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RESEARCH ARTICLE

The Anticenter and Reflection Line of a Cyclic Quadrilateral

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ABSTRACT

The *Simson point* S of a quadrilateral $\mathcal{Q}$ is the point for which the pedal polygon of $\mathcal{Q}$ with respect to S degenerates into a single line, called the *Simson line*. If we reflect the Simson point in the lines containing the sides of $\mathcal{Q}$, then we get another line that is parallel to the Simson line. We refer to this second line as the *Reflection line* of S. Ferrarello, Mammana, and Pennisi have conjectured that if $\mathcal{Q}$ is a cyclic quadrilateral that does not have parallel sides, then the reflection line of S passes through the anticenter of $\mathcal{Q}$. We give a positive answer to this conjecture. We also give characterizations using the reflection line for a convex quadrilateral to be cyclic or to be semi-symmetric.

Keywords: *Cyclic quadrilateral, Simson point, anticenter, Simson line, reflection line, semi-symmetric quadrilateral.*
AMS Subject Classification (2020): *Primary: 51M04 ; Secondary: 51M15; 51M20.*

1. Introduction

Let $\square A_1A_2A_3A_4$ denote a quadrilateral with vertices A_1, A_2, A_3, and A_4. Let P denote a point. Let F_1, F_2, F_3, and F_4 denote the feet of the perpendiculars from P to the lines containing the sides $\overline{A_1A_2}$, $\overline{A_2A_3}$, $\overline{A_3A_4}$, and $\overline{A_4A_1}$, respectively. If the points F_1, F_2, F_3, and F_4 form the vertices of a quadrilateral $\square F_1F_2F_3F_4$, then we call $\square F_1F_2F_3F_4$ the *pedal quadrilateral* of P with respect to $\square A_1A_2A_3A_4$. We call the point P the *pedal point*.

In [2], Ferrarello, Mammana, and Pennisi show that if $\mathcal{Q}$ is a quadrilateral that is not a parallelogram, then there exists exactly one pedal point S with respect to which the points F_1, F_2, F_3, and F_4 are collinear. The point S is called the *Simson point* of $\mathcal{Q}$, and the line t containing the points F_1, F_2, F_3, and F_4 is called the *Simson line* of $\mathcal{Q}$. Ferrarello, Mammana, and Pennisi go on to prove that the reflections of the Simson point S with respect to the lines containing the sides of $\mathcal{Q}$ are collinear and the line m containing these reflections is parallel to the Simson line t. We refer to the line m as the *reflection line* (see Figure 1).

A *cyclic quadrilateral* is a quadrilateral $\mathcal{Q}$ all of whose vertices lie on a common circle $\mathcal{K}$. We call $\mathcal{K}$ the circumcircle of $\mathcal{Q}$, and we call the center of $\mathcal{K}$ the *circumcenter* of $\mathcal{Q}$.

Given a quadrilateral $\square A_1A_2A_3A_4$, then a *maltitude* (midpoint altitude) of $\square A_1A_2A_3A_4$ is the perpendicular that we get by starting with the midpoint of a side of $\square A_1A_2A_3A_4$ and then drawing the perpendicular from that midpoint to the opposite side. Note that any quadrilateral, cyclic or not, always has maltitudes (although in the case of a rectangle or trapezoid, some of these maltitudes overlap). Micale and Pennisi [6] show that the maltitudes of quadrilateral $\square A_1A_2A_3A_4$ are concurrent at a point H if and only if $\square A_1A_2A_3A_4$ is cyclic. The point H of concurrency is called the *anticenter* of $\square A_1A_2A_3A_4$.

Ferrarello, Mammana, and Pennisi [2] conjecture that if $\mathcal{Q}$ is a cyclic quadrilateral without parallel sides, then the reflection line m passes through the anticenter H of $\mathcal{Q}$, and that the Simson line t bisects the segment $\overline{SH}$ connecting the anticenter H with the Simson point S (see Figure 1). We give an affirmative answer to this conjecture.

In [3] and [4], Josefsson gives several characteristics of when a quadrilateral is cyclic. We prove that if a convex quadrilateral $\mathcal{Q}$ is not a parallelogram then $\mathcal{Q}$ is cyclic if and only if the reflection line m passes through the intersection point of the diagonals of $\mathcal{Q}$.

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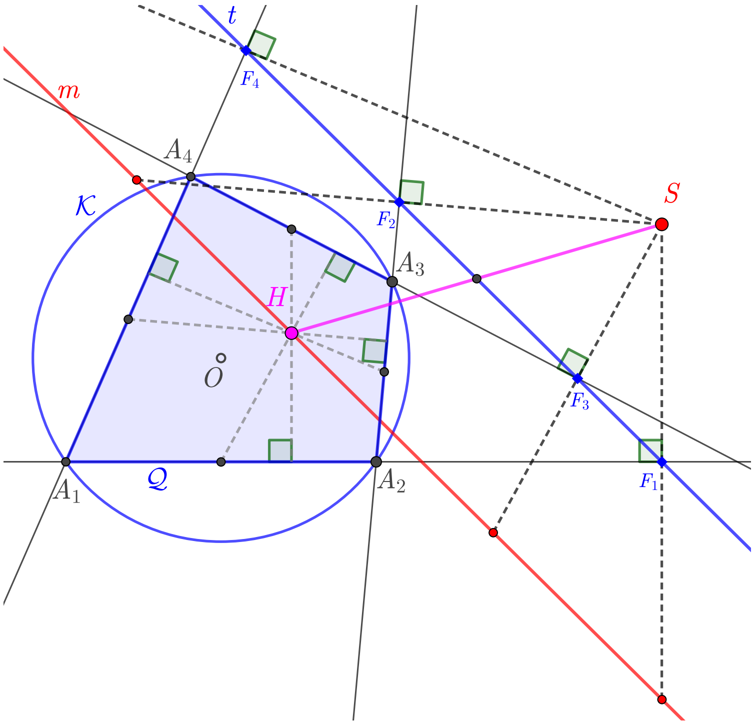

Figure 1. Simson point, Simson line, and reflection line of a quadrilateral

A cyclic quadrilateral $\square A_1A_2A_3A_4$ is called *semi-symmetric* if the vertices A_1 and A_3 are diametrically opposed points. Several characterizations of semi-symmetric cyclic quadrilaterals are given in [1]. In particular, it is shown that $\square A_1A_2A_3A_4$ is semi-symmetric if and only if the angles at vertices A_2 and A_4 are right angles. It is also shown that if $\square A_1A_2A_3A_4$ is a complete semi-symmetric cyclic quadrilateral with right angles at vertices A_2 and A_4, then the reflection line m of $\square A_1A_2A_3A_4$ passes through A_2 and A_4 (see Figure 2). We prove that the converse is also true. In particular, if the reflection line m of a complete quadrilateral $\square A_1A_2A_3A_4$ passes through A_2 and A_4, then $\square A_1A_2A_3A_4$ is a semi-symmetric cyclic quadrilateral with right angles at vertices A_2 and A_4. Consequently, we give a new characterization of semi-symmetric cyclic quadrilaterals.

Throughout the paper, when assuming that $\square A_1A_2A_3A_4$ is a complete quadrilateral, we denote the intersection of lines $\overleftrightarrow{A_1A_2}$ and $\overleftrightarrow{A_3A_4}$ by the point A_5, and we denote the intersection of lines $\overleftrightarrow{A_1A_4}$ and $\overleftrightarrow{A_2A_3}$ by the point A_6. When assuming that $\square A_1A_2A_3A_4$ is a convex quadrilateral, we denote the intersection of the diagonals $\overline{A_1A_3}$ and $\overline{A_2A_4}$ by the point D.

