{\partial x}$$

The differential equation from Newton' s law as

$$m\ddot{x}=-Kx$$

or

$$\ddot{x}+\omega^2x=0$$

where $\omega^2=\frac{K}{m}$, Thus, the potential energy can be expressed as the following:

$$V\left(x\right)=\frac{1}{2}m\omega^2x^2$$

The time-independent Schrödinger equation for the one-dimensional simple harmonic oscillator is given as

$$\frac{-h^2}{2m}\frac{d^2\psi}{d^x2^2}+\frac{1}{2}mw^2x^2\psi=E\psi$$

or

$$\frac{d^2\psi}{\mathrm{dx}^2}+\left[\frac{2mE}{h^2}-\frac{m^2w^2x^2}{h^2}\right]\psi=0.$$

If we take $x:=t\sqrt{\frac{h}{m\omega}},\psi\left(x\right):=y\left(t\right)$ the Schrödinger equation becomes

$$y''\left(t\right)+\left[\lambda-t^2\right]y\left(t\right)=0 \tag{4}$$

where $\lambda=\frac{2E}{hw}$ is the dimensionless energy. Equation (4) is an eigenvalue equation in the form of Equation (1).

The problem Equation (1)-Equation (3) is different from the usual regular eigenvalue problem because eigenvalue parameter λ is held in the second boundary condition. Such problems often arise from physical problems, quantum mechanics and geophysics and are studied by a lot of researchers. Some of them are [1]-[16]. We especially refer to [1], [13] and [16]. In [1], the asymptotic eigenvalues of the problem Equation (1)-Equation (3) with $q\in\mathrm{AC}\left[0,1\right]$ are given and it is shown that the Weyl m-function uniquely determines α, f and q; and is in turn uniquely determined by either two spectra from different values of α or by the Prüfer angle. In [13], the asymptotic eigenfunctions of the problem Equation (1)-Equation (3) with $q\in L_1\left[0,1\right]$ are found. And [16] introduces the inverse problem associated with the problem Equation (1)-Equation (3) by providing the eigenvalues and the corresponding eigenfunctions to uniquely determine the potential $q(t)$ that is be a continuously differentiable function on .

Our aim is to obtain asymptotic expansions of the eigenvalues of the problem Equation (1)-Equation (3) with better error terms than previous works.

MATERIAL AND METHODS
Our method is based on [7]. Let us associate Equation (1) with the Riccati equation

$$v'\left(t,\lambda\right)=-\lambda+q-v^2$$

and define

$$S(t,\lambda):=\mathrm{Re}[v(t,\lambda)], \tag{5}$$

$$T(t,\lambda) := \mathrm{Im}[v(t,\lambda)]. \tag{6}$$

It is shown in [7] that any real-valued solution of Equation (1) is in the form

$$y(t,\lambda) = R(t,\lambda)\cos\theta(t,\lambda)$$

with

$$S(t,\lambda) = \frac{R'(t,\lambda)}{R(t,\lambda)}, \tag{7}$$

$$T(t,\lambda) = \theta'(t,\lambda). \tag{8}$$

Our approach to calculating λ_n is to approximate λ those which are such that

$$\theta(1,\lambda) - \theta(0,\lambda) = \int_0^1 T(x,\lambda)\,dx$$

We suppose that there exist functions $A(t)$ and $\eta(\lambda)$ so that

$$\left|\int_t^1 e^{2i\sqrt{\lambda}x} q(x)\,dx\right| \le A(t)\eta(\lambda),\ t \in [0,1]$$

where

i) $A(t) := \int_t^1 |q(x)|\,dx$ is a decreasing function of

ii) $A(t) \in L[0,1]$,

iii) $\eta(\lambda) \to 0$ as $\lambda \to \infty$.

