Erciyes Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
Cilt 41, Sayı 2, 2025

Erciyes University
Journal of Institue of Science and Technology
Volume 41, Issue 2, 2025

Soft Intersection Weak-interior Ideals of Semigroups

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(Alınış / Received: 01.06.2025, Kabul / Accepted: 08.07.2025, Online Yayınlanma / Published Online: 30.08.2025)

Keywords
Soft sets,
Semigroups,
Weak-interior ideals,
Soft intersection weak-interior ideals,
Simple semigroups

Abstract: The idea of generalization of ideals of the algebraic structures has always shown to be interesting for mathematicians. Within this framework, the notion of a weak-interior ideal has been introduced as a generalization of quasi-ideals, interior ideals, (left/right) ideals in the theory of semigroups. In the present work, we extend this concept into the framework of soft set theory applied to semigroups, and introduce a novel type of the soft intersection (ꞩ-intersection) ideal called soft intersection (ꞩ-intersection) weak-interior ideal. The main aim of this study is to investigate the relations of ꞩ-intersection weak-interior ideals with other types of ꞩ-intersection ideals within semigroups. Our results establish that every ꞩ-intersection weak-interior ideal constitutes an ꞩ-intersection subsemigroup within a regular semigroup. Moreover, every ꞩ-intersection left (right) ideal is an ꞩ-intersection left (right) weak-interior ideal, and every ꞩ-intersection interior ideal is an ꞩ-intersection weak-interior ideal. Consequently, the concept of an ꞩ-intersection weak-interior ideal is a generalization of both ꞩ-intersection ideals and ꞩ-intersection interior ideals. However, the converses do not hold in general, as demonstrated through counterexamples. To satisfy the converses, the semigroup should be a group or the ꞩ-intersection weak-interior ideal should be idempotent. Furthermore, we prove that ꞩ-intersection bi-ideals and ꞩ-intersection quasi-ideals coincide with ꞩ-intersection weak-interior ideals within the framework of group structures. Our key theorem is that if a subsemigroup of a semigroup is a weak-interior ideal, then its soft characteristic function is an ꞩ-intersection weak-interior ideal, and vice versa This allows us to build a bridge between semigroup theory and soft set theory.

Yarıgrupların Esnek Kesişimsel Zayıf-iç İdealleri

Anahtar Kelimeler
Esnek kümeler,
Yarıgruplar,
Zayıf-iç idealler,
Esnek kesişimsel zayıf-iç idealler,
Basit yarıgruplar

Öz: Cebirsel yapıların ideallerin genelleştirilmesi matematikçilere her zaman ilginç gelmiştir. Bu çerçevede, zayıf-iç ideal kavramı, yarıgruplar teorisindeki quasi-idealleri, iç idealleri, (sol/sağ) idealleri kapsayan bir genelleme olarak tanıtılmıştır. Mevcut çalışmada, bu yapıyı yarıgruplara uygulanan esnek küme teorisi çerçevesinde genişletip, esnek kesişim (EK) zayıf-iç ideal adı verilen yarıgrupların yeni bir esnek kesişim (EK) ideal türünü tanıtıyoruz. Bu çalışmanın temel amacı, EK-zayıf-iç ideallerinin yarıgruplar içindeki diğer EK-ideal türleriyle ilişkilerini araştırmaktır. Sonuçlarımız, regüler bir yarıgrubun her EK-zayıf-iç idealinin bir EK-alt yarıgrup oluşturduğunu ortaya koymaktadır. Ayrıca, her EK-sol (sağ) ideal bir EK-sol (sağ) zayıf-iç idealdir ve her EK-iç ideal bir EK-zayıf-iç idealdir. Sonuç olarak, EK-zayıf-iç ideal kavramı, EK-ideallerin hem de EK-iç ideallerin bir genellemesidir. Ancak, karşıtları genel olarak geçerli değildir ve bunu gösteren açık karşıt örnekler sunulmuştur. Karşıtlarının sağlanması için, yarıgrubub grup olması veya EK-zayıf-iç idealin idempotent olması gerekmektedir. Ayrıca, EK-bi-ideallerin ve EK-quasi-ideallerin grup yapıları çerçevesinde EK-zayıf-iç ideallerle çakıştığı kanıtlanmıştır. Bir yarı grubun alt yarıgrubu zayıf-iç ideal ise, esnek karakteristik fonksiyonunun EK-zayıf-iç ideal olduğunu ve karşıtının da doğru olduğunu gösteren temel

teoremimiz, yarıgrup teorisi ile esnek küme teorisi arasında bir köprü kurmamızı sağlamıştır.

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1. Introduction

The concept of ideals is central to the comprehension and analysis of algebraic structures and their applications. Therefore, many studies have been conducted on generalizing the ideals of various algebraic structures. The concept of bi-ideals in semigroup theory was first introduced by Good and Hughes [1] in 1952. The concept of quasi-ideals was initially introduced by Steinfeld [2] in semigroups and was subsequently extended to the ring theory. Bi-ideals are generalizations of quasi-ideals, while quasi-ideals are generalizations of left and right ideals. The concept of interior ideals was originally proposed by Lajos [3] and was later extensively examined in the studies of Szasz [4,5]. Interior ideals are generalizations of ideals. Rao [6] recently expanded the existing framework of ideal structures in semigroups and introduced weak-interior ideals of semigroups as a new type of ideal.

In 1999, Molodtsov [7] introduced the "Soft Set Theory" to provide appropriate solutions for problems involving uncertainty. Subsequently, the foundational concepts and operations of soft sets were refined and extended by Çağman and Enginoğlu [8]. The concept of soft intersection (s-intersection) groups, which laid the groundwork for the systematic investigation of a wide variety of soft algebraic systems, was introduced by Çağman et al. [9]. Sezer et al. [10,11], employing soft sets in semigroup theory, introduced the concepts of s-intersection substructures of semigroups, and conducting a comprehensive examination of the fundamental properties and structural characteristics of these soft algebraic constructs. Furthermore, in this context, Sezgin and Orbay [12] introduced and classified specific types of semigroups by s-intersection substructures of semigroups.

Rao [6] introduced the concept of weak-interior ideals in semigroups as a generalization of quasi-ideals, interior ideals, left (right) ideals, and ideals and studied the properties. The concept of weak-interior ideals has also been studied by Rao [13] for Γ-semirings, Rao and Rao [14] for Γ-semigroups, and Rao [15] for semirings. In the present work, we extend this concept into the framework of soft set theory applied to semigroups, and introduce a novel type of the s-intersection ideal called s-intersection weak-interior ideal. The main objective of this study is to explore how s-intersection weak-interior ideals relate to other types of s-intersection ideals within semigroups. Our results establish that every s-intersection weak-interior ideal constitutes an s-intersection subsemigroup within a regular semigroup. Moreover, every s-intersection left (right) ideal is an s-intersection left (right) weak-interior ideal, and every s-intersection interior ideal is an s-intersection weak-interior ideal. Consequently, the concept of an s-intersection weak-interior ideal is a generalization of both s-intersection ideals and s-intersection interior ideals. However, the converses do not hold in general and we present these with explicit counterexamples. To ensure the converses, it is shown that the semigroup should be group or the s-intersection weak-interior ideal should be idempotent. Furthermore, we prove that s-intersection bi-ideals and s-intersection quasi-ideals coincide with s-intersection weak-interior ideals within the framework of group structures. We also obtain that if a subsemigroup of a semigroup is a weak-interior ideal, then its soft characteristic function is an s-intersection weak-interior ideal, and vice versa.

The paper is structured into four sections. Section 1 offers an introductory overview of the topic. Section 2 explores the foundational concepts of semigroups and soft set ideals, presenting their definitions, the properties and implications. In Section 3, the concept of s-intersection weak-interior ideals was introduced, conduct a comprehensive analysis of their properties, investigate their interrelations with other classes of s-intersection ideals, substantiated with concrete examples. Section 4 concludes with a summary of the main findings and outlines directions for future research.

2. Material and Method

This section provides a brief review of the fundamental concepts related to soft sets and semigroups.

Throughout this paper, let S denote a semigroup. $\emptyset \neq L \subseteq S$ is called a subsemigroup (briefly SS) of S if $LL \subseteq L$, is called a left weak-interior ideal of S if $LL \subseteq L$ and $SLL \subseteq L$, is called a right weak-interior ideal of S if $LL \subseteq L$ and $LLS \subseteq L$, and is called a weak-interior ideal of S if it is both left weak-interior ideal and right weak-interior ideal [6]. A semigroup S is called a regular semigroup, if for all $v \in S$, there exists an element $d \in S$ such that $v = vdv$. A semigroup S is called a left (right) simple (L(R)-simple) if it contains no proper left (right) ideal of S and is called a simple if it contains no proper ideal.

Theorem 2.1 [16,17]. For S, we have the following:

(1) S is L(R)-simple iff $Sk = S$ $(kS = S)$ for all $k \in S$. That is, for every $\mathfrak{n}, \mathfrak{p} \in S$, there exists $\mathfrak{u} \in S$ such that $\mathfrak{p} = \mathfrak{u}\mathfrak{n}$ $(\mathfrak{p} = \mathfrak{n}\mathfrak{u})$

(2) S is both L-simple and R-simple iff S is a group.

Definition 2.2 [7,8]. Let E be a set of parameters, U be a universal set, $P(U)$ be the power set of U, and $\Upsilon \subseteq E$. The soft set f_Υ over U is a function such that $f_\Upsilon: E \to P(U)$, where for all $x \notin \Upsilon$, $f_\Upsilon(x) = \emptyset$. That is,

$$f_\Upsilon = \{(x, f_\Upsilon(x)) : x \in E\}$$

The set of all soft sets over U with the fixed parameter set S, where S is a semigroup, is designated by $S_S(U)$ throughout this paper. For more about the concepts related to soft sets, we refer to [8,9].

Definition 2.3 [10]. Let $\hbar_S, \mathfrak{G}_S \in S_S(U)$. Then, the soft intersection product $\hbar_S \circ \mathfrak{G}_S$ is defined by

$$(\hbar_S \circ \mathfrak{G}_S)(\eta) = \begin{cases} \bigcup_{\eta = \mathfrak{w}\mathfrak{d}} \{\hbar_S(\mathfrak{w}) \cap \mathfrak{G}_S(\mathfrak{d})\}, & if\ \exists \mathfrak{w}, \mathfrak{d} \in S\ such\ that\ \eta = \mathfrak{w}\mathfrak{d} \\ \emptyset, & otherwise \end{cases}$$

Henceforth, we shall use the term "$\mathfrak{s}$-intersection" instead of "soft intersection".

Theorem 2.4 [10]. Let $p_S, \omega_S, \mu_S \in S_S(U)$. Then,

i. $(p_S \circ \omega_S) \circ \mu_S = p_S \circ (\omega_S \circ \mu_S)$
ii. $p_S \circ \omega_S \neq p_S \circ \omega_S$
iii. $p_S \circ (\omega_S \tilde{\cup} \mu_S) = (p_S \circ \omega_S) \tilde{\cup} (p_S \circ \mu_S)$ and $(p_S \tilde{\cup} \omega_S) \circ \mu_S = (p_S \circ \mu_S) \tilde{\cup} (\omega_S \circ \mu_S)$
iv. If $p_S \tilde{\subseteq} \omega_S$, then $p_S \circ \mu_S \tilde{\subseteq} \omega_S \circ \mu_S$ and $\mu_S \circ p_S \tilde{\subseteq} \mu_S \circ \omega_S$
v. If $\mathfrak{H}_S, y_S \in S_S(U)$ such that $\mathfrak{H}_S \tilde{\subseteq} p_S$ and $y_S \tilde{\subseteq} \omega_S$, then $\mathfrak{H}_S \circ y_S \tilde{\subseteq} p_S \circ \omega_S$

Definition 2.5 [10]. Let $\emptyset \neq \mathfrak{B} \subseteq S$. The soft characteristic function of $\mathfrak{B}$, denoted by $S_\mathfrak{B}$, is defined by

$$S_\mathfrak{B}(\mathfrak{z}) = \begin{cases} U, & if\ \mathfrak{z} \in \mathfrak{B} \\ \emptyset, & if\ \mathfrak{z} \in S \backslash \mathfrak{B} \end{cases}$$

Theorem 2.6 [10,18]. Let $\mathcal{H}, \mathcal{M} \subseteq S$. Then,

i. $\mathcal{H} \subseteq \mathcal{M}$ iff $S_\mathcal{H} \tilde{\subseteq} S_\mathcal{M}$
ii. $S_\mathcal{H} \tilde{\cap} S_\mathcal{M} = S_{\mathcal{H} \cap \mathcal{M}}$ and $S_\mathcal{H} \tilde{\cup} S_\mathcal{M} = S_{\mathcal{H} \cup \mathcal{M}}$
iii. $S_\mathcal{H} \circ S_\mathcal{M} = S_{\mathcal{H}\mathcal{M}}$

For more about $\mathfrak{s}$-intersection ideals, we refer to [10,11].