2. The Reflection Line and the Anticenter

In this section we prove that if $\mathcal{Q}$ is a cyclic quadrilateral without parallel sides, and if S denotes the Simson point of $\mathcal{Q}$, then the reflection line m passes through the anticenter H of $\mathcal{Q}$, and the Simson line passes through the midpoint of segment $\overline{SH}$. We begin with the following lemma.

Lemma 2.1. *Let $\square A_1A_2A_3A_4$ be a cyclic quadrilateral without parallel sides. Let S and H denote the Simson point and anticenter of $\square A_1A_2A_3A_4$, respectively. Let M denote the midpoint of segment $\overline{SH}$. Let t and m denote the Simson line and reflection line, respectively. Then the point H is on the reflection line m if and only if the point M is on the Simson line t.*

Proof. Let F_1, F_2, F_3, and F_4 denote the feet of the perpendiculars from S to the lines containing the sides $\overline{A_1A_2}$, $\overline{A_2A_3}$, $\overline{A_3A_4}$, and $\overline{A_4A_1}$, respectively (See Figure 1). Since lines $\overleftrightarrow{SF_1}$, $\overleftrightarrow{SF_2}$, $\overleftrightarrow{SF_3}$, and $\overleftrightarrow{SF_4}$ are distinct and have intersection point S, then H is on at most one of these four lines. We may assume without loss of generality that H is not on $\overleftrightarrow{SF_1}$. Let K denote the reflection of S in the line $\overleftrightarrow{A_1A_2}$.

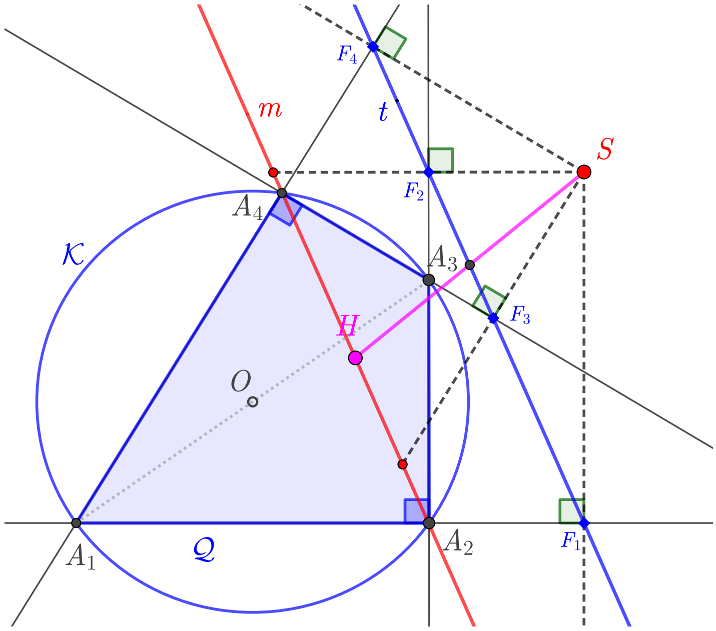

Figure 2. Semi-symmetric cyclic quadrilateral

First assume that H is on the reflection line m. Let l denote the line through M that is parallel to the reflection line m. Since m intersects both lines $\overleftrightarrow{SH}$ and $\overleftrightarrow{SK}$, and since m and l are parallel, then l intersects both lines $\overleftrightarrow{SH}$ and $\overleftrightarrow{SK}$. Let T denote the point of intersection of lines l and $\overleftrightarrow{SK}$. Again, since m and l are parallel, then angles $\angle STM$ and $\angle SKH$ are congruent, and angles $\angle SMT$ and $\angle SHK$ are congruent. Thus, it follows by Angle-Angle-Angle that triangles $\triangle SMT$ and $\triangle SHK$ are similar triangles. Since M is the midpoint of segment $\overline{SH}$, then T is the midpoint of segment $\overline{SK}$. However, F_1 is the midpoint of segment $\overline{SK}$, which implies that $T = F_1$. Hence, l is the line passing through F_1 which is parallel to m. Since t is the unique line passing through F_1 which is parallel to m, then it follows that $l = t$. Consequently, t passes through the midpoint M.

Conversely, assume that the point M is on the line t. Since lines m and t are parallel, then m intersects line $\overleftrightarrow{SH}$ at a point B. Similar to above, we have by Angle-Angle-Angle that triangles $\triangle SMF_1$ and $\triangle SBK$ are similar triangles. This implies that M is the midpoint of segment $\overline{SB}$. Hence, $B = H$, and it follow that m passes through the anticenter H. □

Theorem 2.1. *Let $\square A_1A_2A_3A_4$ be a cyclic quadrilateral without parallel sides. Let S and H denote the Simson point and anticenter of $\square A_1A_2A_3A_4$, respectively. Then the midpoint M of segment $\overline{SH}$ lies on the Simson line.*

Proof. We prove the statement analytically. Since incidence and collinearity are preserved by translations, rotations and dilations, then without loss of generality we can choose a coordinate system with origin at $A_1 = (0, 0)$, and unit point on the x-axis at $A_2 = (1, 0)$.

Let $A_3 = (a, b)$ and $A_5 = (c, 0)$, where $a, b, c \in \mathbb{R}$, are such that $b > 0$, $c \neq 0$ and $c \neq 1$.

The three parameters a, b, and c can be used to express all the other coordinates and equations.

The center O of the circumcircle of triangle $\triangle A_1A_2A_3$ is the intersection of perpendicular bisectors of sides $\overline{A_1A_2}$ and $\overline{A_2A_3}$. Thus, O has coordinates

$$O = \left(\frac{1}{2}, \frac{a^2 - a + b^2}{2b}\right).$$

Thus, the circumcircle $\mathcal{K}$ of triangle $\triangle A_1A_2A_3$ has equation

$$\left(x - \frac{b}{2b}\right)^2 + \left(y - \frac{(a^2 + b^2 - a)}{2b}\right)^2 = \frac{a^4 + b^4 + 2a^2b^2 - 2a^3 - 2ab^2 + a^2 + b^2}{4b^2}.$$

The coordinates of the vertex A_4 can be calculated by computing the intersections of the circle $\mathcal{K}$ and the line $\overleftrightarrow{A_3A_5}$:

$$A_4 = \left(\frac{a^2c - ac^2 - ac + b^2c + c^2}{a^2 - 2ac + b^2 + c^2}, \frac{bc^2 - bc}{a^2 - 2ac + b^2 + c^2} \right).$$

The point A_6 is the intersection of the lines $\overleftrightarrow{A_1A_4}$ and line $\overleftrightarrow{A_2A_3}$. Thus, A_6 has coordinates

$$A_6 = \left(\frac{a^2 - ac - a + b^2 + c}{a^2 - 2ac + b^2 + 2c - 1}, \frac{bc - b}{a^2 - 2ac + b^2 + 2c - 1} \right).$$

The Simson point S of cyclic quadrilateral $\square A_1A_2A_3A_4$ is the common intersection point of the circumcircles of triangles $\triangle A_1A_2A_6$, $\triangle A_2A_3A_5$, $\triangle A_3A_4A_6$, and $\triangle A_1A_4A_5$, respectively (Figure 3).

