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SOME GEOMETRIC CHARACTERIZATIONS OF A FRACTIONAL BANACH SET

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ABSTRACT. This paper is devoted to investigate the modular structure of a fractional Banach set of sequences and prove that this set is reflexive and convex and it possesses uniform Opial, (β), (L) and (H) properties. The convexity of the set is investigated by the notion of extreme points. These properties play an important role both in the study of fixed point theory and in the geometric characterizations of the Banach sets of sequences. This study extends the scope of the fractional calculus and it is related with fixed point and approximation theories.

1. Introduction

The modulus of convexity is a very useful concept in the theory of geometry of Banach spaces. Furthermore, the modulus of convexity plays an important role in the fixed point theory. On the other hand, the modulus of noncompact convexity which is established by Kuratowski or Hausdorff measures of non-compactness is also an important notion in the geometry of Banach and Hilbert spaces [6]. There are also some constants to geometrically characterize the Banach spaces; for instance Gurarii's modulus of convexity is used to determine whether a set is strictly (or uniformly) convex or not [18]. It is important to study the modular structure of a Banach space and investigate the geometric properties such as the Opial property, Kadec-Klee property ((H) property), (L) property, property (β) and drop property since these properties play a crucial role in the different fields of both pure and applied mathematics. For instance, as (H) property is used to establish some results in the ergodic theory, Opial property has many applications in differential and integral equations as well as in the Banach fixed point theory. Note that, (H) property was originally considered by Radon [35]. The drop property in Banach spaces is

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important since every Banach space which has drop property is reflexive. Recall that, the notion of drop property in a Banach space first introduced by Rolewicz in [36]. Montesinos extended this result by proving that the set X has the drop property if and only if X is reflexive and has the property (H) [30]. A Banach space is called rotund when all points in this space are extreme points. This is why the concept of extreme points is the key notion while geometrically characterizing the Banach spaces.

In this study, we geometrically characterize the fractional Banach set of difference sequences $\ell(\Delta^{(\widetilde{\alpha})}, p)$ and investigate the modular structure of this set. We show that this set is k nearly uniformly convex, therefore it is convex and prove that the set possesses uniform Opial, (β), (L) and (H) properties. These properties imply that this set is also reflexive. We also find the necessary and sufficient conditions for an element $x \in \ell(\Delta^{(\widetilde{\alpha})}, p)$ to be an extreme point. We determine the necessary and sufficient condition for the set $\ell(\Delta^{(\widetilde{\alpha})}, p)$ to be rotund using the extreme points concept. The set $\ell(\Delta^{(\widetilde{\alpha})}, p)$ also has drop property. Note that, these properties play an important role in the study of fixed point theory, for instance, a Banach space with Opial property has weak fixed point property.

We now introduce the notions and notations that are used in this study.

The gamma function which can be written by the improper integral is used to construct the fractional difference operators. The gamma function of a real number x (except zero and the negative integers) is defined by an improper integral:

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt.$$

It is known that for any natural number n, $\Gamma(n+1) = n!$ and $\Gamma(n+1) = n\Gamma(n)$ holds for any real number $n \notin \{0, -1, -2, ...\}$. The fractional difference operator for a fraction $\tilde{\alpha}$ have been defined and studied in [4, 5] as

$$\Delta^{(\tilde{\alpha})}(x_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i!\Gamma(\tilde{\alpha}-i+1)} x_{k-i}.$$

It is assumed that this series is convergent. The defined infinite sum becomes a finite sum if $\tilde{\alpha}$ is a nonnegative integer. We use the usual convention that any term with a negative subscript is equal to naught, throughout the paper. For the values $\tilde{\alpha} = 1/2$ and $\tilde{\alpha} = 2/3$ we have

$$\begin{aligned} \Delta^{(1/2)}(x_k) &= x_k - \frac{1}{2}x_{k-1} - \frac{1}{8}x_{k-2} - \frac{1}{16}x_{k-3} - \frac{5}{128}x_{k-4} - \cdots \\ \Delta^{(2/3)}(x_k) &= x_k - \frac{2}{3}x_{k-1} - \frac{1}{9}x_{k-2} - \frac{4}{81}x_{k-3} - \frac{7}{243}x_{k-4} - \cdots . \end{aligned}$$

Let $S(X)$ and $B(X)$ be the unit sphere and unit ball of a Banach space X, respectively.