For $q \in L[0,1]$ the existence of the functions A and η may be established for λ positive as follows. It is clear that $\left|\int_t^1 e^{2i\sqrt{\lambda}x} q(x)\,dx\right| \le \int_t^1 |q(x)|\,dx \le \infty$ hence, if we define

$$F(t,\lambda) := \begin{cases} \dfrac{\left|\int_t^1 e^{2i\lambda^{1/2}x} q(x)\,dx\right|}{\int_t^1 |q(x)|\,dx} & \text{if } \int_t^1 |q(x)|\,dx \ne 0 \\ 0 & \text{if } \int_t^1 |q(x)|\,dx = 0 \end{cases} \tag{9}$$

we gain $0 \le F(t,\lambda) \le 1$. Also, if we set $\eta(\lambda) := \sup_{a \le t \le b} F(t,\lambda)$ we have $\eta(\lambda)$ is well defined by Equation (9) and $\eta(\lambda) \to 0$ as $\lambda \to \infty$ [7].

Our method of approximating a solution of $v'(t,\lambda) = -\lambda + q - v^2$ on [0,1] is similar to [7], so we set

$$v(t,\lambda) := i\lambda^{1/2} + \sum_{n=1}^{\infty} v_n(t,\lambda). \tag{10}$$

When we put this serie into the Riccati equation and solve differential equations, we hold

$$v_1(t,\lambda) = -e^{-2i\lambda^{1/2}t} \int_t^1 e^{2i\lambda^{1/2}x} q(x)\,dx,$$

$$v_2(t,\lambda) = e^{-2i\lambda^{1/2}t} \int_t^1 e^{2i\lambda^{1/2}x} {v_1}^2(x,\lambda)\,dx,$$

for $n \ge 3$

$$v_n(t,\lambda) = e^{-2i\lambda^{1/2}t} \int_t^1 e^{2i\lambda^{1/2}x} \left[{v_{n-1}}^2(x,\lambda) + 2v_{n-1}(x,\lambda)\sum_{m=1}^{n-2} v_m(x,\lambda)\right]dx.$$

Also from Equation (6), Equation (8) and Equation (10), we have $\theta(1,\lambda) - \theta(0,\lambda) = \int_0^1 [\lambda^{1/2} + \mathrm{Im}\sum_{n=1}^{\infty} v_n(x,\lambda)]\,dx$, then $\theta(1,\lambda) - \theta(0,\lambda) = \lambda^{1/2} + \sum_{n=1}^{\infty} \mathrm{Im}\int_0^1 v_n(x,\lambda)\,dx$ and from this equation, [2] proves that

$$\theta(1,\lambda) - \theta(0,\lambda) = \lambda^{1/2} - \tfrac{1}{2}\lambda^{-1/2}\int_0^1 q(x)\,[\cos 2\lambda^{1/2}x]\,dx + O(\lambda^{-1}\eta(\lambda)) \tag{11}$$

Let us consider the following theorem:

Theorem: [2] If $v(t,\lambda)$ as in (2.6), as $\lambda \to \infty$

$$v(t,\lambda) = i\lambda^{1/2} + v_1(t,\lambda) + O(\eta^2(\lambda))$$

where

$$v_1(t,\lambda) = [\mathrm{i}\sin(2\lambda^{1/2}t) - \cos(2\lambda^{1/2}t)]$$
$$\times \int_t^b [\cos(2\lambda^{1/2}x) + \mathrm{i}\sin(2\lambda^{1/2}x)]\,q(x)\,dx + O(\eta^2(\lambda)).$$

After some calculations by using the last theorem, with Equation (5) we gain

$$S(t,\lambda) = -\sin(2\lambda^{1/2}t + \xi_t) + O(\eta^2(\lambda)) \tag{12}$$

where

$$\sin\xi_t := \int_t^1 (\cos 2\lambda^{1/2}x)\,q(x)\,dx,$$

$$\cos\xi_t := \int_t^1 (\sin 2\lambda^{1/2}x)\,q(x)\,dx.$$

Similarly, with Equation (6) we find $T(t,\lambda)$ as

$$T(t,\lambda) = \lambda^{1/2} - \cos(2\lambda^{1/2}t + \xi_t) + O(\eta^2(\lambda)) \tag{13}$$

so we can write

$$S(0,\lambda) = -\int_0^1 (\cos 2\lambda^{1/2}x)\,q(x)\,dx + O(\eta^2(\lambda)), \tag{14}$$