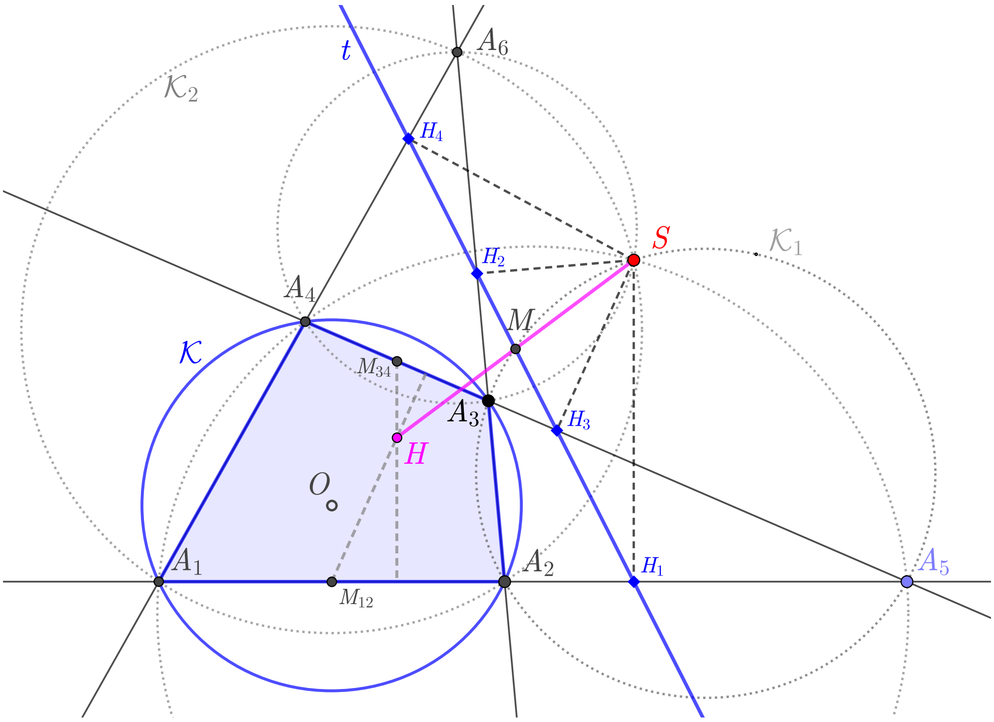

Figure 3. For proof of Theorem 2.1

We calculate the equations of the circumcircles of $\triangle A_2A_3A_5$ and $\triangle A_1A_2A_6$, which we denote by $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively.

We denote the centers of $\mathcal{K}_1$ and $\mathcal{K}_2$ by O_1 and O_2, respectively. We note that O_1 is the intersection of the perpendicular bisectors $\overleftrightarrow{A_3A_5}$ and $\overleftrightarrow{A_2A_5}$, and similarly that O_2 is the intersection of the perpendicular bisectors $\overleftrightarrow{A_1A_2}$ and $\overleftrightarrow{A_2A_6}$. One intersection of these circles is the point A_2, and the other intersection is the Simson point S. Thus, S has coordinates $S = (u, v)$, where

$$u = \frac{-a^3c + 2a^2c^2 + 2a^2c - ab^2c - 4ac^2 - ac + 2b^2c + 2c^2}{a^4 - 4a^3c - 2a^3 + 2a^2b^2 + 4a^2c^2 + 8a^2c + a^2 - 4ab^2c - 2ab^2 - 8ac^2 - 4ac + b^4 + 4b^2c + b^2 + 4c^2}$$

and

$$v = \frac{a^2bc - 2abc^2 + b^3c + 2bc^2 - bc}{a^4 - 4a^3c - 2a^3 + 2a^2b^2 + 4a^2c^2 + 8a^2c + a^2 - 4ab^2c - 2ab^2 - 8ac^2 - 4ac + b^4 + 4b^2c + b^2 + 4c^2}.$$

We denote the pedal point of the Simson point S to the line $\overleftrightarrow{A_iA_{i+1}}$ by H_i. In particular, H_1 is the intersection of the line $\overleftrightarrow{A_1A_2}$ with the perpendicular from S to line $\overleftrightarrow{A_1A_2}$, and H_2 is the intersection of the line $\overleftrightarrow{A_2A_3}$ with the perpendicular from S to line $\overleftrightarrow{A_2A_3}$. Thus, H_1 has coordinates $H_1 = (u_1, v_1)$, where

$$u_1 = \frac{-a^3c + 2a^2c^2 + 2a^2c - ab^2c - 4ac^2 - ac + 2b^2c + 2c^2}{a^4 - 4a^3c - 2a^3 + 2a^2b^2 + 4a^2c^2 + 8a^2c + a^2 - 4ab^2c - 2ab^2 - 8ac^2 - 4ac + b^4 + 4b^2c + b^2 + 4c^2}$$

and $v_1 = 0$. Similarly, H_2 has coordinates $H_2 = (u_2, v_2)$, where

$$u_2 = \frac{-a^3c + a^2b^2 + 2a^2c^2 + 2a^2c - 3ab^2c - 4ac^2 - ac + b^4 + 3b^2c + 2c^2}{a^4 - 4a^3c - 2a^3 + 2a^2b^2 + 4a^2c^2 + 8a^2c + a^2 - 4ab^2c - 2ab^2 - 8ac^2 - 4ac + b^4 + 4b^2c + b^2 + 4c^2}$$

and

$$v_2 = \frac{-a^3b + 3a^2bc + a^2b - ab^3 - 2abc^2 - 3abc + b^3c + b^3 + 2bc^2}{a^4 - 4a^3c - 2a^3 + 2a^2b^2 + 4a^2c^2 + 8a^2c + a^2 - 4ab^2c - 2ab^2 - 8ac^2 - 4ac + b^4 + 4b^2c + b^2 + 4c^2}.$$

Since the points H_i are collinear, we use the points H_1 and H_2 to compute the equation of the Simson line t. Thus, $t = \overleftrightarrow{H_1H_2}$ has equation $y = kx + q$, such that

$$k = \frac{-a^3 - ab^2 - 2ac^2 + 3a^2c + b^2c + a^2 + b^2 + 2c^2 - 3ac}{b^3 + a^2b - 2abc + bc}$$

and

$$q = \frac{(-c)(q_1)(q_2)}{(b)(q_3)(q_4)},$$

where

$$q_1 = 2a^2c - 4ac + 2c - a^3 + 2a^2 - ab^2 - a + 2b^2,$$
$$q_2 = ac^2 - 2c^2 - 3a^2c + 3ac - b^2c + a^3 - a^2 + ab^2 - b^2,$$
$$q_3 = 2ac - c - a^2 - b^2,$$

and

$$q_4 = (a^2)(4c^2 - 4ac + 8c + a^2 - 2a + 2b^2 + 1) - 8ac^2 + 4c^2 - 4ab^2c - 4ac + 4b^2c - 2ab^2 + b^4 + b^2.$$

Let H denote the anticenter of quadrilateral $\square A_1A_2A_3A_4$. Let M_{12} and M_{34} denote the midpoints of segments $\overline{A_1A_2}$ and $\overline{A_3A_4}$, respectively. Let l_{12} denote the line through M_{12} which is perpendicular to $\overline{A_3A_4}$, and let l_{34} denote the line through M_{34} which is perpendicular to $\overline{A_1A_2}$. Thus, H is the point of intersection of lines l_{12} and l_{34}. Therefore, H has coordinates $H = (u_H, v_H)$, where

$$u_H = \frac{a^3 - a^2c + ab^2 - ac + b^2c + c^2}{2a^2 - 4ac + 2b^2 + 2c^2},$$

and

$$v_H = \frac{-a^4 + 2a^3c + a^3 - a^2b^2 - a^2c^2 - 2a^2c + ab^2 + ac^2 + b^2c^2 - b^2c}{2a^2b - 4abc + 2b^3 + 2bc^2}$$

Let M denote the midpoint of segment $\overline{HS}$. Thus, M has coordinates $M = (M_x, M_y)$, where $M_x = \frac{h_x}{k_x}$ and $M_y = \frac{h_y}{k_y}$.