Theorem 2.7 [10]. Let $\varkappa_S \in S_S(U)$. Then,

i) $\tilde{\mathbb{S}} \circ \tilde{\mathbb{S}} \tilde{\subseteq} \tilde{\mathbb{S}}$
ii) $\tilde{\mathbb{S}} \circ \varkappa_S \tilde{\subseteq} \tilde{\mathbb{S}}$ and $\varkappa_S \circ \tilde{\mathbb{S}} \tilde{\subseteq} \tilde{\mathbb{S}}$
iii) $\varkappa_S \tilde{\cup} \tilde{\mathbb{S}} = \tilde{\mathbb{S}}$ and $\varkappa_S \tilde{\cap} \tilde{\mathbb{S}} = \varkappa_S$

Proposition 2.8 [12]. Every $\mathfrak{s}$-intersection bi-ideal is an $\mathfrak{s}$-intersection right ideal of an L-simple semigroup.

For more about soft sets, we refer to [19-25].

3. Results

This section presents the introduction of $\mathfrak{s}$-intersection weak-interior ideals of semigroups, provides illustrative examples, offers a detailed examination of their interrelations with other $\mathfrak{s}$-intersection ideals.

Definition 3.1. A soft set f_S over U is called soft intersection ($\mathfrak{s}$-intersection) left (right) weak-interior ideal of S over U if $f_S(xyz) \supseteq f_S(y) \cap f_S(z)$ $\left(f_S(xyz) \supseteq f_S(x) \cap f_S(y)\right)$ for all $x, y, z \in S$. A soft set over U is called an $\mathfrak{s}$-intersection weak-interior ideal of S if it is both $\mathfrak{s}$-intersection left weak-interior ideal and $\mathfrak{s}$-intersection right weak-interior ideal of S over U.

Hereafter, "weak-interior" is abbreviated by "WI", "ideal" is abbreviated by "*id*", and the phrases such "of S over U" is not used.

Example 3.2. Let $S = \{\varkappa, \mathfrak{j}, \rho, \sigma\}$ be:

Table 1. Cayley Table of "·" binary operation.

·	$\varkappa$	j	ρ	σ
$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$
j	$\varkappa$	$\varkappa$	$\varkappa$	$\varkappa$
ρ	$\varkappa$	$\varkappa$	j	$\varkappa$
σ	$\varkappa$	$\varkappa$	j	j

Let $\mathfrak{f}_S$ and $\mathfrak{v}_S$ be soft sets over $U = S_3$ as follows:

$$\mathfrak{f}_S = \{(\varkappa, \{(12), (13), (23)\}), (\text{j}, \{(12)\}), (\rho, \{(13)\}), (\sigma, \{(23)\})\}$$
$$\mathfrak{v}_S = \{(\varkappa, \{(1), (123), (132)\}), (\text{j}, \{(23)\}), (\rho, \{(13)\}), (\sigma, \{(12)\})\}$$

Here, $\mathfrak{f}_S$ is an $\mathfrak{s}$-intersection left WI-*id*. For example, for a few elements of S,

$$\mathfrak{f}_S(\varkappa \text{j} \rho) = \mathfrak{f}_S(\varkappa) \supseteq \mathfrak{f}_S(\text{j}) \cap \mathfrak{f}_S(\rho)$$
$$\mathfrak{f}_S(\sigma\sigma\sigma) = \mathfrak{f}_S(\varkappa) \supseteq \mathfrak{f}_S(\sigma) \cap \mathfrak{f}_S(\sigma)$$
$$\mathfrak{f}_S(\rho\rho\varkappa) = \mathfrak{f}_S(\varkappa) \supseteq \mathfrak{f}_S(\rho) \cap \mathfrak{f}_S(\varkappa)$$

Similarly, $\mathfrak{f}_S$ is an $\mathfrak{s}$-intersection right WI-*id*. For example, for a few elements of S,

$$\mathfrak{f}_S(\text{j}\rho\sigma) = \mathfrak{f}_S(\varkappa) \supseteq \mathfrak{f}_S(\text{j}) \cap \mathfrak{f}_S(\rho)$$
$$\mathfrak{f}_S(\sigma\sigma\rho) = \mathfrak{f}_S(\varkappa) \supseteq \mathfrak{f}_S(\sigma) \cap \mathfrak{f}_S(\sigma)$$
$$\mathfrak{f}_S(\sigma\text{j}) = \mathfrak{f}_S(\varkappa) \supseteq \mathfrak{f}_S(\sigma) \cap \mathfrak{f}_S(\rho)$$

It can be shown that for the remaining elements of S, the condition is satisfied. Thus, $\mathfrak{f}_S$ is an $\mathfrak{s}$-intersection WI-*id*. However, since

$$\mathfrak{v}_S(\rho\rho\rho) = \mathfrak{v}_S(\varkappa) \not\supseteq \mathfrak{v}_S(\rho) \cap \mathfrak{v}_S(\rho)$$

$\mathfrak{v}_S$ is not an $\mathfrak{s}$-intersection WI-*id*.

Proposition 3.3. Let $\mathfrak{p}_S \in S_S(U)$. Then, $\mathfrak{p}_S$ is an $\mathfrak{s}$-intersection left WI-*id* iff $\widetilde{\mathbb{S}} \circ \mathfrak{p}_S \circ \mathfrak{p}_S \cong \mathfrak{p}_S$.

Proof: Let $\mathfrak{p}_S$ be an $\mathfrak{s}$-intersection left WI-*id* and $\mathfrak{b} \in S$. If $\left(\widetilde{\mathbb{S}} \circ \mathfrak{p}_S \circ \mathfrak{p}_S\right)(\mathfrak{b}) = \emptyset$, then $\widetilde{\mathbb{S}} \circ \mathfrak{p}_S \circ \mathfrak{p}_S \cong \mathfrak{p}_S$. Otherwise, there exist elements $x, y, v, q \in S$ such that $\mathfrak{b} = xy$ and $x = vq$, for $\mathfrak{b} \in S$. Since $\mathfrak{p}_S$ is an $\mathfrak{s}$-intersection left WI-*id*, we have

$$\mathfrak{p}_S(\mathfrak{b}) = \mathfrak{p}_S(xy) = \mathfrak{p}_S\big((vq)y\big) \supseteq \mathfrak{p}_S(q) \cap \mathfrak{p}_S(y)$$

Therefore,

$$\begin{aligned}
\left(\widetilde{\mathbb{S}} \circ \mathfrak{p}_S \circ \mathfrak{p}_S\right)(\mathfrak{b}) &= \left[\left(\widetilde{\mathbb{S}} \circ \mathfrak{p}_S\right) \circ \mathfrak{p}_S\right](\mathfrak{b}) \\
&= \bigcup_{\mathfrak{b}=xy} \left\{\left(\widetilde{\mathbb{S}} \circ \mathfrak{p}_S\right)(x) \cap \mathfrak{p}_S(y)\right\} \\
&= \bigcup_{\mathfrak{b}=xy} \left\{ \bigcup_{x=vq} \left\{\widetilde{\mathbb{S}}(v) \cap \mathfrak{p}_S(q)\right\} \cap \mathfrak{p}_S(y) \right\} \\
&= \bigcup_{\mathfrak{b}=vqy} \left\{\mathfrak{p}_S(q) \cap \mathfrak{p}_S(y)\right\} \\
&\subseteq \bigcup_{\mathfrak{b}=vqy} \left\{\mathfrak{p}_S(vqy)\right\} \\
&= \mathfrak{p}_S(xy) \\
&= \mathfrak{p}_S(\mathfrak{b})
\end{aligned}$$

Thus, we have $\widetilde{\mathbb{S}} \circ \mathfrak{p}_S \circ \mathfrak{p}_S \cong \mathfrak{p}_S$. Moreover, in the case where $\mathfrak{b} = xy$ and $x \neq vq$ for $\mathfrak{b} \in S$, since $\left(\widetilde{\mathbb{S}} \circ \mathfrak{p}_S\right)(x) = \emptyset$, $\widetilde{\mathbb{S}} \circ \mathfrak{p}_S \circ \mathfrak{p}_S \cong \mathfrak{p}_S$ is satisfied.

Conversely, assume that $\widetilde{\mathbb{S}} \circ \mathfrak{p}_S \circ \mathfrak{p}_S \cong \mathfrak{p}_S$. Let $a = xyz$ for $a, x, y, z \in S$. Then, we have

$$\begin{aligned}
\mathfrak{p}_S(xyz) &= \mathfrak{p}_S(a) \\
&\supseteq \left(\widetilde{\mathbb{S}} \circ \mathfrak{p}_S \circ \mathfrak{p}_S\right)(a) \\
&= \left[\left(\widetilde{\mathbb{S}} \circ \mathfrak{p}_S\right) \circ \mathfrak{p}_S\right](a) \\
&= \bigcup_{a=mn} \left\{\left(\widetilde{\mathbb{S}} \circ \mathfrak{p}_S\right)(m) \cap \mathfrak{p}_S(n)\right\} \\
&\supseteq \left(\widetilde{\mathbb{S}} \circ \mathfrak{p}_S\right)(xy) \cap \mathfrak{p}_S(z) \\
&= \bigcup_{xy=pq} \left\{\widetilde{\mathbb{S}}(p) \cap \mathfrak{p}_S(q)\right\} \cap \mathfrak{p}_S(z) \\
&\supseteq \left[\widetilde{\mathbb{S}}(x) \cap \mathfrak{p}_S(y)\right] \cap \mathfrak{p}_S(z) \\
&= [U \cap \mathfrak{p}_S(y)] \cap \mathfrak{p}_S(z) \\
&= \mathfrak{p}_S(y) \cap \mathfrak{p}_S(z)
\end{aligned}$$

Hence, $\mathfrak{p}_S(xyz) \supseteq \mathfrak{p}_S(y) \cap \mathfrak{p}_S(z)$, implying that $\mathfrak{p}_S$ is an $\mathfrak{s}$-intersection left WI-*id*.