Definition 1. *A Banach space X is said to have property (H) if every weakly convergent sequence on $S(X)$ with the weak limit in the sphere is convergent in norm.*

Definition 2. *A Banach space X has the β property if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each element $x \in B(X)$ and each sequence (x_n) in $B(X)$ with $sep(x_n) \geq \varepsilon$ there is an index k for which $\|(x + x_k)/2\| < 1 - \delta$ where $sep(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} > \varepsilon$.*

Definition 3. *A Banach space X is called uniformly convex if, for each $\varepsilon > 0$, there is $\delta > 0$ such that for $x,\ y \in S(X)$ the inequality $\|x - y\| > \varepsilon$ implies $\|(x + y)/2\| < 1 - \delta$. A Banach space X is nearly uniformly convex if for each $\varepsilon > 0$ and every sequence (x_n) in $B(X)$ with $sep(x_n) \geq \varepsilon$ there exists $\delta \in (0, 1)$ such that $conv(x_n) \cap (1 - \delta) B(X) \neq \emptyset$.*

Definition 4. *A Banach space X is said to be k nearly uniformly convex if there exists $\delta > 0$ for any $\varepsilon > 0$ and there are $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that*

$$\|(x_{n_1} + x_{n_2} + \cdots + x_{n_k})/k\| < 1 - \delta \tag{1}$$

holds for any element (x_n) in $B(X)$ with $sep\{x_n\} \geq \varepsilon$.

Every nearly uniformly convex Banach space is reflexive and it has the property (H) by [20]. Rolewicz proved that the β property follows from the uniform convexity and that the property β implies nearly uniform convexity in [37]. It is also known that an k nearly uniformly convex Banach space is nearly uniformly convex but the converse is not true in general.

Definition 5. *A Banach space X has the Opial property if for any weakly null sequence (x_n) in X and any x in $X\backslash\{0\}$, the inequality $\liminf \|x_n\| < \liminf \|x_n + x\|$ holds. A Banach space X has uniform Opial property if for any $\varepsilon > 0$ there exists $\alpha > 0$ such that for any X, the inequality $1 + \alpha < \liminf \|x_n + x\|$ holds.*

Definition 6. *The Hausdorff (or ball) measure of noncompactness of a bounded set $Q \in X$ is*

$\chi(Q) = \inf\{\epsilon > 0 : Q$ *can be covered by finite number of balls radii smaller then* $\varepsilon\}$.

The function Δ defined by

$\Delta(\varepsilon) = \inf\{1 - \inf(\|x\| : x \in Q) : Q$ *is closed convex subset of* $B(X)$ *with* $\chi(Q) \leq \varepsilon\}$

is called the modulus of noncompact convexity. A Banach space X has property (L) if $\lim_{\varepsilon \to 1^-} \Delta(\varepsilon) = 1$.

A Banach space X has property (L) if and only if it is reflexive and has the uniform Opial property.