We have that $h_x = h_{x,1} + h_{x,2} + h_{x,3} + h_{x,4} + h_{x,5}$, where

$$h_{x,1} = (a^2c^2)(8c^2 - 4a^2c - 10ac + 4b^2c + 32c + 8a^3 - 3a^2 + 4ab^2 - 18a),$$
$$h_{x,2} = (ac^2)(10ab^2 - 16c^2 - 14b^2c - 18c + 13a - 4b^4 - 22b^2),$$
$$h_{x,3} = (b^2c)(12c^2 + 5b^2c + 5c - 9a^4 - 3a^2b^2 - 3a + b^4),$$
$$h_{x,4} = (ab^2)(6ac - b^2c + 3a^4 - 4a^3 + 3a^2b^2 + 2a^2 - 2ab^2 + b^4 + b^2 + 6a^2c),$$

and

$$h_{x,5} = 8c^4 - 3a^3c + a^4c + 5b^4c + a^5 - 2a^6 + a^7 - 5a^6c + 7a^5c.$$

We also have that $k_x = 4(c^2 - 2ca + a^2 + b^2)(k_{x,1} + k_{x,2})$, where

$$k_{x,1} = (ac)(4ac - 8c - 4a^2 + 8a - 4b^2 - 4),$$

and

$$k_{x,2} = 4c^2 + 4b^2c + a^4 - 2a^3 + 2a^2b^2 + a^2 - 2ab^2 + b^4 + b^2.$$

We have that $h_y = -(h_{y,1} + h_{y,2})(h_{y,3} + h_{y,4})$, where

$$h_{y,1} = (a)(2c^2 - 3ac + 3c + a^2 - a + b^2),$$

$$h_{y,2} = -(2c^2 + b^2c + b^2),$$

$$h_{y,3} = (ac)(2a^2c - 4ac - 2b^2c + 2c + 6a^2 - 3a^3 - 2ab^2 - 3a + 2b^2),$$

and

$$h_{y,4} = 4b^2c^2 + b^4c - 3b^2c + a^5 - 2a^4 + 2a^3b^2 + a^3 - 2a^2b^2 + ab^4 + ab^2.$$

Finally, we have that $k_y = 4b(c^2 - 2ca + a^2 + b^2)(k_{y,1} + k_{y,2})$, where

$$k_{y,1} = (c)(4a^2c - 8ac + 4c - 4a^3 + 8a^2 - 4ab^2 - 4a + 4b^2)$$

and

$$k_{y,2} = a^4 - 2a^3 + 2a^2b^2 + a^2 - 2ab^2 + b^4 + b^2.$$

We now see by substituting M_x and M_y into the equation of line t, that the point M is a point on t. Thus, the Simson line t bisects segment $\overline{HS}$ at M. $\square$

3. A Characterization of Cyclic Quadrilaterals

In this section we use the reflection line of a convex complete quadrilateral $\square A_1A_2A_3A_4$ to give a necessary and sufficient condition for $\square A_1A_2A_3A_4$ to be cyclic. In particular, we show that if $\square A_1A_2A_3A_4$ is a convex complete quadrilateral without parallel sides, then $\square A_1A_2A_3A_4$ is cyclic if and only if the reflection line passes through the point of intersection D of the diagonals $\overline{A_1A_3}$ and $\overline{A_2A_4}$.

Theorem 3.1. *Let $\square A_1A_2A_3A_4$ be a convex complete quadrilateral. Assume that rays $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_4A_3}$ intersect at a point A_5, and that rays $\overrightarrow{A_4A_1}$ and $\overrightarrow{A_3A_2}$ intersect at a point A_6. Let S denote the Simson point of $\square A_1A_2A_3A_4$. Let D denote the point of intersection of the diagonals $\overline{A_1A_3}$ and $\overline{A_2A_4}$. Let m denote the reflection line of S with respect to $\square A_1A_2A_3A_4$. Then $\square A_1A_2A_3A_4$ is cyclic if and only if m passes through D.*

Proof. For the proof, we again use the tools of analytic geometry. As above, we note that since translations, rotations, and dilations in the plane do not alter incidence, then we may assume that $A_3 = (0, 0)$ and $A_2 = (1, 1)$. Let $A_1 = (c, d)$ and $A_4 = (a, b)$. Since not three of A_1, A_2, A_3, and A_4 are collinear, then it follows that $a \neq b$ and $c \neq d$.

If $b = d - 1$ and $a \leq c - 1$, then rays $\overrightarrow{A_3A_2}$ and $\overrightarrow{A_4A_1}$ do not intersect at A_6, contradicting our hypothesis. If $b = d - 1$ and $a > c - 1$, then rays $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_4A_3}$ do not intersect at A_5, again contradicting our hypothesis. Thus, we see that $b \neq d - 1$.

First assume that $a, c \neq 0$, $a, c \neq 1$, and $a \neq c$.

We see that $\overleftrightarrow{A_1A_2}$ has equation $y = \left(\frac{d-1}{c-1}\right)x + \left(\frac{c-d}{c-1}\right)$, $\overleftrightarrow{A_2A_3}$ has equation $y = x$, $\overleftrightarrow{A_3A_4}$ has equation $y = \left(\frac{b}{a}\right)x$, $\overleftrightarrow{A_1A_4}$ has equation $y = \left(\frac{b-d}{a-c}\right)x + \left(\frac{ad-bc}{a-c}\right)$, $\overleftrightarrow{A_1A_3}$ has equation $y = \left(\frac{d}{c}\right)x$, and $\overleftrightarrow{A_2A_4}$ has equation $y = \left(\frac{b-1}{a-1}\right)x + \left(\frac{a-b}{a-1}\right)$.