$$T(0,\lambda) = \sqrt{\lambda} - \int_0^1 (\sin 2\sqrt{\lambda}x)\,q(x)\,dx + O(\eta^2(\lambda)), \tag{15}$$

$$S(1,\lambda) = O(\eta^2(\lambda)), \tag{16}$$

$$T(1,\lambda) = \lambda^{\frac{1}{2}} + O(\eta^2(\lambda)). \tag{17}$$

RESULTS AND DISCUSSION

We approximate the eigenvalues of the problem Equation (1)-Equation (3), in this section. It is shown in [7] that any real valued solution $y(t,\lambda)$ of Equation (1) is of the form

$$y(t,\lambda) = R(t,\lambda)\cos\theta(t,\lambda), \tag{18}$$

hence

$$y'(t,\lambda) = R'(t,\lambda)\cos\theta(t,\lambda) - R(t,\lambda)\theta'(t,\lambda)\sin\theta(t,\lambda). \tag{19}$$

We now determine the conditions under which the first boundary condition Equation (2) and the second boundary condition Equation (3) are satisfied.

Considering Equation (18) and Equation (19), one observes that Equation (2) holds if

$$R(0,\lambda)\left\{\cos\theta(0,\lambda)\left[\cos\alpha - \frac{R'}{R}(0,\lambda)\sin\alpha\right] + \sin\theta(0,\lambda)\theta'(0,\lambda)\sin\alpha\right\} = 0. \tag{20}$$

i) For $\alpha \neq 0$:
We can write Equation (20) as

$R(0,\lambda)\cdot \sin\left[\theta(0,\lambda)-\gamma_1\right]=0$

where

$$\sin\gamma_1 := \frac{R'(0,\lambda)}{R(0,\lambda)}\cdot \sin\alpha - \cos\alpha,$$

$$\cos\gamma_1 := \theta'(0,\lambda)\cdot\sin\alpha.$$

From Equation (7) and Equation (8)

$$\sin\gamma_1 = S(0,\lambda)\cdot\sin\alpha-\cos\alpha, \tag{21}$$

$$\cos\gamma_1 = T(0,\lambda)\cdot\sin\alpha \tag{22}$$

And we also from Equation (20)

$$\theta(0,\lambda)=\gamma_1. \tag{23}$$

Substituting the values of $S(0,\lambda)$ and $T(0,\lambda)$ given by Equation (14) and Equation (15) into Equation (21) and Equation (22), one obtains

$$\frac{\sin\gamma_1}{\cos\gamma_1}=\frac{-\cos\alpha-\sin\alpha\int_0^1\left(\cos 2\lambda^{1/2}x\right)q\left(x\right)dx+O\left(\eta^2\left(\lambda\right)\right)}{\lambda^{1/2}\sin\alpha-\sin\alpha\int_0^1\left(\sin 2\lambda^{1/2}x\right)q\left(x\right)dx+O\left(\eta^2\left(\lambda\right)\right)}$$

$$=\frac{-\cos\alpha-\sin\alpha\int_0^1\cos 2\sqrt{\lambda}xq\left(x\right)dx+O\left(\eta^2\left(\lambda\right)\right)}{\sqrt{\lambda}\sin\alpha\cdot\left[1-\lambda^{-\frac{1}{2}}\int_0^1\sin 2\sqrt{\lambda}xq\left(x\right)dx+\mathrm{O}\left(\lambda^{-\frac{1}{2}}\cdot\eta^2\left(\lambda\right)\right)\right]}$$

and

$$\frac{\sin\gamma_1}{\cos\gamma_1}=\left\{-\lambda^{-\frac{1}{2}}\cot\alpha-\lambda^{-\frac{1}{2}}\int_0^1\cos 2\sqrt{\lambda}xq\left(x\right)dx+\mathrm{O}\left(\lambda^{-\frac{1}{2}}\eta^2\left(\lambda\right)\right)\right\}$$

$$\times\left\{1+\lambda^{-\frac{1}{2}}\int_0^1\sin 2\sqrt{\lambda}xq\left(x\right)dx+\mathrm{O}\left(\lambda^{-\frac{1}{2}}\eta^2\left(\lambda\right)\right)\right\}$$