Definition 7. *The drop determined by x is $D(x, B(X)) = conv(\{x\} \cup B(X))$ for any $x \notin B(X)$. If for every closed set C disjoint with $B(X)$, there exists an element $x \in C$ such that $D(x, B(X)) \cap C = \{x\}$ we say Banach space X has drop property.*

2. The Modular structure of the set $\ell(\Delta^{(\widetilde{\alpha})},p)$ and related characterizations

For a real vector space X, a functional $\sigma : X \to [0,\infty]$ is called a *modular* if it satisfies the following conditions:

(a) $\sigma(x) = 0$ if and only if $x = 0$,
(b) $\sigma(\alpha x) = \sigma(x)$ for all scalars α with $|\alpha| = 1$,
(c) $\sigma(\alpha x + \beta y) \leq \sigma(x) + \sigma(y)$ for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

Moreover, the modular σ is called *convex* if

(d) $\sigma(\alpha x + \beta y) \leq \alpha\sigma(x) + \beta\sigma(y)$ holds for all $x, y \in X$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

The space

$$X_\sigma = \{x \in X : \sigma(\lambda x) < \infty \text{ for some } \sigma > 0\}$$

is called the *modular space* for any modular σ on X.

A modular σ is said to satisfy the δ_2 condition, if for any $\varepsilon > 0$ there exists constant $K \geq 2$ and $a > 0$ such that $\sigma(2u) \leq K\sigma(u) + \varepsilon$ for all $u \in X_\sigma$ with $\sigma(u) \leq a$. If σ satisfies the δ_2 condition for any $a > 0$ with $K \geq 2$ depending on a, we say that σ satisfies the strong δ_2 condition that is $\sigma \in \delta_2^s$.

A sequence (x_n) in X_σ is called modular convergent to $x \in X_\sigma$ if there exists a $\alpha > 0$ such that $\sigma(\alpha(x_n - x)) \to 0$ as $n \to \infty$. Convergence in norm and in modular are equivalent in X_σ if $\sigma \in \delta_2$ [15, Lemma 2.3].

Let (p_k) be a sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$. We now define the modular set of sequences $\ell(\Delta^{(\widetilde{\alpha})},p)$ of fractional orders by

$$\ell(\Delta^{(\widetilde{\alpha})},p) := \{x : \sigma_p(\lambda x) < \infty \text{ for some } \lambda > 0\}, \text{ where } \sigma_p(x) = \sum_{k=0}^{\infty} \left|\Delta^{(\widetilde{\alpha})} x(k)\right|^{p_k}.$$

Here $\sigma_p : \ell(\Delta^{(\widetilde{\alpha})},p) \to [0,\infty]$ is called a *modular* on the set $\ell(\Delta^{(\widetilde{\alpha})},p)$. Note that the modular σ_p on $\ell(\Delta^{(\widetilde{\alpha})},p)$ is a continuous and convex modular.

The *Luxemburg norm* on the set $\ell(\Delta^{(\widetilde{\alpha})},p)$ is defined by

$$\|x\| = \inf\left\{\lambda > 0 : \sigma_p\left(\frac{x}{\lambda}\right) \leq 1\right\}$$

for all $x \in \ell(\Delta^{(\widetilde{\alpha})},p)$. It is not difficult to show that the space $\ell(\Delta^{(\widetilde{\alpha})},p)$ is a Banach space with defined Luxemburg norm and a complete paranormed space with

$$\upsilon(x) = \left(\sum_{k=0}^{\infty} \left|\Delta^{(\widetilde{\alpha})} x(k)\right|^{p_k}\right)^{1/\Lambda}, \text{ where } \Lambda = \max\{1, \sup p_k\}.$$

We assume that the sequence (p_k) is bounded with $p_k > 1$ for all $k \in \mathbb{N}$ throughout the paper and introduce the sequences $x^k = (0, 0, \ldots, 0, x(k), x(k+1), x(k+2), \ldots)$ and $x_n^k = (0, 0, \ldots, 0, x_n(k), x_n(k+1), x_n(k+2), \ldots)$. We now state, mostly lacking in proofs, some primary relations and basic properties of modular $\sigma_p(x)$ on the space $\ell(\Delta^{(\widetilde{\alpha})},p)$. Detailed proofs, that we leave to the reader, can be obtained by using standard techniques in [15] and by direct application of the definition of modular $\sigma_p(x)$.