Since $\square A_1A_2A_3A_4$ is convex, then the diagonals $\overline{A_1A_3}$ and $\overline{A_2A_4}$ intersect at a point D [5]. We see that D has coordinates

$$D = \left(\frac{ac - bc}{ad - bc + c - d}, \frac{ad - bd}{ad - bc + c - d}\right).$$

The points A_5 and A_6 have coordinates

$$A_5 = \left(\frac{ad - ac}{ad - a - bc + b}, \frac{bd - bc}{ad - a - bc + b}\right) \text{ and } A_6 = \left(\frac{ad - bc}{a - b - c + d}, \frac{ad - bc}{a - b - c + d}\right).$$

Let H_1 and H_2 denote the points on the circumcircles of triangles $\triangle A_2A_3A_5$ and $\triangle A_3A_4A_6$, respectively, that are diametrically opposed to A_3. Let H_3 and H_4 denote the points on the circumcircles of triangles $\triangle A_1A_4A_5$ and $\triangle A_1A_2A_6$, respectively, that are diametrically opposed to A_1 (Figure 4). It is shown in [1] that S is the intersection of lines $\overleftrightarrow{H_1H_2}$ and $\overleftrightarrow{H_3H_4}$. The points H_1 and H_2 have coordinates

$$H_1 = \left(\frac{-ac + ad - bc - bd + 2b}{ad - a - bc + b}, \frac{ac + ad - 2a - bc + bd}{ad - a - bc + b}\right)$$

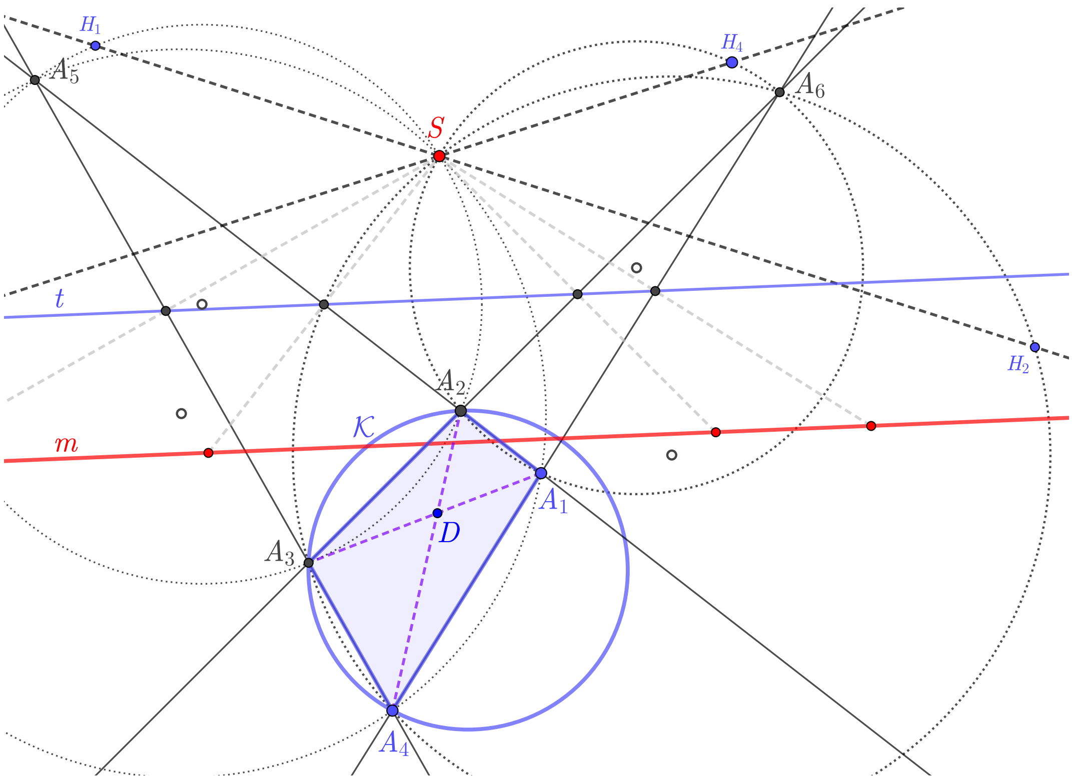

Figure 4. When $\mathcal{Q}$ is not cyclic

and

$$H_2 = \left(\frac{a^2 - ac + ad + b^2 - bc - bd}{a - b - c + d}, \frac{-a^2 + ac + ad - b^2 - bc + bd}{a - b - c + d} \right).$$

Thus, the line $\overleftrightarrow{H_1H_2}$ has equation

$$y = \left(\frac{-a^2 - b^2 + ac + ad - bc + bd - 2a}{a^2 + b^2 - ac + ad - bc - bd + 2b} \right) x + \left(\frac{2a^2 + 2b^2}{a^2 + b^2 - ac + ad - bc - bd + 2b} \right).$$

The points H_3 and H_4 have coordinates

$$H_3 = \left(\frac{a^2d - a^2 - acd + ad + b^2d - b^2 - bd^2 + bd}{ad - a - bc + b}, \frac{-a^2c + a^2 + ac^2 - ac - b^2c + b^2 + bcd - bc}{ad - a - bc + b} \right)$$

and

$$H_4 = \left(\frac{ad + bd - 2b - cd - d^2 + 2d}{a - b - c + d}, \frac{-ac + 2a - bc + c^2 + cd - 2c}{a - b - c + d} \right).$$

Thus, the line $\overleftrightarrow{H_3H_4}$ has equation $y = ux + v$, where

$$u = \frac{-c^3 + 2ac^2 - cd^2 - a^2c - b^2c + 2bcd + a^2 + b^2 + 2c^2 - 3ac - ad - bc - bd + 2cd + 2a - 2c}{d^3 - 2bd^2 + a^2d + b^2d + c^2d - 2acd - a^2 - b^2 - 2d^2 + ac + ad + bc + 3bd - 2cd - 2b + 2d}$$

and

$$v = \frac{-ac^2 + ad^2 + bc^2 - bd^2 + a^2c + a^2d + b^2c + b^2d - 2acd - 2bcd - 2a^2 - 2b^2 + 2ac + 2bd}{d^3 - 2bd^2 + a^2d + b^2d + c^2d - 2acd - a^2 - b^2 - 2d^2 + ac + ad + bc + 3bd - 2cd - 2b + 2d}.$$

Therefore, the point S has coordinates (S_x, S_y), where

$$S_x = \frac{a^2 - ac - ad + 2a + b^2 + bc - bd}{a^2 - 2ac + 2a + b^2 - 2bd + 2b + c^2 - 2c + d^2 - 2d + 2}$$

and
$$S_y = \frac{a^2 - ac + ad + b^2 - bc - bd + 2b}{a^2 - 2ac + 2a + b^2 - 2bd + 2b + c^2 - 2c + d^2 - 2d + 2}.$$
The reflection line m of S has equation
$$y = \left(\frac{-a + c - 1}{b - d + 1}\right) x + \left(\frac{a + b}{b - d + 1}\right).$$
The circumcircle $\mathcal{K}$ of triangle $\triangle A_2A_3A_4$ has equation
$$x^2 - \left(\frac{a^2 + b^2 - 2b}{a - b}\right) x + y^2 - \left(\frac{a^2 + b^2 - 2a}{b - a}\right) y = 0.$$
We have that $\square A_1A_2A_3A_4$ is cyclic if and only if $A_1 = (c, d)$ is on $\mathcal{K}$. The point A_1 is on $\mathcal{K}$ if and only if
$$c^2 - \left(\frac{a^2 + b^2 - 2b}{a - b}\right)(c) + d^2 + \left(\frac{a^2 + b^2 - 2a}{a - b}\right)(d) = 0.$$
Equivalently, A_1 is on $\mathcal{K}$ if and only if
$$\frac{ac^2 - a^2c + a^2d + ad^2 + b^2d - bd^2 - bc^2 - b^2c + 2bc - 2ad}{a - b} = 0$$
which is true if and only if $ac^2 - a^2c + a^2d + ad^2 + b^2d - bd^2 - bc^2 - b^2c + 2bc - 2ad = 0$.