Proposition 3.4. Let $\mathfrak{p}_S \in S_S(U)$. Then, $\mathfrak{p}_S$ is an $\mathfrak{s}$-intersection right WI-*id* iff $\mathfrak{p}_S \circ \mathfrak{p}_S \circ \widetilde{\mathbb{S}} \cong \mathfrak{p}_S$.
Proof: Let $\mathfrak{p}_S$ be an $\mathfrak{s}$-intersection right WI-*id* and $\mathfrak{v} \in S$. If $(\mathfrak{p}_S \circ \mathfrak{p}_S \circ \widetilde{\mathbb{S}})(\mathfrak{v}) = \emptyset$, then $\mathfrak{p}_S \circ \mathfrak{p}_S \circ \widetilde{\mathbb{S}} \cong \mathfrak{p}_S$. Otherwise, there exist elements $x, y, r, q \in S$ such that $\mathfrak{v} = xy$ and $y = rq$, for $\mathfrak{v} \in S$. Since $\mathfrak{p}_S$ is an $\mathfrak{s}$-intersection right WI-*id*, we have

$$\mathfrak{p}_S(\mathfrak{v}) = \mathfrak{p}_S(xy) = \mathfrak{p}_S(x(rq)) \supseteq \mathfrak{p}_S(x) \cap \mathfrak{p}_S(r)$$

Thus,

$$\begin{aligned}
(\mathfrak{p}_S \circ \mathfrak{p}_S \circ \widetilde{\mathbb{S}})(\mathfrak{v}) &= [\mathfrak{p}_S \circ (\mathfrak{p}_S \circ \widetilde{\mathbb{S}})](\mathfrak{v}) \\
&= \bigcup_{\mathfrak{v}=xy} \{\mathfrak{p}_S(x) \cap (\mathfrak{p}_S \circ \widetilde{\mathbb{S}})(y)\} \\
&= \bigcup_{\mathfrak{v}=xy} \left\{ \mathfrak{p}_S(x) \cap \bigcup_{y=rq} \{\mathfrak{p}_S(r) \cap \widetilde{\mathbb{S}}(q)\} \right\} \\
&= \bigcup_{\mathfrak{v}=xrq} \{\mathfrak{p}_S(x) \cap \mathfrak{p}_S(r)\} \\
&\subseteq \bigcup_{\mathfrak{v}=xrq} \{\mathfrak{p}_S(xrq)\} \\
&= \mathfrak{p}_S(xy) \\
&= \mathfrak{p}_S(\mathfrak{v})
\end{aligned}$$

Hence, we have $\mathfrak{p}_S \circ \mathfrak{p}_S \circ \widetilde{\mathbb{S}} \cong \mathfrak{p}_S$. Moreover, in the case where $\mathfrak{v} = xy$ and $y \neq rq$ for $\mathfrak{v} \in S$, since $(\mathfrak{p}_S \circ \widetilde{\mathbb{S}})(y) = \emptyset$, $\mathfrak{p}_S \circ \mathfrak{p}_S \circ \widetilde{\mathbb{S}} \cong \mathfrak{p}_S$ is satisfied.
Conversely, let $\mathfrak{p}_S \circ \mathfrak{p}_S \circ \widetilde{\mathbb{S}} \cong \mathfrak{p}_S$. Let $\mathfrak{v} = \mathfrak{net}$ for $\mathfrak{v}, \mathfrak{n}, \mathfrak{e}, \mathfrak{t} \in S$. Then, we have

$$\begin{aligned}
\mathfrak{p}_S(\mathfrak{net}) &= \mathfrak{p}_S(\mathfrak{v}) \\
&\supseteq (\mathfrak{p}_S \circ \mathfrak{p}_S \circ \widetilde{\mathbb{S}})(\mathfrak{v}) \\
&= [\mathfrak{p}_S \circ (\mathfrak{p}_S \circ \widetilde{\mathbb{S}})](\mathfrak{v}) \\
&= \bigcup_{\mathfrak{v}=mn} \{\mathfrak{p}_S(m) \cap (\mathfrak{p}_S \circ \widetilde{\mathbb{S}})(n)\} \\
&\supseteq \mathfrak{p}_S(\mathfrak{n}) \cap (\mathfrak{p}_S \circ \widetilde{\mathbb{S}})(\mathfrak{et}) \\
&= \mathfrak{p}_S(\mathfrak{n}) \cap \bigcup_{\mathfrak{et}=pq} \{\mathfrak{p}_S(p) \cap \widetilde{\mathbb{S}}(q)\} \\
&\supseteq \mathfrak{p}_S(\mathfrak{n}) \cap [\mathfrak{p}_S(\mathfrak{e}) \cap \widetilde{\mathbb{S}}(\mathfrak{t})] \\
&= \mathfrak{p}_S(\mathfrak{n}) \cap [\mathfrak{p}_S(\mathfrak{e}) \cap U] \\
&= \mathfrak{p}_S(\mathfrak{n}) \cap \mathfrak{p}_S(\mathfrak{e})
\end{aligned}$$

Therefore, $\mathfrak{p}_S(\mathfrak{net}) \supseteq \mathfrak{p}_S(\mathfrak{n}) \cap \mathfrak{p}_S(\mathfrak{e})$, implying that $\mathfrak{p}_S$ is an $\mathfrak{s}$-intersection right WI-*id*.

Theorem 3.5. Let $\mathfrak{p}_S \in S_S(U)$. Then, $\mathfrak{p}_S$ is an $\mathfrak{s}$-intersection WI-*id* iff $\widetilde{\mathbb{S}} \circ \mathfrak{p}_S \circ \mathfrak{p}_S \cong \mathfrak{p}_S$ and $\mathfrak{p}_S \circ \mathfrak{p}_S \circ \widetilde{\mathbb{S}} \cong \mathfrak{p}_S$.
Proof: It is followed by Proposition 3.3 and Proposition 3.4.

Corollary 3.6. $\widetilde{\mathbb{S}}$ and $\emptyset_S$ are $\mathfrak{s}$-intersection WI-*id*s.

Theorem 3.7. Let $\mathfrak{B}$ be an $\mathfrak{SS}$. Then, $\mathfrak{B}$ is a WI-*id* iff $S_{\mathfrak{B}}$ is an $\mathfrak{s}$-intersection WI-*id*.
Proof: Let $\mathfrak{B}$ be a WI-*id*. Then, $S\mathfrak{B}\mathfrak{B} \subseteq \mathfrak{B}$ and $\mathfrak{B}\mathfrak{B}S \subseteq \mathfrak{B}$. By Theorem 2.12, we have

$$\widetilde{\mathbb{S}} \circ S_{\mathfrak{B}} \circ S_{\mathfrak{B}} = S_S \circ S_{\mathfrak{B}} \circ S_{\mathfrak{B}} = S_{S\mathfrak{B}\mathfrak{B}} \cong S_{\mathfrak{B}} \text{ and } S_{\mathfrak{B}} \circ S_{\mathfrak{B}} \circ \widetilde{\mathbb{S}} = S_{\mathfrak{B}} \circ S_{\mathfrak{B}} \circ S_S = S_{\mathfrak{B}\mathfrak{B}S} \cong S_{\mathfrak{B}}$$

Hence, $S_{\mathfrak{B}}$ is an $\mathfrak{s}$-intersection WI-*id*.
Conversely, let $S_{\mathfrak{B}}$ be an $\mathfrak{s}$-intersection WI-*id* and $\mathfrak{B}$ be an $\mathfrak{SS}$. Then, $\widetilde{\mathbb{S}} \circ S_{\mathfrak{B}} \circ S_{\mathfrak{B}} \cong S_{\mathfrak{B}}$ and $S_{\mathfrak{B}} \circ S_{\mathfrak{B}} \circ \widetilde{\mathbb{S}} \cong S_{\mathfrak{B}}$. Let $x \in S\mathfrak{B}\mathfrak{B}$. Then,

$$S_{\mathfrak{B}}(x) \supseteq (\widetilde{\mathbb{S}} \circ S_{\mathfrak{B}} \circ S_{\mathfrak{B}})(x) = (S_S \circ S_{\mathfrak{B}} \circ S_{\mathfrak{B}})(x) = S_{S\mathfrak{B}\mathfrak{B}}(x) = U$$

Thus, $S_{\mathfrak{B}}(x) = U$ and so $x \in \mathfrak{B}$, implying that $S\mathfrak{B}\mathfrak{B} \subseteq \mathfrak{B}$. Hence, $\mathfrak{B}$ is a left WI-*id*. Similarly, let $y \in \mathfrak{B}\mathfrak{B}S$. Then,

$$S_{\mathfrak{B}}(y) \supseteq (S_{\mathfrak{B}} \circ S_{\mathfrak{B}} \circ \widetilde{\mathbb{S}})(y) = (S_{\mathfrak{B}} \circ S_{\mathfrak{B}} \circ S_S)(y) = S_{\mathfrak{B}\mathfrak{B}S}(y) = U$$

Thus, $S_{\mathfrak{B}}(y) = U$, and so $y \in \mathfrak{B}$, implying that $\mathfrak{B}\mathfrak{B}S \subseteq \mathfrak{B}$. Hence, $\mathfrak{B}$ is a right WI-*id*. Therefore, $\mathfrak{B}$ is a WI-*id*.

Example 3.8. Consider the semigroup in Example 3.2. Then, $\mathfrak{P} = \{\mathfrak{n}, \mathfrak{j}, \mathfrak{p}\}$ is a WI-*id*. Hence, $S_{\mathfrak{P}} = \{(\mathfrak{n}, U), (\mathfrak{j}, U), (\mathfrak{p}, U), (\mathfrak{o}, \emptyset)\}$ is an $\mathfrak{s}$-intersection WI-*id*. Conversely, by choosing the $\mathfrak{s}$-intersection WI-*id* as $f_S = \{(\mathfrak{n}, U), (\mathfrak{j}, U), (\mathfrak{p}, U), (\mathfrak{o}, \emptyset)\}$, which is the soft characteristic function of $\mathfrak{R} = \{\mathfrak{n}, \mathfrak{j}, \mathfrak{p}\}$, one can show that $\mathfrak{R}$ is a WI-*id*.

Proposition 3.9. Every $\mathfrak{s}$-intersection left WI-*id* is an $\mathfrak{s}$-intersection $\mathfrak{SS}$ of a regular semigroup.

Proof: Let $\mathfrak{f}_S$ be an s-intersection WI-*id* of a regular S and $u, \varrho \in S$. By assumption, for all $u \in S$, there exists $\hbar \in S$ such that $u = u\hbar u$. Thus,

$$\mathfrak{f}_S(u\varrho) = \mathfrak{f}_S\big((u\hbar u)\varrho\big) = \mathfrak{f}_S\big((u\hbar)u\varrho\big) \supseteq \mathfrak{f}_S(u) \cap \mathfrak{f}_S(\varrho)$$

Hence, $\mathfrak{f}_S$ is an s-intersection $\mathbb{SS}$.

Proposition 3.10. Every s-intersection right WI-*id* is an s-intersection $\mathbb{SS}$ of a regular semigroup.
Proof: Let $\mathfrak{f}_S$ be an s-intersection right WI-*id* of a regular S and $\upsilon, \eta \in S$. Then, for all $\eta \in S$, there exists $x \in S$ such that $\eta = \eta x \eta$. Thus,

$$\mathfrak{f}_S(\upsilon\eta) = \mathfrak{f}_S\big(\upsilon(\eta x\eta)\big) = \mathfrak{f}_S\big(\upsilon\eta(x\eta)\big) \supseteq \mathfrak{f}_S(\upsilon) \cap \mathfrak{f}_S(\eta)$$

Hence, $\mathfrak{f}_S$ is an s-intersection $\mathbb{SS}$.

Theorem 3.11. Every s-intersection WI-*id* is an s-intersection $\mathbb{SS}$ of a regular semigroup.
Proof: It is followed by Proposition 3.9 and Proposition 3.10.

Proposition 3.12. Every s-intersection left *id* is an s-intersection left WI-*id*.
Proof: Let $\mathfrak{f}_S$ be an s-intersection left *id*. Then, $\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \cong \mathfrak{f}_S$ and $\mathfrak{f}_S \circ \mathfrak{f}_S \cong \mathfrak{f}_S$. Thus,

$$\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \mathfrak{f}_S \cong \mathfrak{f}_S \circ \mathfrak{f}_S \cong \mathfrak{f}_S$$

Hence, $\mathfrak{f}_S$ is an s-intersection left WI-*id*.

Example 3.13. Consider the soft set $\mathfrak{f}_S$ in Example 3.2. As seen in Example 3.2, $\mathfrak{f}_S$ is an s-intersection left WI-*id*. However, since $\mathfrak{f}_S(\rho\rho) = \mathfrak{f}_S(\mathrm{j}) \not\supseteq \mathfrak{f}_S(\rho)$, $\mathfrak{f}_S$ is not an s-intersection left *id*.

Proposition 3.14. Let $\mathfrak{f}_S \in S_S(U)$ and S be an L-simple. Then, the following conditions are equivalent:
1. $\mathfrak{f}_S$ is an s-intersection left *id*.
2. $\mathfrak{f}_S$ is an s-intersection left WI-*id*.

Proof: (1) implies (2) is by Proposition 3.12. Let $\mathfrak{f}_S$ be an s-intersection left WI-*id* and $\hbar, \eta \in S$. By assumption, there exists $x \in S$ such that $\hbar = x\eta$. Thus,

$$\mathfrak{f}_S(\hbar\eta) = \mathfrak{f}_S\big((x\eta)\eta\big) = \mathfrak{f}_S(x\eta\eta) \supseteq \mathfrak{f}_S(\eta) \cap \mathfrak{f}_S(\eta) = \mathfrak{f}_S(\eta)$$

Thus, $\mathfrak{f}_S$ is an s-intersection left *id*.

Proposition 3.15. Let $\mathfrak{f}_S$ be an idempotent soft set. Then, the following conditions are equivalent:
1. $\mathfrak{f}_S$ is an s-intersection left *id*.
2. $\mathfrak{f}_S$ is an s-intersection left WI-*id*.