Proposition 8. *Let (x_n) be a sequence in $\ell(\Delta^{(\widetilde{\alpha})}, p)$. Then, we have*

(a) $\|x\| < 1$ *implies* $\sigma_p(x) \leq \|x\|$,
(b) $\|x\| > 1$ *implies* $\sigma_p(x) \geq \|x\|$,
(c) $\|x\| = 1$ *if and only if* $\sigma_p(x) = 1$,
(d) *If* $0 < \alpha < 1$ *and* $\|x\| > \alpha$, *then* $\sigma_p(x) > \alpha^M$,
(e) *If* $\alpha \geq 1$ *and* $\|x\| < \alpha$, *then* $\sigma_p(x) < \alpha^M$.

Proposition 9. *Let (x_n) be a sequence in $\ell(\Delta^{(\widetilde{\alpha})}, p)$. Then, we have*

(a) $\lim_{n\to\infty} \|x_n\| = 1$ *implies* $\lim_{n\to\infty} \sigma_p(x_n) = 1$,
(b) $\lim_{n\to\infty} \sigma_p(x_n) = 0$ *implies* $\lim_{n\to\infty} \|x_n\| = 0$.

Proof. Let $\lim_{n\to\infty} \|x_n\| = 1$ and $\varepsilon \in (0,1)$ then for all $n \geq m$ there exists $m \in \mathbb{N}$ such that $|\|x_n\| - 1| < \varepsilon$. Since $(1-\varepsilon)^M < \|x_n\| < (1+\varepsilon)^M$ via Parts (d-e) of Proposition 8 we get $\sigma_p(x_n) \geq (1-\varepsilon)^M$ and $\sigma_p(x_n) \leq (1-\varepsilon)^M$. Whence $\lim_{n\to\infty} \sigma_p(x_n) = 1$.

On the other hand, let $\lim_{n\to\infty} \|x_n\| \neq 0$ then there exist a number $\varepsilon \in (0,1)$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \varepsilon$ for all $k \in \mathbb{N}$. With the aid of Part (d) of Proposition 8 we deduce $\sigma_p(x_{n_k}) > \varepsilon^M$ for all $k \in \mathbb{N}$. This implies $\lim_{n\to\infty} \sigma_p(x_{n_k}) \neq 0$. Hence $\lim_{n\to\infty} \sigma_p(x_n) \neq 0$. □

A point $x \in S(X)$ is called an extreme point if $2x = y + z$ implies $y = z$ for every $y, z \in S(X)$. A Banach space X is said to be rotund (strictly convex) if every point of $S(X)$ is an extreme point, that is, for any two points $y, z \in S(X)$ the equality $\|y + z\| = 2$ implies $y = z$. We now give the necessary and sufficient condition for the set $\ell(\Delta^{(\widetilde{\alpha})}, p)$ to be rotund.

Theorem 10. *The fractional Banach set $\ell(\Delta^{(\widetilde{\alpha})}, p)$ is rotund if and only if $p_k > 1$ for all $k \in \mathbb{N}$.*

Proof. Let the set $\ell(\Delta^{(\widetilde{\alpha})}, p)$ be rotund and $m \in \mathbb{N}$ so that $p_m = 1$. Consider the following different sequences α and β given by

$$\begin{aligned}
\alpha &= \left(\frac{1}{2\sum_i (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i!\Gamma(\tilde{\alpha}-i+1)}}, \frac{1}{2\sum_i (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i!\Gamma(\tilde{\alpha}-i+1)}}, 0, 0, \ldots \right), \\
\beta &= \left(\frac{-1}{2\sum_i (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i!\Gamma(\tilde{\alpha}-i+1)}}, \frac{-1}{2\sum_i (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i!\Gamma(\tilde{\alpha}-i+1)}}, \frac{2}{\sum_i (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i!\Gamma(\tilde{\alpha}-i+1)}}, 0, 0, \ldots \right).
\end{aligned}$$