Also, m passes through D if and only if the coordinates of D satisfy the equation of m. Thus, m passes through D if and only if
$$\frac{ad - bd}{ad - bc + c - d} = \left(\frac{-a + c - 1}{b - d + 1}\right)\left(\frac{ac - bc}{ad - bc + c - d}\right) + \left(\frac{a + b}{b - d + 1}\right).$$
Equivalently, m passes through D if and only if
$$\frac{ac^2 - a^2c + a^2d + ad^2 + b^2d - bd^2 - bc^2 - b^2c + 2bc - 2ad}{(b - d + 1)(ad - bc + c - d)} = 0$$
which is true if and only if $ac^2 - a^2c + a^2d + ad^2 + b^2d - bd^2 - bc^2 - b^2c + 2bc - 2ad = 0$.

Hence, quadrilateral $\square A_1A_2A_3A_4$ is cyclic if and only if m passes through D if and only if $ac^2 - a^2c + a^2d + ad^2 + b^2d - bd^2 - bc^2 - b^2c + 2bc - 2ad = 0$. Figure 5 shows when the quadrilateral $\square A_1A_2A_3A_4$ is cyclic.

Now assume that $a = 0$. Since A_1, A_3, and A_4 are not collinear, then it follows that $c \neq 0$. In this case, line $\overleftrightarrow{A_3A_4}$ has equation $x = 0$, line $\overleftrightarrow{A_1A_4}$ has equation $y = \left(\frac{d-b}{c}\right)x + b$, and line $\overleftrightarrow{A_2A_4}$ has equation $y = (1 - b)x + b$. Using an argument similar to the one just given, we have that A_1 is on $\mathcal{K}$ if and only if $b^2d - bd^2 - bc^2 - b^2c + 2bc = 0$, and that m passes through D if and only if $b^2d - bd^2 - bc^2 - b^2c + 2bc = 0$. Thus, we again have that quadrilateral $\square A_1A_2A_3A_4$ is cyclic if and only if m passes through D. The cases where $c = 0$, $a = 1$, $c = 1$, or $a = c$ are similar, and are left to the reader. □

Many of the above calculations were performed using a computer algebra system (the CAS view of GeoGebra).

4. A Characterization of Semi-Symmetric Quadrilaterals

Several characterizations of semi-symmetric cyclic quadrilaterals are given in [1]. It is also shown that if $\square A_1A_2A_3A_4$ is a complete semi-symmetric cyclic quadrilateral with right angles at vertices A_2 and A_4, then the reflection line m of $\square A_1A_2A_3A_4$ passes through A_2 and A_4 (see Figure 2). In this section, we extend this result. In particular, given complete quadrilateral $\square A_1A_2A_3A_4$, we prove that A_2 and A_4 are the midpoints of the reflections of the Simson point S in the sides of $\square A_1A_2A_3A_4$ if and only if $\square A_1A_2A_3A_4$ is a semi-symmetric cyclic quadrilateral with right angles at vertices A_2 and A_4. Thus, Theorem 4.1 below gives a new characterization of semi-symmetric cyclic quadrilaterals.

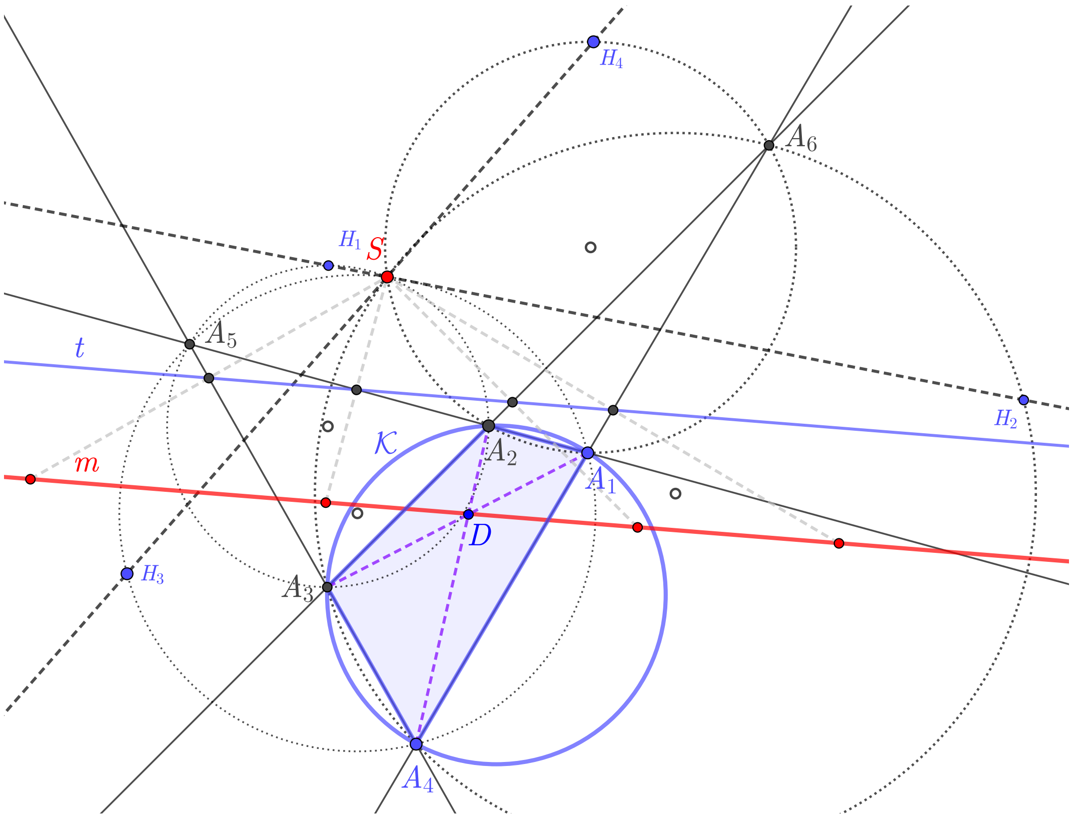

Figure 5. When $\mathcal{Q}$ is cyclic

Lemma 4.1. *Let $\square A_1A_2A_3A_4$ be a complete quadrilateral. Assume that rays $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_4A_3}$ intersect at a point A_5, and that rays $\overrightarrow{A_1A_4}$ and $\overrightarrow{A_2A_3}$ intersect at a point A_6. Let E be a point in the interior of angle $\angle A_5A_3A_6$. Let F_1, F_2, F_3, and F_4 denote the feet of the perpendiculars from E to the lines $\overleftrightarrow{A_1A_2}$, $\overleftrightarrow{A_2A_3}$, $\overleftrightarrow{A_1A_4}$, and $\overleftrightarrow{A_3A_4}$, respectively. Let R_1, R_2, R_3, and R_4 denote the reflections of E in the lines $\overleftrightarrow{A_1A_2}$, $\overleftrightarrow{A_2A_3}$, $\overleftrightarrow{A_1A_4}$, and $\overleftrightarrow{A_3A_4}$, respectively. Then the vertex A_2 is the midpoint of segment $\overline{R_1R_2}$ if and only if $\angle A_1A_2A_3$ is a right angle. Similarly, the vertex A_4 is the midpoint of segment $\overline{R_3R_4}$ if and only if $\angle A_1A_4A_3$ is a right angle.*

Note that the point E is not necessarily the Simson point of $\square A_1A_2A_3A_4$, and that $\square A_1A_2A_3A_4$ is not necessarily cyclic.

Proof. We prove the lemma for the vertex A_2 and the segment $\overline{R_1R_2}$. The proof for the vertex A_4 and the segment $\overline{R_3R_4}$ is similar, and is left to the reader.