Proof: (1) implies (2) is by Proposition 3.12. Let $\mathfrak{f}_S$ be an s-intersection left WI-*id*. Since $\mathfrak{f}_S$ is an idempotent s-intersection left WI-*id*, we have

$$\widetilde{\mathbb{S}} \circ \mathfrak{f}_S = \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \mathfrak{f}_S \cong \mathfrak{f}_S$$

Hence, $\mathfrak{f}_S$ is an s-intersection left *id*.

From here, it is obvious that any idempotent s-intersection left WI-*id* coincides with the s-intersection left *id*.

Proposition 3.16. Every s-intersection right *id* is an s-intersection right WI-*id*.
Proof: Let $\mathfrak{f}_S$ be an s-intersection right *id*. Then, $\mathfrak{f}_S \circ \widetilde{\mathbb{S}} \cong \mathfrak{f}_S$ and $\mathfrak{f}_S \circ \mathfrak{f}_S \cong \mathfrak{f}_S$. Thus,

$$\mathfrak{f}_S \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \cong \mathfrak{f}_S \circ \mathfrak{f}_S \cong \mathfrak{f}_S$$

Therefore, $\mathfrak{f}_S$ is an s-intersection right WI-*id*.

Example 3.17. Consider the soft set $\mathfrak{f}_S$ in Example 3.2. As seen in Example 3.2, $\mathfrak{f}_S$ is an s-intersection right WI-*id*. However, since $\mathfrak{f}_S(\rho\rho) = \mathfrak{f}_S(\mathrm{j}) \not\supseteq \mathfrak{f}_S(\rho)$, $\mathfrak{f}_S$ is not an s-intersection right *id*.

Proposition 3.18. Let $\mathfrak{f}_S \in S_S(U)$ and S be an R-simple. Then, the following conditions are equivalent:
1. $\mathfrak{f}_S$ is an s-intersection right *id*.
2. $\mathfrak{f}_S$ is an s-intersection right WI-*id*.

Proof: (1) implies (2) is by Proposition 3.16. Let $\mathfrak{f}_S$ be an s-intersection right WI-*id* and $\upsilon, \hbar \in S$. By assumption, there exists $x \in S$ such that $\hbar = \upsilon x$. Then,

$$\mathfrak{f}_S(\upsilon\hbar) = \mathfrak{f}_S\big(\upsilon(\upsilon x)\big) = \mathfrak{f}_S(\upsilon\upsilon x) \supseteq \mathfrak{f}_S(\upsilon) \cap \mathfrak{f}_S(\upsilon) = \mathfrak{f}_S(\upsilon)$$

Thereby, $\mathfrak{f}_S$ is an s-intersection right *id*.

Proposition 3.19. Let $\mathfrak{f}_S$ be an idempotent soft set. Then, the following conditions are equivalent:
1. $\mathfrak{f}_S$ is an s-intersection right *id*.
2. $\mathfrak{f}_S$ is an s-intersection right WI-*id*.

Proof: (1) implies (2) is by Proposition 3.16. Let $\mathfrak{f}_S$ be an $\mathfrak{s}$-intersection right WI-*id*. Since $\mathfrak{f}_S$ is an idempotent $\mathfrak{s}$-intersection right WI-*id*, we have

$$\mathfrak{f}_S \circ \widetilde{\mathbb{S}} = \mathfrak{f}_S \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \cong \mathfrak{f}_S$$

Thus, $\mathfrak{f}_S$ is an $\mathfrak{s}$-intersection right *id*.

From here, it is obvious that any idempotent $\mathfrak{s}$-intersection right WI-*id* coincides with the $\mathfrak{s}$-intersection right *id*.

Theorem 3.20. Every $\mathfrak{s}$-intersection *id* is an $\mathfrak{s}$-intersection WI-*id*.
Proof: It is followed by Proposition 3.12 and Proposition 3.16.

Theorem 3.21. Let $\mathfrak{f}_S \in S_S(U)$. For a group, the following conditions are equivalent:

1. $\mathfrak{f}_S$ is an $\mathfrak{s}$-intersection *id*.
2. $\mathfrak{f}_S$ is an $\mathfrak{s}$-intersection WI-*id*.

Proof: (1) implies (2) is by Theorem 3.20. Let $\mathfrak{f}_S$ be an $\mathfrak{s}$-intersection WI-*id* of a group. The remainder of the proof is completed based on Theorem 2.1 (2), Proposition 3.37 and Proposition 3.38.

Theorem 3.22. Let $\mathfrak{f}_S$ be an idempotent soft set. Then, the following conditions are equivalent:

1. $\mathfrak{f}_S$ is an $\mathfrak{s}$-intersection *id*.
2. $\mathfrak{f}_S$ is an $\mathfrak{s}$-intersection WI-*id*.

Proof: It is followed by Proposition 3.15 and Proposition 3.19.

In the remainder of this section, for brevity, we will use "$\mathfrak{s}$-intersection I-*id*" instead of "$\mathfrak{s}$-intersection interior *id*", "$\mathfrak{s}$-intersection B-*id*" instead of "$\mathfrak{s}$-intersection bi-*id*", and "$\mathfrak{s}$-intersection Q-*id*" instead of "$\mathfrak{s}$-intersection quasi-*id*".

Proposition 3.23. Every $\mathfrak{s}$-intersection I-*id* is an $\mathfrak{s}$-intersection left WI-*id*.
Proof: Let $\mathfrak{f}_S$ be an $\mathfrak{s}$-intersection I-*id*. Then, $\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \cong \mathfrak{f}_S$. Thus,

$$\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \mathfrak{f}_S \cong \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \cong \mathfrak{f}_S$$

Hence, $\mathfrak{f}_S$ is an $\mathfrak{s}$-intersection left WI-*id*.

Example 3.24. Let $S = \{\mathfrak{Ł}, \mathfrak{w}, \mathfrak{E}\}$ be:

Table 2. Cayley Table of "$\circledast$" binary operation.

$\circledast$	$\mathfrak{Ł}$	$\mathfrak{w}$	$\mathfrak{E}$
$\mathfrak{Ł}$	$\mathfrak{Ł}$	$\mathfrak{w}$	$\mathfrak{E}$
$\mathfrak{w}$	$\mathfrak{Ł}$	$\mathfrak{w}$	$\mathfrak{E}$
$\mathfrak{E}$	$\mathfrak{Ł}$	$\mathfrak{w}$	$\mathfrak{E}$

Let $\mathfrak{p}_S$ be a soft set over $U = \{\Gamma, \Theta, \mathcal{V}, \Pi, \Sigma\}$ as follows:

$$\mathfrak{p}_S = \{(\mathfrak{Ł}, \{\Gamma, \Theta, \mathcal{V}\}), (\mathfrak{w}, \{\Pi\}), (\mathfrak{E}, \{\Sigma\})\}$$

Here, $\mathfrak{p}_S$ is an $\mathfrak{s}$-intersection left WI-*id*. In fact,

$$\left(\widetilde{\mathbb{S}} \circ \mathfrak{p}_S \circ \mathfrak{p}_S\right)(\mathfrak{Ł}) = \{\Gamma, \Theta, \mathcal{V}\} \subseteq \mathfrak{p}_S(\mathfrak{Ł}) = \{\Gamma, \Theta, \mathcal{V}\}$$
$$\left(\widetilde{\mathbb{S}} \circ \mathfrak{p}_S \circ \mathfrak{p}_S\right)(\mathfrak{w}) = \{\Pi\} \subseteq \mathfrak{p}_S(\mathfrak{w}) = \{\Pi\}$$
$$\left(\widetilde{\mathbb{S}} \circ \mathfrak{p}_S \circ \mathfrak{p}_S\right)(\mathfrak{E}) = \{\Sigma\} \subseteq \mathfrak{p}_S(\mathfrak{E}) = \{\Sigma\}$$

Thus, $\mathfrak{p}_S$ is an $\mathfrak{s}$-intersection left WI-*id*. However, since $\mathfrak{p}_S(\mathfrak{Ł}\mathfrak{w}\mathfrak{E}) = \mathfrak{p}_S(\mathfrak{E}) \not\supseteq \mathfrak{p}_S(\mathfrak{w})$, $\mathfrak{p}_S$ is not an $\mathfrak{s}$-intersection I-*id*.

Proposition 3.25. Let $\mathfrak{f}_S \in S_S(U)$ and S be an L-simple. Then, the following conditions are equivalent:

1. $\mathfrak{f}_S$ is an $\mathfrak{s}$-intersection I-*id*.
2. $\mathfrak{f}_S$ is an $\mathfrak{s}$-intersection left WI-*id*.

Proof: (1) implies (2) is by Proposition 3.23. Let $\mathfrak{f}_S$ be an $\mathfrak{s}$-intersection left WI-*id*. Then, by Proposition 3.14, $\mathfrak{f}_S$ is an $\mathfrak{s}$-intersection left *id*. Let $\mathfrak{d}, \mathfrak{h}, \eta \in S$. By assumption, there exists $\mathfrak{j} \in S$ such that $\eta = x\mathfrak{h}$. Thus,

$$\mathfrak{f}_S(\mathfrak{d}\mathfrak{h}\eta) = \mathfrak{f}_S\left(\mathfrak{d}\mathfrak{h}(\mathfrak{j}\mathfrak{h})\right) = \mathfrak{f}_S\left((\mathfrak{d}\mathfrak{h}\mathfrak{j})\mathfrak{h}\right) \supseteq \mathfrak{f}_S(\mathfrak{h})$$

Hence, $\mathfrak{f}_S$ is an $\mathfrak{s}$-intersection I-*id*.

Proposition 3.26. Every $\mathfrak{s}$-intersection I-*id* is an $\mathfrak{s}$-intersection right WI-*id*.
Proof: Let $\mathfrak{f}_S$ be an $\mathfrak{s}$-intersection I-*id*. Then, $\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \cong \mathfrak{f}_S$. Thus,

$$\mathfrak{f}_S \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \cong \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \cong \mathfrak{f}_S$$

Hence, $\mathfrak{f}_S$ is an $\mathfrak{s}$-intersection right WI-*id*.

Example 3.27. Let $S = \{\rho, \nu, \tau\}$ be:

Table 3. Cayley Table of "$\ominus$" binary operation.

$\ominus$	ρ	ν	τ
ρ	ρ	ρ	ρ
ν	ν	ν	ν
τ	τ	τ	τ

Let $\mathfrak{G}_S$ be a soft set over $U = \{\Gamma, \Theta, \mathcal{V}, \Pi, \Sigma\}$ as follows:

$$\mathfrak{G}_S = \{(\rho, \{\Gamma, \Theta\}), (\nu, \{\mathcal{V}\}), (\tau, \{\Pi, \Sigma\})\}$$

Here, $\mathfrak{G}_S$ is an s-intersection right WI-*id*. In fact,

$$\left(\mathfrak{G}_S \circ \mathfrak{G}_S \circ \widetilde{\mathbb{S}}\right)(\rho) = \{\Gamma, \Theta\} \subseteq \mathfrak{G}_S(\rho) = \{\Gamma, \Theta\}$$
$$\left(\mathfrak{G}_S \circ \mathfrak{G}_S \circ \widetilde{\mathbb{S}}\right)(\nu) = \{\mathcal{V}\} \subseteq \mathfrak{G}_S(\nu) = \{\mathcal{V}\}$$
$$\left(\mathfrak{G}_S \circ \mathfrak{G}_S \circ \widetilde{\mathbb{S}}\right)(\tau) = \{\Pi, \Sigma\} \subseteq \mathfrak{G}_S(\tau) = \{\Pi, \Sigma\}$$

Thus, $\mathfrak{G}_S$ is an s-intersection right WI-*id*. However, since $\mathfrak{G}_S(\tau\nu\rho) = \mathfrak{G}_S(\tau) \not\supseteq \mathfrak{G}_S(\nu)$, $\mathfrak{G}_S$ is not an s-intersection I-*id*.

Proposition 3.28. Let $\mathfrak{f}_S \in S_S(U)$ and S be an R-simple. Then, the following conditions are equivalent:

1. $\mathfrak{f}_S$ is an s-intersection I-*id*.
2. $\mathfrak{f}_S$ is an s-intersection right WI-*id*.