Then we have

$$\sigma_p(\alpha) = \sigma_p(\beta) = \sigma_p\left(\frac{\alpha+\beta}{2}\right) = 1.$$

By means of Part (a) of Proposition 8, for the points α, β, $(\alpha+\beta)/2 \in S(\ell(\Delta^{(\widetilde{\alpha})}, p))$ we find a point on $S(\ell(\Delta^{(\widetilde{\alpha})}, p))$ which is not an extreme point, contradiction. Hence the set $\ell(\Delta^{(\widetilde{\alpha})}, p)$ is not rotund when the inequality $p_k > 1$ does not satisfy.

On the other hand, let $x \in S[\ell(\Delta^{(\widetilde{\alpha})}, p)]$ and $y, z \in B[\ell(\Delta^{(\widetilde{\alpha})}, p)]$ with $(z+y)/2 = x$. Since the modular σ_p is convex and the condition in Part (c) of Proposition 8 holds, we have

$$1 = \sigma_p(x) \leq \frac{1}{2}(\sigma_p(y) + \sigma_p(z)) \leq \frac{1}{2} + \frac{1}{2} = 1,$$

which gives that $\sigma_p(y) = \sigma_p(z) = 1$ and

$$\sigma_p(x) = \frac{1}{2}(\sigma_p(y) + \sigma_p(z)).$$

Moreover, we have by this fact that

$$\sum_k \left|\left(\Delta^{(\widetilde{\alpha})}x\right)_k\right|^{p_k} = \frac{1}{2}\sum_k \left|\left(\Delta^{(\widetilde{\alpha})}y\right)_k\right|^{p_k} + \frac{1}{2}\sum_k \left|\left(\Delta^{(\widetilde{\alpha})}z\right)_k\right|^{p_k}.$$

Using the equality $(y+z)/2 = x$, we have

$$\sum_k \left|\frac{1}{2}\left[\Delta^{(\widetilde{\alpha})}(y+z)\right]_k\right|^{p_k} = \frac{1}{2}\sum_k \left|\left(\Delta^{(\widetilde{\alpha})}y\right)_k\right|^{p_k} + \frac{1}{2}\sum_k \left|\left(\Delta^{(\widetilde{\alpha})}z\right)_k\right|^{p_k}.$$

This implies

$$\left|\frac{1}{2}\left(\Delta^{(\widetilde{\alpha})}y\right)_k + \frac{1}{2}\left(\Delta^{(\widetilde{\alpha})}z\right)_k\right|^{p_k} = \frac{1}{2}\left|\left(\Delta^{(\widetilde{\alpha})}y\right)_k\right|^{p_k} + \frac{1}{2}\left|\left(\Delta^{(\widetilde{\alpha})}z\right)_k\right|^{p_k} \text{ for all } k \in \mathbb{N}.$$

Then we have

$$\left(\Delta^{(\widetilde{\alpha})}y\right)_k = \left(\Delta^{(\widetilde{\alpha})}z\right)_k \text{ for all } k \in \mathbb{N}. \tag{2}$$

It follows by (2) that $y_k = z_k$ for all $k \in \mathbb{N}$ since the function $t \longrightarrow |t|^{p_k}$ is strictly convex for all $k \in \mathbb{N}$. Hence $y = z$, that is, the space $\ell(\Delta^{(\widetilde{\alpha})}, p)$ is rotund. □

We now give the following result to prove that defined modular fractional set of sequences has (H) property.