so

$$\tan\gamma_1=-\lambda^{-1/2}\cot\alpha-\lambda^{-1/2}\int_0^1\left(\cos 2\lambda^{1/2}x\right)q\left(x\right)dx+O\left(\left(\lambda^{-1/2}\eta\right)^2\left(\lambda\right)\right) \tag{24}$$

ii) For $\alpha=0$
In this case, Equation (20) reduces
$R(0,\lambda)\cos\theta(0,\lambda)=0$

and from this equation, the first boundary condition Equation (2) is satisfied for

$$\theta\left(0,\lambda\right)=\frac{\pi}{2}.$$

Considering Equation (18) and Equation (19), one observes that the second boundary condition Equation (3) holds if

$$R\left(1,\lambda\right)\left\{\cos\theta\left(1,\lambda\right)\left[h\left(\lambda\right)-\frac{R\prime(1,\lambda)}{R(1,\lambda)}g\left(\lambda\right)\right]+\sin\theta\left(1,\lambda\right)\theta\prime\left(1,\lambda\right)g\left(\lambda\right)\right\}=0 \tag{26}$$

iii) For $\deg(h)\leq\deg(g)=M$:

The Equation (26) is expressed as

$$R(1,\lambda)\sin\left[\theta(1,\lambda)-\gamma\_2\right]=0 \tag{27}$$

where

$$\sin\gamma_2 := \frac{R'(1,\lambda)}{R(1,\lambda)}g\left(\lambda\right)-h\left(\lambda\right),$$

$$\cos\gamma_2 := \theta'(1,\lambda)g(\lambda).$$

From the definitions of $g(\lambda)$ and $h(\lambda)$ one writes

$$\sin\gamma_2=S(1,\lambda)\left[\lambda^M+B_{M-1}\lambda^{M-1}+\cdots+B_0\right]-\left[A_M\lambda^M+A_{M-1}\lambda^{M-1}+\cdots+A_0\right]$$
$$\cos\gamma_2=T(1,\lambda)\left[\lambda^M+B_{M-1}\lambda^{M-1}+\cdots+B_0\right]$$

and substitution of Equation (16) and Equation (17) into the last equations gives

$$\sin\gamma_2=-A_M\lambda^M+\mathrm{O}\left(\lambda^M\eta^2(\lambda)\right),$$
$$\cos\gamma_2=\lambda^{M+\frac{1}{2}}+\mathrm{O}\left(\lambda^M\eta^2(\lambda)\right)$$

hence

$$\frac{\sin\gamma_2}{\cos\gamma_2}=\frac{-A_M\lambda^M+\mathrm{O}\left(\lambda^M\eta^2\right)}{\lambda^{M+\frac{1}{2}}+\mathrm{O}\left(\lambda^M\eta^2\right)}=\frac{-A_M\lambda^M+\mathrm{O}\left(\lambda^M\eta^2\right)}{\lambda^{M+\frac{1}{2}}\left[1+\mathrm{O}\left(\lambda^{-\frac{1}{2}}\eta^2\right)\right]}$$

$$=\left\{-A_M\lambda^{-\frac{1}{2}}+\mathrm{O}\left(\lambda^{-\frac{1}{2}}\eta^2\left(\lambda\right)\right)\right\}\cdot\left\{1-\mathrm{O}\left(\lambda^{-\frac{1}{2}}\eta^2\left(\lambda\right)\right)\right\}$$

then

$$\tan\gamma_2=-A_M\lambda^{-\frac{1}{2}}+\mathrm{O}\left(\lambda^{-\frac{1}{2}}\eta^2\left(\lambda\right)\right). \tag{28}$$

Also from Equation (27), we have

$$\theta(1,\lambda)=(n+1)\pi+\gamma_2 \tag{29}$$

iv) For $\deg(g)<\deg(h)=M$:

The Equation (26) is obtained as
$R(1,\lambda)\sin\left[\theta(1,\lambda)-\gamma_3\right]=0$

where

$$\sin\gamma_3 := \frac{R\prime(1,\lambda)}{R(1,\lambda)}g\left(\lambda\right)-h\left(\lambda\right),$$

$$\cos\gamma_3 := \theta'(1,\lambda)g(\lambda)$$

so that

$$\theta(1,\lambda)=(n+1)\pi+\gamma_3 \tag{30}$$

From the definitions of and one writes

$$\sin\gamma_3=S\left(1,\lambda\right)\left[A_{M-1}\lambda^{M-1}+A_{M-2}\lambda^{M-2}+\cdots+A_0\right]$$

$$-\left[\lambda^M+B_{M-1}\lambda^{M-1}+\cdots+B_0\right]$$

$$\cos\gamma_3=T\left(1,\lambda\right)\left[A_{M-1}\lambda^{M-1}+A_{M-2}\lambda^{M-2}+\cdots+A_0\right]$$

and substitution of Equation (16) and Equation (17) into the last equations gives

$$\sin\gamma_3=-\lambda^M-B_{M-1}\lambda^{M-1}+O\left(\lambda^{M-1}\eta^2(\lambda)\right)$$

$$\cos\gamma_3=A_{M-1}\lambda^{M-\frac{1}{2}}+O\left(\lambda^{M-1}\eta^2\left(\lambda\right)\right),$$

thus

$$\frac{\cos\gamma_3}{\sin\gamma_3}=\frac{A_{M-1}\lambda^{M-\frac{1}{2}}+O\left(\lambda^{M-1}\eta^2\left(\lambda\right)\right)}{-\lambda^M-B_{M-1}\lambda^{M-1}+O(\lambda^{M-1}\eta^2(\lambda))}$$

$$=\frac{A_{M-1}\lambda^{M-\frac{1}{2}}+O\left(\lambda^{M-1}\eta^2\left(\lambda\right)\right)}{-\lambda^M(1+B_{M-1}\lambda^{-1}+O(\lambda^{-1}\eta^2(\lambda)))}$$

then

$$\cot\gamma_3=-A_{M-1}\lambda^{-\frac{1}{2}}+O\left(\lambda^{-1}\eta^2\left(\lambda\right)\right) \tag{31}$$

Theorem: The asymptotic formulae for the eigenvalues of the problem Equation (1)-Equation (3) satisfy, as $n\to\infty$

(i) if $\alpha\neq 0$ and $\deg(h)\leq\deg(g)=M$,

$$\lambda_n^{1/2}=\left(n+1\right)\pi+\frac{1}{(n+1)\pi}\left[-A_M+\cot\alpha+\frac{3}{2}\int_0^1 q\left(x\right)\cos\left(2\left(n+1\right)\pi x\right)dx\right]$$
$$+O\left(n^{-1}\eta^2\left(n\right)\right)+O\left(n^{-2}\eta\left(n\right)\right)$$

(ii) if $\alpha\neq 0$ and $\deg(g)<\deg(h)=M$,

$$\lambda_n^{1/2}=\frac{2n+3}{2}\pi+\frac{2}{2n+3\pi}\left[A_{M-1}+\cot\alpha+\frac{3}{2}\int_0^1 q\left(x\right)\cos\left((2n+3)\pi x\right)dx\right]$$
$$+O\left(n^{-1}\eta^2\left(n\right)\right)+O\left(n^{-2}\eta\left(n\right)\right)$$

(iii) if $\alpha=0$ and $\deg(h)\leq\deg(g)=M$,

$$\lambda_n^{1/2}=\frac{2n+1}{2}\pi+\frac{2}{2n+1\pi}\left[-A_M+\frac{1}{2}\int_0^1 q\left(x\right)\cos\left((2n+1)\pi x\right)dx\right]$$
$$+O\left(n^{-1}\eta^2\left(n\right)\right)+O\left(n^{-2}\eta\left(n\right)\right)$$

(iv) if $\alpha=0$ and $\deg(g)<\deg(h)=M$,

$$\lambda_n^{1/2}=\left(n+1\right)\pi+\frac{1}{(n+1)\pi}\left[A_{M-1}+\frac{1}{2}\int_0^1 q\left(x\right)\cos\left(2\left(n+1\right)\pi x\right)dx\right]$$
$$+O\left(n^{-1}\eta^2\left(n\right)\right)+O\left(n^{-2}\eta\left(n\right)\right)$$