Proof: (1) implies (2) is by Proposition 3.26. Let $\mathfrak{f}_S$ be an s-intersection right WI-*id*. Then, by Proposition 3.18, $\mathfrak{f}_S$ is an s-intersection right *id*. Let $\mathfrak{v}, \mathfrak{b}, \hbar \in S$. By assumption, there exists $\mathfrak{j} \in S$ such that $\mathfrak{v} = \mathfrak{b}\mathfrak{j}$. Thus,

$$\mathfrak{f}_S(\mathfrak{v}\mathfrak{b}\hbar) = \mathfrak{f}_S\left((\mathfrak{b}\mathfrak{j})\mathfrak{b}\hbar\right) = \mathfrak{f}_S\left(\mathfrak{b}(\mathfrak{j}\mathfrak{b}\hbar)\right) \supseteq \mathfrak{f}_S(\mathfrak{b})$$

Hence, $\mathfrak{f}_S$ is an s-intersection I-*id*.

Theorem 3.29. Every s-intersection I-*id* is an s-intersection WI-*id*.
Proof: It is followed by 3.23 and Proposition 3.26.

Theorem 3.30. Let $\mathfrak{f}_S \in S_S(U)$. For a group, the following conditions are equivalent:

1. $\mathfrak{f}_S$ is an s-intersection I-*id*.
2. $\mathfrak{f}_S$ is an s-intersection WI-*id*.

Proof: (1) implies (2) is by Theorem 3.29. Let $\mathfrak{f}_S$ be an s-intersection WI-*id* of a group. The remainder of the proof is completed based on Theorem 2.1 (2), Proposition 3.25 and Proposition 3.28.

Proposition 3.31. Every s-intersection B-*id* is an s-intersection right WI-*id* of an L-simple semigroup.
Proof: Let $\mathfrak{f}_S$ be an s-intersection B-*id* of an L-simple semigroup. Then, by Proposition 2.8, $\mathfrak{f}_S$ is an s-intersection right *id*. The rest of the proof is obvious by Proposition 3.16.

Proposition 3.32. Every s-intersection B-*id* is an s-intersection left WI-*id* of an R-simple semigroup.
Proof: Let $\mathfrak{f}_S$ be an s-intersection B-*id* of an R-simple S and $\mathfrak{u}, \mathfrak{d}, \mathfrak{r} \in S$. By assumption, there exists $\mathfrak{y} \in S$ such that $\mathfrak{u} = \mathfrak{d}\mathfrak{y}$. Then,

$$\mathfrak{f}_S(\mathfrak{u}\mathfrak{d}\mathfrak{r}) = \mathfrak{f}_S\left((\mathfrak{d}\mathfrak{y})\mathfrak{d}\mathfrak{r}\right) = \mathfrak{f}_S\left((\mathfrak{d}\mathfrak{y}\mathfrak{d})\mathfrak{r}\right) \supseteq \mathfrak{f}_S(\mathfrak{d}\mathfrak{y}\mathfrak{d}) \cap \mathfrak{f}_S(\mathfrak{r}) \supseteq \left(\mathfrak{f}_S(\mathfrak{d}) \cap \mathfrak{f}_S(\mathfrak{d})\right) \cap \mathfrak{f}_S(\mathfrak{r}) = \mathfrak{f}_S(\mathfrak{d}) \cap \mathfrak{f}_S(\mathfrak{r})$$

implying that $\mathfrak{f}_S$ is an s-intersection left WI-*id*.

Theorem 3.33. Every s-intersection B-*id* is an s-intersection WI-*id* of a group.
Proof: It is followed by Theorem 2.1 (2), Proposition 3.31 and Proposition 3.32.

Proposition 3.34. Every s-intersection Q-*id* is an s-intersection right WI-*id* of an L-simple semigroup.
Proof: Let $\mathfrak{f}_S$ be an s-intersection Q-*id* of an L-simple semigroup. Then, $\mathfrak{f}_S$ is an s-intersection B-*id*. The remainder of the proof is completed based on Proposition 3.31.

Proposition 3.35. Every s-intersection Q-*id* is an s-intersection left WI-*id* of an R-simple semigroup.
Proof: Let $\mathfrak{f}_S$ be an s-intersection Q-*id* of an R-simple semigroup. Then, $\mathfrak{f}_S$ is an s-intersection B-*id*. The remainder of the proof is completed based on Proposition 3.32.

Theorem 3.36. Every s-intersection Q-*id* is an s-intersection WI-*id* of a group.
Proof: It is followed by Theorem 2.1 (2), Proposition 3.34 and Proposition 3.35.

Proposition 3.37. Every s-intersection left WI-*id* is an s-intersection Q-*id* of an L-simple semigroup.
Proof: Let $\mathfrak{f}_S$ be an s-intersection left WI-*id* of an L-simple semigroup. Then, $\mathfrak{f}_S$ is an s-intersection left *id* by Proposition 3.14. Thus, $\mathfrak{f}_S$ is an s-intersection Q-*id*.

Moreover, it is obvious that every idempotent s-intersection left WI-*id* is an s-intersection Q-*id*.

Proposition 3.38. Every s-intersection right WI-*id* is an s-intersection Q-*id* of an R-simple semigroup.
Proof: Let $\mathfrak{f}_S$ be an s-intersection right WI-*id* of an R-simple semigroup. Then, $\mathfrak{f}_S$ is an s-intersection right *id* by Proposition 3.18. Thus, $\mathfrak{f}_S$ is an s-intersection Q-*id*.

Moreover, it is obvious that every idempotent s-intersection right WI-*id* is an s-intersection Q-*id*.

Theorem 3.39. Every s-intersection WI-*id* is an s-intersection Q-*id* of a group.
Proof: It is followed by Theorem 2.1 (2), Proposition 3.37 and Proposition 3.38.

Moreover, it is obvious that every idempotent s-intersection WI-*id* is an s-intersection Q-*id*.

Proposition 3.40. Every s-intersection left WI-*id* is an s-intersection B-*id* of an L-simple semigroup.
Proof: Let $\mathfrak{f}_S$ be an s-intersection left WI-*id* of an L-simple semigroup. Then, $\mathfrak{f}_S$ is an s-intersection Q-*id* by Proposition 3.37. Thus, $\mathfrak{f}_S$ is an s-intersection B-*id*.

Moreover, it is obvious that every idempotent s-intersection left WI-*id* is an s-intersection B-*id*.

Proposition 3.41. Every s-intersection right WI-*id* is an s-intersection B-*id* of an R-simple semigroup.
Proof: Let $\mathfrak{f}_S$ be an s-intersection right WI-*id* of an R-simple semigroup. Then, $\mathfrak{f}_S$ is an s-intersection Q-*id* by Proposition 3.34. Thus, $\mathfrak{f}_S$ is an s-intersection B-*id*.

Moreover, it is obvious that every idempotent s-intersection right WI-*id* is an s-intersection B-*id*.

Theorem 3.42. Every s-intersection WI-*id* is an s-intersection B-*id* of a group.
Proof: It is followed by Theorem 2.1 (2), Proposition 3.40 and Proposition 3.41.

Moreover, it is obvious that every idempotent s-intersection WI-*id* is an s-intersection B-*id*.

Proposition 3.43. Let $\varrho_{\mathfrak{R}}$ and $\varrho_{\mathfrak{T}}$ be s-intersection left (right) WI-*id*s of $\mathfrak{R}$ and $\mathfrak{T}$, respectively. Then, $\varrho_{\mathfrak{R}} \wedge \varrho_{\mathfrak{T}}$ is an s-intersection left (right) WI-*id* of $\mathfrak{R} \times \mathfrak{T}$.
Proof: The proof is given exclusively for s-intersection left WI-*id*, as the proof for s-intersection right WI-*id* can be demonstrated analogously. Let $(\varsigma_1, \mathfrak{t}_1), (\varsigma_2, \mathfrak{t}_2), (\varsigma_3, \mathfrak{t}_3) \in \mathfrak{R} \times \mathfrak{T}$. Then,

$$\begin{aligned}\varrho_{\mathfrak{R}\wedge\mathfrak{T}}\big((\varsigma_1,\mathfrak{t}_1)(\varsigma_2,\mathfrak{t}_2)(\varsigma_3,\mathfrak{t}_3)\big) &= \varrho_{\mathfrak{R}\wedge\mathfrak{T}}(\varsigma_1\varsigma_2\varsigma_3, \mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_3)\\ &= \varrho_{\mathfrak{R}}(\varsigma_1\varsigma_2\varsigma_3) \cap \varrho_{\mathfrak{T}}(\mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_3)\\ &\supseteq \big(\varrho_{\mathfrak{R}}(\varsigma_2) \cap \varrho_S(\varsigma_3)\big) \cap \big(\varrho_{\mathfrak{T}}(\mathfrak{t}_2) \cap \varrho_{\mathfrak{T}}(\mathfrak{t}_3)\big)\\ &= \big(\varrho_{\mathfrak{R}}(\varsigma_2) \cap \varrho_{\mathfrak{T}}(\mathfrak{t}_2)\big) \cap \big(\varrho_S(\varsigma_3) \cap \varrho_{\mathfrak{T}}(\mathfrak{t}_3)\big)\\ &= \varrho_{\mathfrak{R}\wedge\mathfrak{T}}(\varsigma_2,\mathfrak{t}_2) \cap \varrho_{\mathfrak{R}\wedge\mathfrak{T}}(\varsigma_3,\mathfrak{t}_3)\end{aligned}$$

Thus, $\varrho_{\mathfrak{R}} \wedge \varrho_{\mathfrak{T}}$ is an s-intersection left WI-*id* of $\mathfrak{R} \times \mathfrak{T}$.

Theorem 3.44. Let $\varrho_{\mathfrak{R}}$ and $\varrho_{\mathfrak{T}}$ be s-intersection WI-*id*s of $\mathfrak{R}$ and $\mathfrak{T}$, respectively. Then, $\varrho_{\mathfrak{R}} \wedge \varrho_{\mathfrak{T}}$ is an s-intersection WI-*id* of $\mathfrak{R} \times \mathfrak{T}$.

Proposition 3.45. Let $\varrho_{\mathfrak{R}}$ and $\varrho_{\mathfrak{T}}$ be s-intersection left (right) WI-*id*s of $\mathfrak{R}$ and $\mathfrak{T}$ over U, respectively. Then, $\varrho_{\mathfrak{R}} \times \varrho_{\mathfrak{T}}$ is an s-intersection left (right) WI-*id* of $\mathfrak{R} \times \mathfrak{T}$ over $U \times U$.
Proof: The proof is given exclusively for s-intersection left WI-*id*, as the proof for s-intersection right WI-*id* can be demonstrated analogously. Let $\varrho_{\mathfrak{R}} \times \varrho_{\mathfrak{T}} = \varrho_{\mathfrak{R}\times\mathfrak{T}}$, where $\varrho_{\mathfrak{R}\times\mathfrak{T}}(\varsigma, \mathfrak{t}) = \varrho_{\mathfrak{R}}(\varsigma) \times \varrho_{\mathfrak{T}}(\mathfrak{t})$ for all $(\varsigma, \mathfrak{t}) \in \mathfrak{R} \times \mathfrak{T}$. Then, for all $(\varsigma_1, \mathfrak{t}_1), (\varsigma_2, \mathfrak{t}_2), (\varsigma_3, \mathfrak{t}_3) \in \mathfrak{R} \times \mathfrak{T}$,

$$\begin{aligned}\varrho_{\mathfrak{R}\times\mathfrak{T}}\big((\varsigma_1,\mathfrak{t}_1)(\varsigma_2,\mathfrak{t}_2)(\varsigma_3,\mathfrak{t}_3)\big) &= \varrho_{\mathfrak{R}\times\mathfrak{T}}(\varsigma_1\varsigma_2\varsigma_3, \mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_3)\\ &= \varrho_{\mathfrak{R}}(\varsigma_1\varsigma_2\varsigma_3) \times \varrho_{\mathfrak{T}}(\mathfrak{t}_1\mathfrak{t}_2\mathfrak{t}_3)\\ &\supseteq \big(\varrho_{\mathfrak{R}}(\varsigma_2) \cap \varrho_S(\varsigma_3)\big) \times \big(\varrho_{\mathfrak{T}}(\mathfrak{t}_2) \cap \varrho_{\mathfrak{T}}(\mathfrak{t}_3)\big)\\ &= \big(\varrho_{\mathfrak{R}}(\varsigma_2) \times \varrho_{\mathfrak{T}}(\mathfrak{t}_2)\big) \cap \big(\varrho_{\mathfrak{R}}(\varsigma_3) \times \varrho_{\mathfrak{T}}(\mathfrak{t}_3)\big)\\ &= \varrho_{\mathfrak{R}\times\mathfrak{T}}(\varsigma_2,\mathfrak{t}_2) \cap \varrho_{\mathfrak{R}\times\mathfrak{T}}(\varsigma_3,\mathfrak{t}_3)\end{aligned}$$

Hence, $\varrho_{\mathfrak{R}} \times \varrho_{\mathfrak{T}} = \varrho_{\mathfrak{R}\times\mathfrak{T}}$ is an s-intersection left WI-*id* of $\mathfrak{R} \times \mathfrak{T}$ over $U \times U$.