Proposition 11. *Let x be a point in $\ell(\Delta^{(\widetilde{\alpha})}, p)$ and the sequence (x_n) be an element of the space $\ell(\Delta^{(\widetilde{\alpha})}, p)$. If $\sigma_p(x_n) \to \sigma_p(x)$ as $n \to \infty$ and $x_n(j) \to x(j)$ as $n \to \infty$ for all $j \in \mathbb{N}$, then $x_n \to x$ as $n \to \infty$.*

Proof. Let $\Lambda = \max\left\{1, 2^{M-1}\right\}$ and $\varepsilon > 0$. There exists $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty} \left|\sum_{i=0}^{\infty}(-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i!\Gamma(\tilde{\alpha}-i+1)} x(k-i)\right|^{p_k} < \frac{\varepsilon}{6\Lambda} \tag{3}$$

since the modular $\sigma_p(x) = \sum_{k=0}^{\infty} \left| \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\tilde{\alpha}+1)}{i!\Gamma(\tilde{\alpha}-i+1)} x(k-i) \right|^{p_k}$ is bounded. The following equality

$$\lim_{n\to\infty} \left[\sigma_p(x_n) - \sum_{k=1}^{k_0} \left| \Delta^{(\widetilde{\alpha})} x_n(k) \right|^{p_k} \right] = \sigma_p(x) - \sum_{k=1}^{k_0} \left| \Delta^{(\widetilde{\alpha})} x(k) \right|^{p_k}$$

holds since $\Delta^{(\widetilde{\alpha})} x_n(k) \to \Delta^{(\widetilde{\alpha})} x(k)$ as $n \to \infty$ for all $k \in \mathbb{N}$ and $\sigma_p(x_n) \to \sigma_p(x)$. Therefore, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $k \in \mathbb{N}$ the following inequalities hold:

$$\sigma_p(x_n) - \sum_{k=1}^{k_0} \left| \Delta^{(\widetilde{\alpha})} x_n(k) \right|^{p_k} < \sigma_p(x) - \sum_{k=1}^{k_0} \left| \Delta^{(\widetilde{\alpha})} x(k) \right|^{p_k} + \frac{\varepsilon}{3\Lambda}, \tag{4}$$

$$\sum_{k=1}^{k_0} \left| \Delta^{(\widetilde{\alpha})} \{x_n(k) - x(k)\} \right|^{p_k} < \frac{\varepsilon}{3}. \tag{5}$$

Then, combining (3), (4) and (5) it follows for all $n \geq n_0$ that

$$\begin{aligned}
\sigma_p(x_n - x) &= \sum_{k=1}^{\infty} \left| \Delta^{(\widetilde{\alpha})} \{x_n(k) - x(k)\} \right|^{p_k} \\
&< \sum_{k=1}^{k_0} \left| \Delta^{(\widetilde{\alpha})} \{x_n(k) - x(k)\} \right|^{p_k} + \sum_{k=k_0+1}^{\infty} \left| \Delta^{(\widetilde{\alpha})} \{x_n(k) - x(k)\} \right|^{p_k} \\
&< \Lambda \left[\sum_{k=k_0+1}^{\infty} \left| \Delta^{(\widetilde{\alpha})} x_n(k) \right|^{p_k} + \sum_{k=k_0+1}^{\infty} \left| \Delta^{(\widetilde{\alpha})} x(k) \right|^{p_k} \right] + \frac{\varepsilon}{3} \\
&= \Lambda \left[\sigma_p(x_n) - \sum_{k=1}^{k_0} \left| \Delta^{(\widetilde{\alpha})} x_n(k) \right|^{p_k} + \sum_{k=k_0+1}^{\infty} \left| \Delta^{(\widetilde{\alpha})} x(k) \right|^{p_k} \right] + \frac{\varepsilon}{3} \\
&< \Lambda \left[\sigma_p(x) - \sum_{k=1}^{k_0} \left| \Delta^{(\widetilde{\alpha})} x(k) \right|^{p_k} + \sum_{k=k_0+1}^{\infty} \left| \Delta^{(\widetilde{\alpha})} x(k) \right|^{p_k} + \frac{\varepsilon}{3\Lambda} \right] + \frac{\varepsilon}{3} \\
&= \Lambda \left[2 \sum_{k=k_0+1}^{\infty} \left| \Delta^{(\widetilde{\alpha})} x(k) \right|^{p_k} + \frac{\varepsilon}{3\Lambda} \right] + \frac{\varepsilon}{3} \\
&< \Lambda \left(2\frac{\varepsilon}{6\Lambda} + \frac{\varepsilon}{3\Lambda} \right) + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