First assume that $\angle A_1A_2A_3$ is a right angle (see Figure 6). Since $\angle A_1A_2A_3$ is a right angle and since F_1 and F_2 are the feet of the perpendiculars from E to lines $\overleftrightarrow{A_1A_2}$ and $\overleftrightarrow{A_2A_3}$, respectively, then quadrilateral $\square A_2F_1EF_2$ is a rectangle.

This implies that $\angle R_1ER_2 = \angle F_1EF_2$ is a right angle.

Let M denote the midpoint of side $\overline{R_1R_2}$ of $\triangle ER_1R_2$. Also, we have that F_1 and F_2 are the midpoints of sides $\overline{ER_1}$ and $\overline{ER_2}$, respectively. It follows that $\triangle MF_1R_1$ and $\triangle R_2ER_1$ are similar, which implies that $\angle MF_1R_1$ is a right angle. Also, $\triangle MF_2R_2$ and $\triangle R_1ER_2$ are similar. This implies that $\angle MF_2R_2$ is a right angle. Thus, $\overleftrightarrow{MF_1}$ is the perpendicular from M to $\overleftrightarrow{ER_1}$. Since lines $\overleftrightarrow{A_1A_2}$ and $\overleftrightarrow{MF_1}$ are perpendicular to line $\overleftrightarrow{ER_1}$ at F_1, then $\overleftrightarrow{MF_1}$ = $\overleftrightarrow{A_1A_2}$, which implies that M is on $\overleftrightarrow{A_1A_2}$. Similarly, $\overleftrightarrow{MF_2}$ is the perpendicular from M to $\overleftrightarrow{ER_2}$, which implies that M is on $\overleftrightarrow{A_2A_3}$. Therefore, M is the unique point of intersection of lines $\overleftrightarrow{A_1A_2}$ and $\overleftrightarrow{A_2A_3}$, which implies that $M = A_2$.

Conversely, assume that A_2 is the midpoint of segment $\overline{R_1R_2}$. Suppose to the contrary that $\angle A_1A_2A_3$ is not a right angle. We may assume that $\angle A_1A_2A_3$ is acute. The argument when $\angle A_1A_2A_3$ is obtuse is similar.

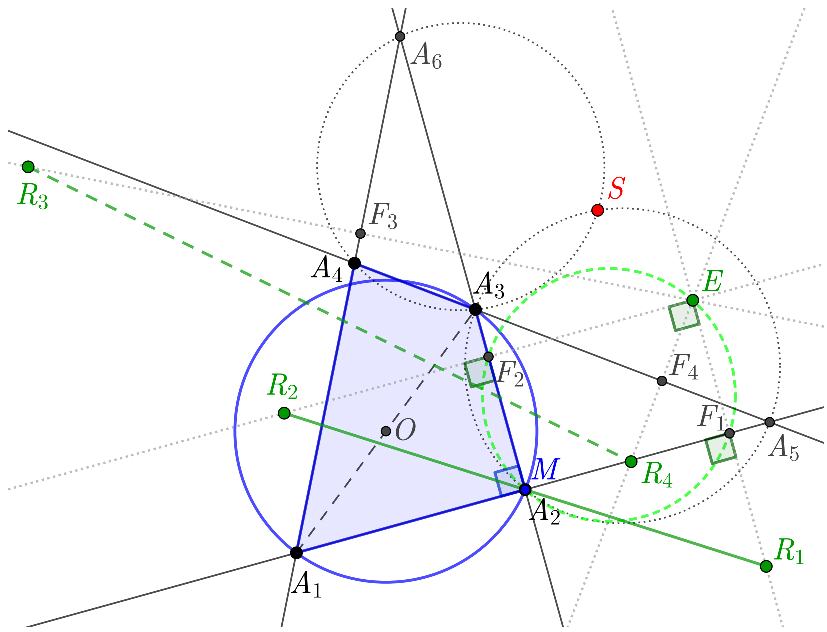

Figure 6. $\angle A_1A_2A_3$ is a right angle

Let A_7 be a point on the same side of line $\overleftrightarrow{A_1A_2}$ as E such that $\angle A_1A_2A_7$ is a right angle (see Figure 7). We may assume without loss of generality that A_7 is a point on ray $\overrightarrow{A_4A_3}$. Since $\angle A_1A_2A_3$ is acute and $\angle A_1A_2A_7$ is a right angle, then we have that $A_4 - A_3 - A_7 - A_5$. In particular, points A_1, A_3, A_4, F_2, A_6, and R_2 are on the same side of line $\overleftrightarrow{A_2A_7}$. Let F_5 denote the foot of the perpendicular from E to $\overleftrightarrow{A_2A_7}$. Let R_5 denote the reflection of E in the line $\overleftrightarrow{A_2A_7}$.

Suppose that E is on ray $\overrightarrow{A_2A_7}$. This implies that A_2 is the foot of the perpendicular from E to line $\overleftrightarrow{A_1A_2}$, which in turn implies that $A_2 = F_1$. It follows that $E - A_2 - R_1$, which implies that lines $\overleftrightarrow{R_1R_2}$ and $\overleftrightarrow{EA_2}$ cross at the point R_1, not at the point A_2. Since this point of intersection is unique, then it follows that $\overleftrightarrow{R_1R_2}$ does not pass through A_2, a contradiction.

Now suppose that A_2 and E are on the same side of line $\overleftrightarrow{A_2A_7}$. Since angles $\angle EF_1A_1$ and $\angle A_7A_2A_1$ are right angles, then lines $\overleftrightarrow{ER_1}$ and $\overleftrightarrow{A_2A_7}$ are parallel. In this case, it follows that R_1 and R_2 are on the same side of line $\overleftrightarrow{A_2A_7}$. This implies that segment $\overline{R_1R_2}$ does not pass through A_2, again a contradiction. In either case, we have a contradiction, and it follows that E and A_3 are on opposite sides of line $\overleftrightarrow{A_2A_7}$.

Let A_8 denote a point on ray $\overrightarrow{A_2A_7}$ such that $A_2 - A_7 - A_8$, and such that A_8 is in the interior of angle $\angle A_4A_1A_3$. Let A_9 denote the point of intersection of rays $\overrightarrow{A_1A_8}$ and segment $\overline{A_4A_3}$. It follows immediately that E is in the interior of $\angle A_5A_7A_8$, and also that $\square A_1A_2A_7A_9$ is a complete quadrilateral. Since $\angle A_1A_2A_7$ is a right angle, then it follows by the argument given above that A_2 is the midpoint of segment $\overline{R_1R_5}$. Moreover, we have that A_3, R_2 and R_5 are all on the same side of line $\overleftrightarrow{A_2A_7}$.