Theorem 3.46. Let $\varrho_{\mathfrak{R}}$ and $\varrho_{\mathfrak{T}}$ be s-intersection WI-*id*s of $\mathfrak{R}$ and $\mathfrak{T}$ over U, respectively. Then, $\varrho_{\mathfrak{R}} \times \varrho_{\mathfrak{T}}$ is an s-intersection WI-*id* of $\mathfrak{R} \times \mathfrak{T}$ over $U \times U$.

Proposition 3.47. Let $\mathfrak{f}_S$ and $\wp_S$ be s-intersection left (right) WI-*id*s. Then, $\mathfrak{f}_S \widetilde{\cap} \wp_S$ is an s-intersection left (right) WI-*id*.
Proof: The proof is given exclusively for s-intersection left WI-*id*, as the proof for s-intersection right WI-*id* can be demonstrated analogously. Let $\mathfrak{f}_S$ and $\wp_S$ be s-intersection left WI-*id*s. Then, $\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \mathfrak{f}_S \widetilde{\subseteq} \mathfrak{f}_S$ and $\widetilde{\mathbb{S}} \circ \wp_S \circ \wp_S \widetilde{\subseteq} \wp_S$. Thus, we have

$$\widetilde{\mathbb{S}} \circ (\mathfrak{f}_S \widetilde{\cap} \wp_S) \circ (\mathfrak{f}_S \widetilde{\cap} \wp_S) \widetilde{\subseteq} \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \mathfrak{f}_S \widetilde{\subseteq} \mathfrak{f}_S \text{ and } \widetilde{\mathbb{S}} \circ (\mathfrak{f}_S \widetilde{\cap} \wp_S) \circ (\mathfrak{f}_S \widetilde{\cap} \wp_S) \widetilde{\subseteq} \widetilde{\mathbb{S}} \circ \wp_S \circ \wp_S \widetilde{\subseteq} \wp_S$$

Hence,

$$\widetilde{\mathbb{S}} \circ (\mathfrak{f}_S \widetilde{\cap} \wp_S) \circ (\mathfrak{f}_S \widetilde{\cap} \wp_S) \widetilde{\subseteq} \mathfrak{f}_S \widetilde{\cap} \wp_S$$

Therefore, $\mathfrak{f}_S \widetilde{\cap} \wp_S$ is an s-intersection left WI-*id*.

Theorem 3.48. Let $\mathfrak{f}_S$ and $\wp_S$ be s-intersection WI-*id*s. Then, $\mathfrak{f}_S \widetilde{\cap} \wp_S$ is an s-intersection WI-*id*.

Proposition 3.49. Let $\mathfrak{f}_S$ be an s-intersection left *id* and $\mathfrak{n}_S$ be an s-intersection right *id*. Then, $\mathfrak{f}_S \circ \mathfrak{n}_S$ is an s-intersection WI-*id*.
Proof: Let $\mathfrak{f}_S$ be an s-intersection left *id* and $\mathfrak{n}_S$ be an s-intersection right *id*. Then, we have $\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \widetilde{\subseteq} \mathfrak{f}_S$, $\mathfrak{n}_S \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} \mathfrak{n}_S$, $\mathfrak{f}_S \circ \mathfrak{f}_S \widetilde{\subseteq} \mathfrak{f}_S$ and $\mathfrak{n}_S \circ \mathfrak{n}_S \widetilde{\subseteq} \mathfrak{n}_S$. Thus,

$$\widetilde{\mathbb{S}} \circ (\mathfrak{f}_S \circ \mathfrak{n}_S) \circ (\mathfrak{f}_S \circ \mathfrak{n}_S) \widetilde{\subseteq} \mathfrak{f}_S \circ \mathfrak{n}_S \circ \mathfrak{f}_S \circ \mathfrak{n}_S \widetilde{\subseteq} \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \mathfrak{n}_S \widetilde{\subseteq} \mathfrak{f}_S \circ \mathfrak{f}_S \circ \mathfrak{n}_S \widetilde{\subseteq} \mathfrak{f}_S \circ \mathfrak{n}_S$$

Hence, $\mathfrak{f}_S \circ \mathfrak{n}_S$ is an s-intersection left WI-*id*. Similarly, since

$$(\mathfrak{f}_S \circ \mathfrak{n}_S) \circ (\mathfrak{f}_S \circ \mathfrak{n}_S) \circ \widetilde{\mathbb{S}} \widetilde{\subseteq} \mathfrak{f}_S \circ \mathfrak{n}_S \circ \mathfrak{f}_S \circ \mathfrak{n}_S \widetilde{\subseteq} \mathfrak{f}_S \circ \mathfrak{n}_S \circ \widetilde{\mathbb{S}} \circ \mathfrak{n}_S \widetilde{\subseteq} \mathfrak{f}_S \circ \mathfrak{n}_S \circ \mathfrak{n}_S \widetilde{\subseteq} \mathfrak{f}_S \circ \mathfrak{n}_S$$

$\mathfrak{f}_S \circ \mathfrak{n}_S$ is an s-intersection right WI-*id*. Therefore, $\mathfrak{f}_S \circ \mathfrak{n}_S$ is an s-intersection WI-*id*.

Corollary 3.50. Let $\mathfrak{f}_S$ and $\mathfrak{h}_S$ be s-intersection *id*s. Then, $\mathfrak{f}_S \circ \mathfrak{h}_S$ is an s-intersection WI-*id*.

Proposition 3.51. Let $\mathfrak{f}_S$ be an s-intersection $\mathbb{SS}$ over U, $\sigma \subseteq U$, $Im(\mathfrak{f}_S)$ be the image of $\mathfrak{f}_S$ such that $\sigma \in Im(\mathfrak{f}_S)$. If $\mathfrak{f}_S$ is an s-intersection left (right) WI-*id*, then $\mathfrak{U}(\mathfrak{f}_S; \sigma)$ is a left (right) WI-*id*.
Proof: The proof is given exclusively for s-intersection left WI-*id*, as the proof for s-intersection right WI-*id* can be demonstrated analogously. Since $\mathfrak{f}_S(r) = \sigma$ for some $r \in S$, $\emptyset \neq \mathfrak{U}(\mathfrak{f}_S; \sigma) \subseteq S$. Let $\mathfrak{k} \in [S \cdot \mathfrak{U}(\mathfrak{f}_S; \sigma) \cdot \mathfrak{U}(\mathfrak{f}_S; \sigma)]$. Then, there exist $\upsilon, \hbar \in \mathfrak{U}(\mathfrak{f}_S; \sigma)$ and $\mathfrak{b} \in S$ such that $\mathfrak{k} = \mathfrak{b}\upsilon\hbar$. Thus, $\mathfrak{f}_S(\upsilon) \supseteq \sigma$ and $\mathfrak{f}_S(\hbar) \supseteq \sigma$. Since $\mathfrak{f}_S$ is an s-intersection left WI-*id*, $\mathfrak{f}_S(\mathfrak{k}) = \mathfrak{f}_S(\mathfrak{u}\upsilon\hbar) \supseteq \mathfrak{f}_S(\upsilon) \cap \mathfrak{f}_S(\hbar) \supseteq \sigma \cap \sigma = \sigma$. Hence, $\mathfrak{f}_S(\mathfrak{k}) \supseteq \sigma$, implying that $\mathfrak{k} \in \mathfrak{U}(\mathfrak{f}_S; \sigma)$. Therefore, $S \cdot \mathfrak{U}(\mathfrak{f}_S; \sigma) \cdot \mathfrak{U}(\mathfrak{f}_S; \sigma) \subseteq \mathfrak{U}(\mathfrak{f}_S; \sigma)$. Moreover, since $\mathfrak{f}_S$ is an s-intersection $\mathbb{SS}$, $\mathfrak{U}(\mathfrak{f}_S; \sigma)$ is an $\mathbb{SS}$. Thus, $\mathfrak{U}(\mathfrak{f}_S; \sigma)$ is a left WI-*id*.

Theorem 3.52. Let $\mathfrak{f}_S$ be an s-intersection $\mathbb{SS}$ over U, $\sigma \subseteq U$, $Im(\mathfrak{f}_S)$ be the image of $\mathfrak{f}_S$ such that $\sigma \in Im(\mathfrak{f}_S)$. If $\mathfrak{f}_S$ is an s-intersection WI-*id*, then $\mathfrak{U}(\mathfrak{f}_S; \sigma)$ is a WI-*id*.

Example 3.53. Consider the S in Example 3.2. Let $\wp_S$ be a soft set over $U = S_3$ as follows:

$$\wp_S = \{(\mathfrak{n}, \{(1), (12), (123), (132)\}), (\mathfrak{j}, \{(12), (123), (132)\}), (\mathfrak{p}, \{(12)\}), (\boldsymbol{\sigma}, \{(132)\})\}$$

Here, $\wp_S$ is an s-intersection WI-*id*. Firstly, one can shown that $\wp_S$ is an s-intersection $\mathbb{SS}$. Indeed,

$$(\wp_S \circ \wp_S)(\mathfrak{n}) = \{(1), (12), (123), (132)\} \subseteq \wp_S(\mathfrak{n}) = \{(1), (12), (123), (132)\}$$
$$(\wp_S \circ \wp_S)(\mathfrak{j}) = \{(12), (132)\} \subseteq \wp_S(\mathfrak{j}) = \{(12), (123), (132)\}$$
$$(\wp_S \circ \wp_S)(\mathfrak{p}) = \emptyset \subseteq \wp_S(\mathfrak{p}) = \{(12)\}$$
$$(\wp_S \circ \wp_S)(\boldsymbol{\sigma}) = \emptyset \subseteq \wp_S(\boldsymbol{\sigma}) = \{(132)\}$$

Thus, $\wp_S$ is an s-intersection $\mathbb{SS}$. Moreover, $\wp_S$ is an s-intersection left WI-*id*. Indeed,

$$(\widetilde{\mathbb{S}} \circ \wp_S \circ \wp_S)(\mathfrak{n}) = \{(1), (12), (123), (132)\} \subseteq \wp_S(\mathfrak{n}) = \{(1), (12), (123), (132)\}$$
$$(\widetilde{\mathbb{S}} \circ \wp_S \circ \wp_S)(\mathfrak{j}) = \emptyset \subseteq \wp_S(\mathfrak{j}) = \{(12), (123), (132)\}$$
$$(\widetilde{\mathbb{S}} \circ \wp_S \circ \wp_S)(\mathfrak{p}) = \emptyset \subseteq \wp_S(\mathfrak{p}) = \{(12)\}$$
$$(\widetilde{\mathbb{S}} \circ \wp_S \circ \wp_S)(\boldsymbol{\sigma}) = \emptyset \subseteq \wp_S(\boldsymbol{\sigma}) = \{(132)\}$$

Furthermore, $\wp_S$ is an s-intersection right WI-*id*. Indeed,

$$(\wp_S \circ \wp_S \circ \widetilde{\mathbb{S}})(\mathfrak{n}) = \{(1), (12), (123), (132)\} \subseteq \wp_S(\mathfrak{n}) = \{(1), (12), (123), (132)\}$$
$$(\wp_S \circ \wp_S \circ \widetilde{\mathbb{S}})(\mathfrak{j}) = \emptyset \subseteq \wp_S(\mathfrak{j}) = \{(12), (123), (132)\}$$
$$(\wp_S \circ \wp_S \circ \widetilde{\mathbb{S}})(\mathfrak{p}) = \emptyset \subseteq \wp_S(\mathfrak{p}) = \{(12)\}$$
$$(\wp_S \circ \wp_S \circ \widetilde{\mathbb{S}})(\boldsymbol{\sigma}) = \emptyset \subseteq \wp_S(\boldsymbol{\sigma}) = \{(132)\}$$