These relations imply $\lim_n \sigma_p(x_n - x) = 0$. Finally, it follows by Part (b) of Proposition 9 that $\lim_n \|x_n - x\| = 0$. □

Theorem 12. *The set $\ell(\Delta^{(\widetilde{\alpha})}, p)$ has (H) property.*

Proof. Let x be a point in $S[\ell(\Delta^{(\widetilde{\alpha})}, p)]$ and the sequence (x_n) be included by the set $B[\ell(\Delta^{(\widetilde{\alpha})}, p)]$ such that $\|x_n\| \to 1$ and $x_n \to x$ weakly as $n \to \infty$. We have

$\sigma_p(x) = 1$ by Part (c) of Proposition 8 and it follows from Part (a) of Proposition 9 that $\sigma_p(x_n) \to \sigma_p(x)$ as $n \to \infty$. Since $x_n \to x$ weakly as $n \to \infty$ and the $m^{th}-$coordinate mapping $\theta_m : \ell(\Delta^{(\widetilde{\alpha})}, p) \to \mathbb{R}$ defined by $\theta_m(x) = \Delta^{(\widetilde{\alpha})} x(m)$ is a continuous linear function on $\ell(\Delta^{(\widetilde{\alpha})}, p)$, it follows that $\Delta^{(\widetilde{\alpha})} x_n(m) \to \Delta^{(\widetilde{\alpha})} x(m)$ as $n \to \infty$ for all $m \in \mathbb{N}$. Finally, we have $x_n \to x$ as $n \to \infty$ by Proposition 11. Since any weakly convergent sequence in $\ell(\tilde{B}, p)$ is convergent, the set $\ell(\tilde{B}, p)$ has (H) property. □

Lemma 13. *[29, Lemma 2.2] There exists $\varepsilon \in (0,1)$ such that $\sigma_p(x) \le 1 - \lambda$ implies $\|x\| \le 1 - \varepsilon$ for any $x \in X$ and $\lambda \in (0,1)$.*

Lemma 14. *[15, Lemma 2.1] There exists $\zeta = \zeta(L, \varepsilon)$ such that $|\sigma(u+v) - \sigma(u)| < \varepsilon$ for any $L > 0$ and $\varepsilon > 0$ whenever u, $v \in X$ with $\sigma(u) \le L$ and $\sigma(v) \le \zeta$ if $\sigma \in \delta_2^s$.*

Theorem 15. *The fractional Banach set $\ell(\Delta^{(\widetilde{\alpha})}, p)$ is k nearly uniformly convex.*

Proof. Let the sequence (x_n) be included by $B[\ell(\Delta^{(\widetilde{\alpha})}, p)]$ and $\text{sep}(x_n) \ge \varepsilon$ with $\varepsilon > 0$. Then, there exists a subsequence (x_{n_j}) of (x_n) such that $(x_{n_j}(i))$ converges for all $i \in \mathbb{N}$, $1 \le i \le m$ since the sequence $(x_n(i))_{n=1}^{\infty}$ is bounded for all $i \in \mathbb{N}$. It means there exists an increasing sequence (θ_m) such that $\text{sep}\left\{(x_{n_j}^m)_{i>\theta_m}\right\} \ge \varepsilon$ for any $i \in \mathbb{N}$. So there exists an increasing sequence of positive integers $(\tau_m)_{m=1}^{\infty}$ for any $m \in \mathbb{N}$ such that $\left\|x_{\tau_m}^m\right\| \ge \varepsilon/2$. We may suppose that there exists $\mu > 0$ such that