Proof: Theorem (i) is proved by using Equation (11), Equation (23), Equation (24), Equation (28), Equation (29) together with inverse trigonometric series and reversion. Theorem (ii) is proved by using Equation (11), Equation (23), Equation (24), Equation (30), Equation (31) together with inverse trigonometric series and reversion. Theorem (iii) is proved by using Equation (11), Equation (25), Equation (28), Equation (29) together with inverse trigonometric serie and reversion. Theorem (iv) is proved by using Equation (11), Equation (25), Equation (30), Equation (31) together with inverse trigonometric serie and reversion.

CONCLUSION

In this work, asymptotic expansions of the eigenvalues of an eigenvalue problem are calculated with better error terms than previous works.

References

1. Binding P. A., Browne P. J. and Watson B. A., 2004. Equivalence of Inverse Sturm-Liouville Problems with Boundary Conditions Rationally Dependent on the Eigenparameter. Journal of Mathematical Analysis and Applications, 291, 246-261.
2. Coşkun H. and Başkaya E., 2010. Asymptotics of eigenvalues for regular Sturm-Liouville problems with eigenvalue parameter in the boundary condition for integrable potential. Mathematica Scandinavica, 107, 209-223.
3. Coşkun H., Başkaya E. and Kabataş A., 2019. Instability intervals for Hill's equation with symmetric single well potential. Ukrainian Mathematical Journal, 71 (6), 977-983.
4. Coşkun H. and Kabataş A., 2013. Asymptotic approximations of eigenfunctions for regular Sturm-Liouville problems with eigenvalue parameter in the boundary condition for integrable potential. Mathematica Scandinavica, 113 (1), 143-160.
5. Coşkun H. and Kabataş A., 2016. Green's function of regular Sturm-Liouville problem having eigenparameter in one boundary condition. Turkish Journal of Mathematics and Computer Science, 4, 1–9.
6. Coşkun H., Kabataş A. and Başkaya E., 2017. On Green's function for boundary value problem with eigenvalue dependent quadratic boundary condition. Boundary Value Problems, 71.
7. Harris B. J., 1997. The Form of the Spectral Functions Associated with Sturm-Liouville Problems with Continuous Spectrum. Mathematika, 44 (1), 149-161.
8. Kabataş A., 2022. On eigenfunctions of Hill' s equation with symmetric double well potential. Communications Faculty of Sciences University of Ankara-Series A1 Mathematics and Statistics, 71 (3), 634-649.
9. Kabataş A., 2022. Eigenfunction and Green' s function asymptotics for Hill' s equation with symmetric single-well potential. Ukrainian Mathematical Journal, 74 (2), 218-231.
10. Kabataş A., 2023. One boundary value problem including a spectral parameter in all boundary conditions. Opuscula Mathematica, 43 (5), 651-661.
11. Kabataş A., 2023. Sturm-Liouville problems with polynomially eigenparameter dependent boundary conditions. Sakarya University Journal of Science, 27 (6), 1235-1242.
12. Kabataş A., 2023. Sturm-Liouville problems including boundary conditions polynomially dependent on the eigenparameter with differentiable potential. International Research and Reviews in Science and Mathematics, Serüven Publishing, Ankara, 1-16.
13. Kabataş A., 2023. On eigenfunctions of the Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter. International Theory, Research and Reviews in Science and Mathematics, Serüven Publishing, Ankara, 263-281.
14. Kabataş A., 2023. On Sturm-Liouville problems including the square-root of the eigenvalue parameter in one boundary condition. New Frontiers in Natural Science and Mathematics, Duvar Publishing, İzmir, 233-251.
15. Kabataş A., 2024. Green's function asymptotics for a boundary value problem contained eigenvalue parameter in the boundary condition. Advances in Mathematics Research Volume 35, NOVA Science Publishers Inc. , New York, 43-53.
16. Mosazadeh S. and Akbarfam A. J., 2020. Inverse and Expansion Problems with Boundary Conditions Rationally Dependent on the Eigenparameter. Bulletin of the Iranian Mathematical Society, 46, 67-78.