Since both $\overleftrightarrow{R_1R_2}$ and $\overleftrightarrow{R_1R_5}$ pass through A_2, then $\overleftrightarrow{R_1R_2} = \overleftrightarrow{R_1A_2} = \overleftrightarrow{R_1R_5} = \overleftrightarrow{R_2R_5}$. This implies that R_1, R_2, R_5, and A_2 are all collinear. Since A_2 is the midpoint of segment $\overline{R_1R_2}$, then $R_1 - A_2 - R_2$ and $R_1A_2 = R_2A_2$. Similarly, since A_2 is the midpoint of segment $\overline{R_1R_5}$, then $R_1 - A_2 - R_5$ and $R_1A_2 = R_5A_2$. Thus, $R_2 = R_5$, which implies that $\overline{ER_2} = \overline{ER_5}$. Since E is in the interior of $\angle A_5A_3A_6$, then F_2 is on ray $\overrightarrow{A_2A_3}$ and F_5 is on ray $\overrightarrow{A_2A_7}$. Since ray $\overrightarrow{A_2A_3}$ is in the interior of angle $\angle A_1A_2A_7$, then $E - F_5 - F_2 - R_2$. Thus, we have right triangle $\triangle A_2F_2F_5$ with right angles at both vertices F_2 and F_5. Hence, we have a contradiction and it follows that $\angle A_1A_2A_3$ is a right angle. $\square$

Theorem 4.1. *Let $\square A_1A_2A_3A_4$ be a complete quadrilateral. Assume that rays $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_4A_3}$ intersect at a point A_5, and that rays $\overrightarrow{A_1A_4}$ and $\overrightarrow{A_2A_3}$ intersect at a point A_6. Let S denote the Simson point of $\square A_1A_2A_3A_4$. Let F_1, F_2, F_3, and F_4 denote the feet of the perpendiculars from S to the lines $\overleftrightarrow{A_1A_2}$, $\overleftrightarrow{A_2A_3}$, $\overleftrightarrow{A_1A_4}$, and $\overleftrightarrow{A_3A_4}$, respectively. Let R_1, R_2,*

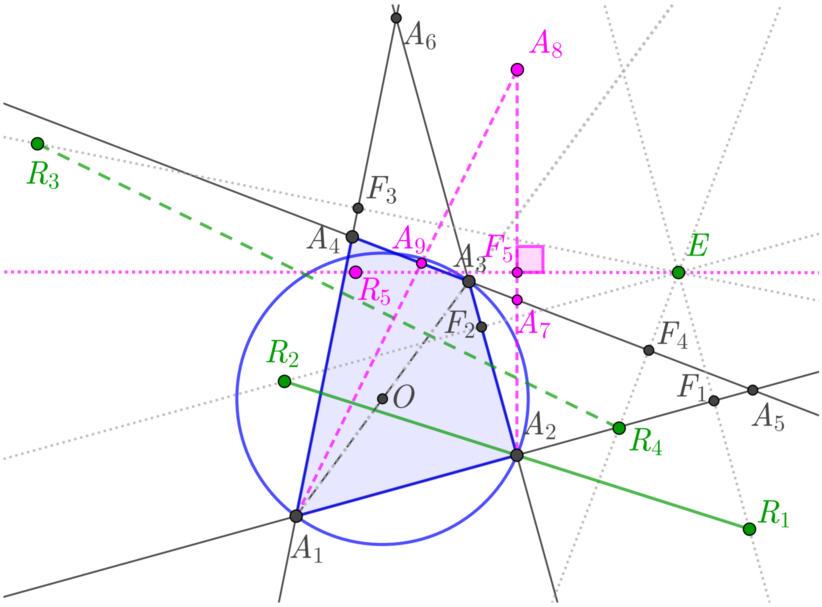

Figure 7. Suppose that $\angle A_1A_2A_3$ is not a right angle

R_3, and R_4 denote the reflections of S in the lines $\overleftrightarrow{A_1A_2}$, $\overleftrightarrow{A_2A_3}$, $\overleftrightarrow{A_1A_4}$, and $\overleftrightarrow{A_3A_4}$, respectively. Then $\square A_1A_2A_3A_4$ is a semi-symmetric cyclic quadrilateral if and only if A_2 and A_4 are the midpoints of segments $\overline{R_1R_2}$ and $\overline{R_3R_4}$, respectively.

Proof. First assume that A_2 and A_4 are the midpoints of segments $\overline{R_1R_2}$ and $\overline{R_3R_4}$, respectively.

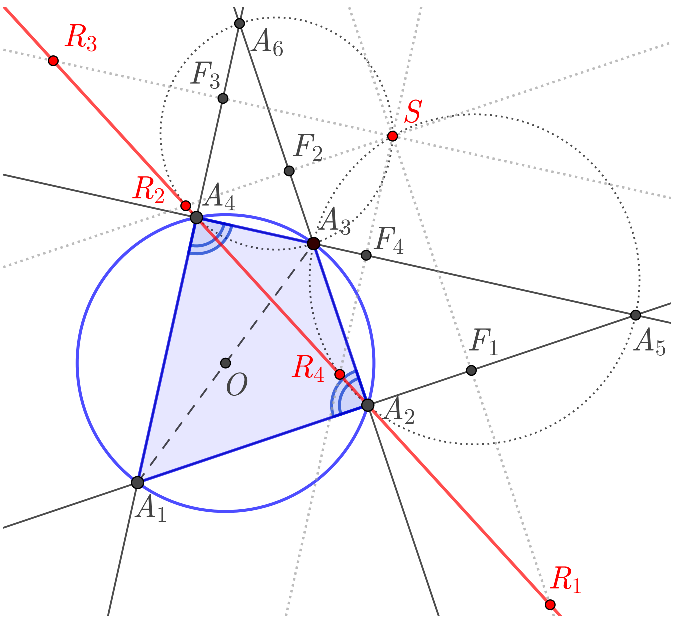

Figure 8. $\square A_1A_2A_3A_4$ is a semi-symmetric cyclic quadrilateral

Since A_2 is the midpoint of segment $\overline{R_1R_2}$, then it follows by Lemma 4.1 that angle $\angle A_1A_2A_3$ is a right angle. Similarly, since A_4 is the midpoint of segment $\overline{R_3R_4}$, then it follows by Lemma 4.1 that angle $\angle A_1A_4A_3$ is a right angle (See Figure 8). Since angles $\angle A_1A_2A_3$ and $\angle A_1A_4A_3$ are right angles, then it follows that $\square A_1A_2A_3A_4$ is a semi-symmetric cyclic quadrilateral [1].

Conversely, assume that $\square A_1A_2A_3A_4$ is a semi-symmetric cyclic quadrilateral. Since $\square A_1A_2A_3A_4$ is a semi-symmetric cyclic quadrilateral, then $\angle A_1A_2A_3$ and $\angle A_1A_4A_3$ are right angles [1]. It follows by Lemma 4.1 that A_2 and A_4 are the midpoints of segments $\overline{R_1R_2}$ and $\overline{R_3R_4}$, respectively. $\square$

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Availability of data and materials

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] Donnelly, J.: *The Reflection Line and Miquel Point of a Cyclic Quadrilateral*. International Journal of Geometry **13** (3), 47-64 (2024).
[2] Ferrarello, D., Mammana, M.F., and Pennisi, M.: *Pedal Polygons*. Forum. Geom. **13**, 153-164 (2013).
[3] Josefsson, M.: *Characterizations of Cyclic Quadrilaterals*. International Journal of Geometry **8** (1), 5-21 (2019).
[4] Josefsson, M.: *More Characterizations of Cyclic Quadrilaterals*. International Journal of Geometry **8** (2), 14-32 (2019).
[5] Martin, G. E.: *The Foundations of Geometry and the Non-Euclidean Plane*. Springer-Verlag, New York, 149-151 (1986).
[6] Micale, B. and Pennisi, M.: *On the altitudes of quadrilaterals*. International Journal of Mathematical Education in Science and Technology **36**, 15-24 (2005). https://doi.org/10.1080/00207390412331283688

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