Thus, $\wp_S$ is an s-intersection WI-*id*. Since

$$Im(\wp_S) = \{\{(1), (12), (123), (132)\}, \{(12), (123), (132)\}, \{(12)\}, \{(132)\}\}$$

we obtain the following:

$$\mathfrak{U}(\wp_S; \sigma) = \begin{cases} \{\mathfrak{n}, \mathfrak{j}, \mathfrak{p}\}, & \sigma = \{(12)\} \\ \{\mathfrak{n}, \mathfrak{j}, \boldsymbol{\sigma}\}, & \sigma = \{(132)\} \\ \{\mathfrak{n}, \mathfrak{j}\}, & \sigma = \{(12), (123), (132)\} \\ \{\mathfrak{n}\}, & \sigma = \{(1), (12), (123), (132)\} \end{cases}$$

Here, $\{\mathfrak{n}, \mathfrak{j}, \mathfrak{p}\}$, $\{\mathfrak{n}, \mathfrak{j}, \boldsymbol{\sigma}\}$, $\{\mathfrak{n}, \mathfrak{j}\}$ and $\{\mathfrak{n}\}$ are all WI-*id*s. In fact, since

$$\{\mathcal{H}, \mathfrak{j}, \rho\} \cdot \{\mathcal{H}, \mathfrak{j}, \rho\} = \{\mathcal{H}, \mathfrak{j}\} \subseteq \{\mathcal{H}, \mathfrak{j}, \rho\}$$
$$\{\mathcal{H}, \mathfrak{j}, \boldsymbol{\sigma}\} \cdot \{\mathcal{H}, \mathfrak{j}, \boldsymbol{\sigma}\} = \{\mathcal{H}, \mathfrak{j}\} \subseteq \{\mathcal{H}, \mathfrak{j}, \boldsymbol{\sigma}\}$$
$$\{\mathcal{H}, \mathfrak{j}\} \cdot \{\mathcal{H}, \mathfrak{j}\} = \{\mathcal{H}\} \subseteq \{\mathcal{H}, \mathfrak{j}\}$$
$$\{\mathcal{H}\} \cdot \{\mathcal{H}\} = \{\mathcal{H}\} \subseteq \{\mathcal{H}\}$$

each $\mathfrak{U}(\wp_S; \sigma)$ is an $\mathbb{SS}$. Similarly, since

$$S \cdot \{\mathcal{H}, \mathfrak{j}, \rho\} \cdot \{\mathcal{H}, \mathfrak{j}, \rho\} = \{\mathcal{H}\} \subseteq \{\mathcal{H}, \mathfrak{j}, \rho\}$$
$$S \cdot \{\mathcal{H}, \mathfrak{j}, \boldsymbol{\sigma}\} \cdot \{\mathcal{H}, \mathfrak{j}, \boldsymbol{\sigma}\} = \{\mathcal{H}\} \subseteq \{\mathcal{H}, \mathfrak{j}, \boldsymbol{\sigma}\}$$
$$S \cdot \{\mathcal{H}, \mathfrak{j}\} \cdot \{\mathcal{H}, \mathfrak{j}\} = \{\mathcal{H}\} \subseteq \{\mathcal{H}, \mathfrak{j}\}$$
$$S \cdot \{\mathcal{H}\} \cdot \{\mathcal{H}\} = \{\mathcal{H}\} \subseteq \{\mathcal{H}\}$$

each $\mathfrak{U}(\wp_S; \sigma)$ is a left WI-*id*. Similarly, since

$$\{\mathcal{H}, \mathfrak{j}, \rho\} \cdot \{\mathcal{H}, \mathfrak{j}, \rho\} \cdot S = \{\mathcal{H}\} \subseteq \{\mathcal{H}, \mathfrak{j}, \rho\}$$
$$\{\mathcal{H}, \mathfrak{j}, \boldsymbol{\sigma}\} \cdot \{\mathcal{H}, \mathfrak{j}, \boldsymbol{\sigma}\} \cdot S = \{\mathcal{H}\} \subseteq \{\mathcal{H}, \mathfrak{j}, \boldsymbol{\sigma}\}$$
$$\{\mathcal{H}, \mathfrak{j}\} \cdot \{\mathcal{H}, \mathfrak{j}\} \cdot S = \{\mathcal{H}\} \subseteq \{\mathcal{H}, \mathfrak{j}\}$$
$$\{\mathcal{H}\} \cdot \{\mathcal{H}\} \cdot S = \{\mathcal{H}\} \subseteq \{\mathcal{H}\}$$

each $\mathfrak{U}(\wp_S; \sigma)$ is a right WI-*id*, and thus each of $\mathfrak{U}(\wp_S; \sigma)$ is a WI-*id*.
Now, consider the soft set $\mathcal{U}_S$ in Example 3.2. As

$$Im(\mathcal{U}_S) = \{\{(12)\}, \{(13)\}, \{(23)\}, \{(1), (123), (132)\}\},$$

we obtain the following:

$$\mathfrak{U}(\mathcal{U}_S; \sigma) = \begin{cases} \{\boldsymbol{\sigma}\}, & \sigma = \{(12)\} \\ \{\rho\}, & \sigma = \{(13)\} \\ \{\mathfrak{j}\}, & \sigma = \{(23)\} \\ \{\mathcal{H}\}, & \sigma = \{(1), (123), (132)\} \end{cases}$$

Here, $\{\boldsymbol{\sigma}\}$ is not a WI-*id*. Indeed, since $\{\boldsymbol{\sigma}\} \cdot \{\boldsymbol{\sigma}\} = \{\mathfrak{j}\} \not\subseteq \{\boldsymbol{\sigma}\}$, one of the $\mathfrak{U}(\mathcal{U}_S; \sigma)$ is not an $\mathbb{SS}$, hence it is not a WI-*id*. It is seen that each of $\mathfrak{U}(\mathcal{U}_S; \sigma)$ is not a WI-*id*. It was shown in Example 3.2 that $\mathcal{U}_S$ is not an $\mathfrak{s}$-intersection WI-*id*.

Definition 3.54. Let $\mathfrak{f}_S$ be an $\mathfrak{s}$-intersection $\mathbb{SS}$ and $\mathfrak{s}$-intersection left (right) WI-*id*. Then, the left (right) WI-*id* $\mathfrak{U}(\mathfrak{f}_S; \sigma)$ are called upper σ-left (right) WI-*id*s of $\mathfrak{f}_S$.

Proposition 3.55. Let $\varrho_S \in S_S(U)$, $\mathfrak{U}(\varrho_S; \sigma)$ be the upper σ-left (right) WI-*id* of ϱ_S for each $\sigma \subseteq U$ and $Im(\varrho_S)$ be an ordered set by inclusion. Then, ϱ_S is an $\mathfrak{s}$-intersection left (right) WI-*id*.
Proof: The proof is given exclusively for $\mathfrak{s}$-intersection left WI-*id*, as the proof for $\mathfrak{s}$-intersection right WI-*id* can be demonstrated analogously. Let $\mathfrak{s}, \mathfrak{h}, \mathfrak{a} \in S$ and $\varrho_S(\mathfrak{h}) = \sigma_1$ and $\varrho_S(\mathfrak{a}) = \sigma_2$. Suppose that $\sigma_1 \subseteq \sigma_2$. It is obvious that $\mathfrak{h} \in \mathfrak{U}(\varrho_S; \sigma_1)$ and $\mathfrak{a} \in \mathfrak{U}(\varrho_S; \sigma_2)$. Since $\sigma_1 \subseteq \sigma_2$, $\mathfrak{h}, \mathfrak{a} \in \mathfrak{U}(\varrho_S; \sigma_1)$ and since $\mathfrak{U}(\varrho_S; \sigma)$ is a left WI-*id* for all $\sigma \subseteq U$, it follows that $\mathfrak{s}\mathfrak{h}\mathfrak{a} \in \mathfrak{U}(\varrho_S; \sigma_1)$. Hence, $\varrho_S(\mathfrak{s}\mathfrak{h}\mathfrak{a}) \supseteq \sigma_1 = \sigma_1 \cap \sigma_2 = \varrho_S(\mathfrak{h}) \cap \varrho_S(\mathfrak{a})$. Thus, ϱ_S is an $\mathfrak{s}$-intersection left WI-*id*.

Theorem 3.56. Let $\varrho_S \in S_S(U)$, $\mathfrak{U}(\varrho_S; \sigma)$ be upper σ-WI-*id* of ϱ_S for each $\sigma \subseteq U$ and $Im(\varrho_S)$ be an ordered set by inclusion. Then, ϱ_S is an $\mathfrak{s}$-intersection WI-*id*.

Proposition 3.57. Let $\varrho_S, \varrho_{\mathfrak{T}} \in S_E(U)$, and ψ be a semigroup isomorphism from S to $\mathfrak{T}$. If ϱ_S is an $\mathfrak{s}$-intersection left (right) WI-*id* of S, then $\psi(\varrho_S)$ is an $\mathfrak{s}$-intersection left (right) WI-*id* of $\mathfrak{T}$.
Proof: The proof is given exclusively for $\mathfrak{s}$-intersection left WI-*id*, as the proof for $\mathfrak{s}$-intersection right WI-*id* can be demonstrated analogously. Let $\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3 \in \mathfrak{T}$. Since ψ is surjective, there exist $\mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3 \in S$ such that $\psi(\mathfrak{b}_1) = \mathfrak{z}_1$, $\psi(\mathfrak{b}_2) = \mathfrak{z}_2$ and $\psi(\mathfrak{b}_3) = \mathfrak{z}_3$. Then,

$$\begin{aligned}
\big(\psi(\varrho_S)\big)(\mathfrak{z}_1\mathfrak{z}_2\mathfrak{z}_3) &= \bigcup \{\varrho_S(\mathfrak{b}) : \mathfrak{b} \in S, \psi(\mathfrak{b}) = \mathfrak{z}_1\mathfrak{z}_2\mathfrak{z}_3\} \\
&= \bigcup \{\varrho_S(\mathfrak{b}) : \mathfrak{b} \in S, \mathfrak{b} = \psi^{-1}(\mathfrak{z}_1\mathfrak{z}_2\mathfrak{z}_3)\} \\
&= \bigcup \big\{\varrho_S(\mathfrak{b}) : \mathfrak{b} \in S, \mathfrak{b} = \psi^{-1}\big(\psi(\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3)\big) = \mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3\big\} \\
&= \bigcup \{\varrho_S(\mathfrak{b}_1\mathfrak{b}_2\mathfrak{b}_3) : \mathfrak{b}_i \in S, \psi(\mathfrak{b}_i) = \mathfrak{z}_i, i = 1,2,3\} \\
&\supseteq \bigcup \{\varrho_S(\mathfrak{b}_2) \cap \varrho_S(\mathfrak{b}_3) : \mathfrak{b}_2, \mathfrak{b}_3 \in S, \psi(\mathfrak{b}_2) = \mathfrak{z}_2 \text{ and } \psi(\mathfrak{b}_3) = \mathfrak{z}_3\} \\
&= \big(\psi(\varrho_S)\big)(\mathfrak{z}_2) \cap \big(\psi(\varrho_S)\big)(\mathfrak{z}_3)
\end{aligned}$$

Hence, $\psi(\varrho_S)$ is an $\mathfrak{s}$-intersection left WI-*id* of $\mathfrak{T}$.

Theorem 3.58. Let $\varrho_S, \varrho_{\mathfrak{T}} \in S_E(U)$, and ψ be a semigroup isomorphism from S to $\mathfrak{T}$. If ϱ_S is an $\mathfrak{s}$-intersection WI-*id* of S, then $\psi(\varrho_S)$ is an $\mathfrak{s}$-intersection WI-*id* of $\mathfrak{T}$.

Proposition 3.59. Let $\varrho_S, \varrho_{\mathfrak{T}} \in S_E(U)$, and ψ be a semigroup isomorphism from S to $\mathfrak{T}$. If $\varrho_{\mathfrak{T}}$ is an $\mathfrak{s}$-intersection left (right) WI-*id* of $\mathfrak{T}$, then $\psi^{-1}(\varrho_{\mathfrak{T}})$ is an $\mathfrak{s}$-intersection left (right) WI-*id* of S.

Proof: The proof is given exclusively for s-intersection left WI-*id*, as the proof for s-intersection right WI-*id* can be demonstrated analogously. Let $\eta_1, \eta_2, \eta_3 \in S$. Then,

$$\begin{aligned}\left(\psi^{-1}(\varrho_{\mathfrak{T}})\right)(\eta_1\eta_2\eta_3) &= \varrho_{\mathfrak{T}}\left(\psi(\eta_1\eta_2\eta_3)\right)\\ &= \varrho_{\mathfrak{T}}\left(\psi(\eta_1)\psi(\eta_2)\psi(\eta_3)\right)\\ &\supseteq \varrho_{\mathfrak{T}}\left(\psi(\eta_2)\right) \cap \varrho_{\mathfrak{T}}\left(\psi(\eta_3)\right)\\ &= \left(\psi^{-1}(\varrho_{\mathfrak{T}})\right)(\eta_2) \cap \left(\psi^{-1}(\varrho_{\mathfrak{T}})\right)(\eta_3)\end{aligned}$$

Thus, $\psi^{-1}(\varrho_{\mathfrak{T}})$ is an s-intersection left WI-*id* of S.

Theorem 3.60. Let $\varrho_S, \varrho_{\mathfrak{T}} \in S_E(U)$, and ψ be a semigroup isomorphism from S to $\mathfrak{T}$. If $\varrho_{\mathfrak{T}}$ is an s-intersection WI-*id* of $\mathfrak{T}$, then $\psi^{-1}(\varrho_{\mathfrak{T}})$ is an s-intersection WI-*id* of S.

Proposition 3.61. For S, the following conditions are equivalent:

1. S is regular.
2. $\mathfrak{f}_S = \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \mathfrak{f}_S$ for every idempotent s-intersection left WI-*id* $\mathfrak{f}_S$.

Proof: First, let S be regular, $\mathfrak{f}_S$ be an idempotent s-intersection left WI-*id*, and $\mathfrak{m} \in S$. Then, $\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \mathfrak{f}_S \cong \mathfrak{f}_S, \mathfrak{f}_S \circ \mathfrak{f}_S = \mathfrak{f}_S$ and there exists an element $\mathfrak{u} \in S$ such that $\mathfrak{m} = \mathfrak{m}\mathfrak{u}\mathfrak{m}$. Thus,

$$\begin{aligned}\left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \mathfrak{f}_S\right)(\mathfrak{m}) &= \left(\widetilde{\mathbb{S}} \circ \mathfrak{f}_S\right)(\mathfrak{m})\\ &= \bigcup_{\mathfrak{m}=k\mathfrak{q}} \left\{\widetilde{\mathbb{S}}(k) \cap \mathfrak{f}_S(\mathfrak{q})\right\}\\ &\supseteq \widetilde{\mathbb{S}}(\mathfrak{m}\mathfrak{u}) \cap \mathfrak{f}_S(\mathfrak{m})\\ &= U \cap \mathfrak{f}_S(\mathfrak{m})\\ &= \mathfrak{f}_S(\mathfrak{m})\end{aligned}$$

Therefore, $\mathfrak{f}_S \cong \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \mathfrak{f}_S$ implying that $\mathfrak{f}_S = \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \mathfrak{f}_S$.
Conversely, let $\mathfrak{f}_S = \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \mathfrak{f}_S$, where $\mathfrak{f}_S$ is an s-intersection left WI-*id*. We need to show that $\mathfrak{B} = S\mathfrak{B}\mathfrak{B}$ for every left WI-*id* $\mathfrak{B}$. It is obvious that $S\mathfrak{B}\mathfrak{B} \subseteq \mathfrak{B}$. Thus, it is enough to show that $\mathfrak{B} \subseteq S\mathfrak{B}\mathfrak{B}$. Let $\mathrm{n} \in \mathfrak{B}$ and $\mathfrak{B}$ be any left WI-*id*. Then, $S_{\mathfrak{B}}$ is an s-intersection left WI-*id*. Hence,

$$S_{\mathfrak{B}}(\mathrm{n}) = \left(\widetilde{\mathbb{S}} \circ S_{\mathfrak{B}} \circ S_{\mathfrak{B}}\right)(\mathrm{n}) = (S_S \circ S_{\mathfrak{B}} \circ S_{\mathfrak{B}})(\mathrm{n}) = S_{S\mathfrak{B}\mathfrak{B}}(\mathrm{n}) = U$$

implying that $\mathrm{n} \in S\mathfrak{B}\mathfrak{B}$. Thus, $\mathfrak{B} \subseteq S\mathfrak{B}\mathfrak{B}$. Therefore, $\mathfrak{B} = S\mathfrak{B}\mathfrak{B}$, so S is regular.

Proposition 3.62. For S, the following conditions are equivalent:

1. S is regular.
2. $\mathfrak{f}_S = \mathfrak{f}_S \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}$ for every idempotent s-intersection right WI-*id* $\mathfrak{f}_S$.

Proof: First, let S be regular, $\mathfrak{f}_S$ be an idempotent s-intersection right WI-*id*, and $\mathfrak{m} \in S$. Then, $\mathfrak{f}_S \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}} \cong \mathfrak{f}_S, \mathfrak{f}_S \circ \mathfrak{f}_S = \mathfrak{f}_S$, and there exists an element $\mathfrak{u} \in S$ such that $\mathfrak{m} = \mathfrak{m}\mathfrak{u}\mathfrak{m}$. Thus,

$$\begin{aligned}\left(\mathfrak{f}_S \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right)(\mathfrak{m}) &= \left(\mathfrak{f}_S \circ \widetilde{\mathbb{S}}\right)(\mathfrak{m})\\ &= \bigcup_{\mathfrak{m}=k\mathfrak{q}} \left\{\mathfrak{f}_S(k) \cap \widetilde{\mathbb{S}}(\mathfrak{q})\right\}\\ &\supseteq \mathfrak{f}_S(\mathfrak{m}) \cap \widetilde{\mathbb{S}}(\mathfrak{u}\mathfrak{m})\\ &= \mathfrak{f}_S(\mathfrak{m}) \cap U\\ &= \mathfrak{f}_S(\mathfrak{m})\end{aligned}$$

Therefore, $\mathfrak{f}_S \cong \mathfrak{f}_S \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}$ implying that $\mathfrak{f}_S = \mathfrak{f}_S \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}$.
Conversely, let $\mathfrak{f}_S \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}} = \mathfrak{f}_S$, where $\mathfrak{f}_S$ is an s-intersection right WI-*id*. We need to show that $\mathfrak{B} = \mathfrak{B}\mathfrak{B}S$ for every right WI-*id* $\mathfrak{B}$. It is obvious that $\mathfrak{B}\mathfrak{B}S \subseteq \mathfrak{B}$. Thus, it is enough to show that $\mathfrak{B} \subseteq \mathfrak{B}\mathfrak{B}S$. Let $\mathrm{n} \in \mathfrak{B}$ and $\mathfrak{B}$ be any right WI-*id*. Then, $S_{\mathfrak{B}}$ is an s-intersection right WI-*id*. Hence,

$$S_{\mathfrak{B}}(\mathrm{n}) = \left(S_{\mathfrak{B}} \circ S_{\mathfrak{B}} \circ \widetilde{\mathbb{S}}\right)(\mathrm{n}) = (S_{\mathfrak{B}} \circ S_{\mathfrak{B}} \circ S_S)(\mathrm{n}) = S_{\mathfrak{B}\mathfrak{B}S}(\mathrm{n}) = U$$

implying that $\mathrm{n} \in \mathfrak{B}\mathfrak{B}S$. Thus, $\mathfrak{B} \subseteq \mathfrak{B}\mathfrak{B}S$. Therefore, $\mathfrak{B} = \mathfrak{B}\mathfrak{B}S$, so S is regular.

Theorem 3.63. For S, the following conditions are equivalent:

1. S is regular.
2. $\mathfrak{f}_S = \widetilde{\mathbb{S}} \circ \mathfrak{f}_S \circ \mathfrak{f}_S = \mathfrak{f}_S \circ \mathfrak{f}_S \circ \widetilde{\mathbb{S}}$ for every idempotent s-intersection WI-*id* $\mathfrak{f}_S$.

4. Discussion and Conclusion

Rao [6] introduced the concept of weak-interior ideals in semigroups as a generalization of quasi-ideal, interior ideal, left (right) ideal, and ideal and studied the properties. In the present work, we extend this construct into the framework of soft set theory applied to semigroups, and introduce a novel type of the s-intersection ideal called s-intersection weak-interior ideal (abbreviated by s-intersection WI-*id* throughout the text). The main objective of this study is to explore how s-intersection weak-interior ideals relate to other types of s-intersection ideals within

semigroups. Our results establish that every s-intersection weak-interior ideal constitutes an s-intersection subsemigroup within a regular semigroup. Moreover, every s-intersection left (right) ideal is an s-intersection left (right) weak-interior ideal, and every s-intersection interior ideal is an s-intersection weak-interior ideal. Consequently, the concept of an s-intersection weak-interior ideal is a generalization of both s-intersection ideals and s-intersection interior ideals. However, the converses do not hold in general with explicit counterexamples. To satisfy the converses, it is shown that semigroup should be group or the s-intersection weak-interior ideal should be idempotent. Furthermore, we prove that s-intersection bi-ideals and s-intersection quasi-ideals coincide with s-intersection weak-interior ideals within the framework of group structures. Our key theorem, which shows that if a subsemigroup of a semigroup is a weak-interior ideal, then its soft characteristic function is an s-intersection weak-interior ideal, and vice versa, establishes a connection between semigroup theory and soft set theory

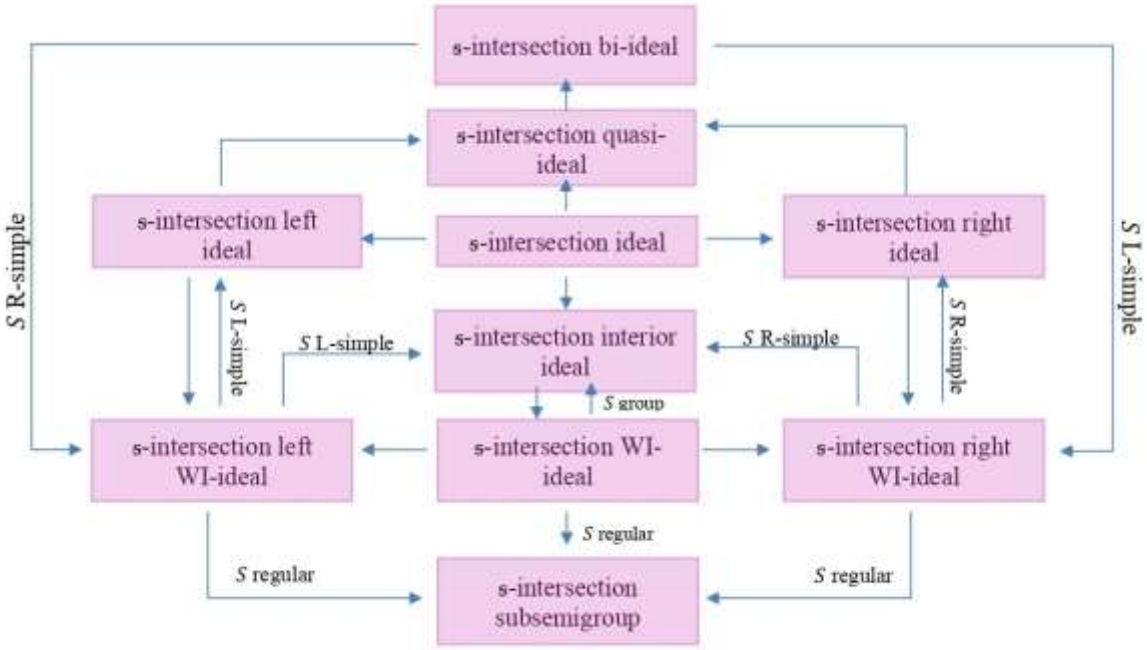

Figure 1. Diagram showing the relationships between some s-intersection ideals

Acknowledgment

This paper is derived from the second author's Master's Thesis supervised by the first author. They all read and approved the final version of the paper.

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