$$\sigma_p(x_{\tau_m}^m) \ge \mu \text{ for all } m \in \mathbb{N} \tag{6}$$

by Proposition 9. Let $\varOmega = \dfrac{\mu}{2}\left\{\dfrac{k^{\alpha} - k}{(k-1)k^{\alpha+1}}\right\}$ for fixed integers $k \ge 2$, where $1 < \alpha < \liminf p_n$ with $\alpha > 0$. Taking into account Lemma 14 there exists $\zeta > 0$ such that

$$|\sigma_p(u+v) - \sigma_p(u)| \le \varOmega \tag{7}$$

whenever $\sigma_p(u) \le 1$ and $\sigma_p(v) \le \zeta$. There exists positive integers m_i $(i = 1, 2, \ldots, k-1)$ with $m_1 < m_2 < \cdots < m_{k-1}$ such that $\sigma_p\left(x_{p_i}^{m_i}\right) \le \zeta$ and $\alpha \le p_i$ for all $j \ge m_{k-1}$ by Part (a) of Proposition 8. Assume that $s_i = i(1 \le i \le k-1)$, $s_k = \tau_{m_k}$ and $m_k = m_{k-1} + 1$, then $\sigma_p(x_{\tau_{m_k}}^{m_k}) \ge \mu$ holds by (6). Since the function $\theta \longrightarrow |\theta|^{p_k}$ is convex for all $k \in \mathbb{N}$ and (6), (7) hold, we get

$$\begin{aligned}
\sigma_p\left(\frac{x_{\tau_1} + x_{\tau_2} + \ldots + x_{\tau_k}}{k}\right) &= \sum_{n=1}^{\infty}\left|\frac{\Delta^{(\widetilde{\alpha})}\left\{x_{\tau_1}(n) + x_{\tau_2}(n) + \ldots + x_{\tau_k}(n)\right\}}{k}\right|^{p_n} \\
&= \sum_{n=1}^{m_1}\sum_{j=1}^{k}\left|\frac{\Delta^{(\widetilde{\alpha})}\left\{x_{\tau_j}(n)\right\}}{k}\right|^{p_n}
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{n=m_1+1}^{m_2} \left| \frac{\Delta^{(\widetilde{\alpha})}\left\{x_{\tau_1}(n)+x_{\tau_2}(n)+\ldots+x_{\tau_k}(n)\right\}}{k} \right|^{p_n} \\
& + \sum_{n=m_2}^{\infty} \left| \frac{\Delta^{(\widetilde{\alpha})}\left\{x_{\tau_1}(n)+x_{\tau_2}(n)+\ldots+x_{\tau_k}(n)\right\}}{k} \right|^{p_n} + \Omega \\
\leq\; & \sum_{n=1}^{m_1} \sum_{j=1}^{k} \left| \Delta^{(\widetilde{\alpha})}\left\{x_{\tau_j}(n)\right\} \right|^{p_n} + \sum_{n=m_1+1}^{m_2} \sum_{j=2}^{k} \left| \Delta^{(\widetilde{\alpha})}\left\{x_{\tau_j}(n)\right\} \right|^{p_n} \\
& + \sum_{n=m_2+1}^{m_3} \sum_{j=3}^{k} \left| \Delta^{(\widetilde{\alpha})}\left\{x_{\tau_j}(n)\right\} \right|^{p_n} + \cdots \\
& + \sum_{n=m_{k-1}+1}^{m_k} \sum_{j=k-1}^{k} \left| \Delta^{(\widetilde{\alpha})}\left\{x_{\tau_j}(n)\right\} \right|^{p_n} \\
& + \sum_{n=m_k+1}^{\infty} \left| \frac{\Delta^{(\widetilde{\alpha})}\left\{x_{\tau_k}(n)\right\}}{k} \right|^{p_n} + \Omega(k-1) \\
\leq\; & \frac{1}{k} \sum_{n=1}^{m_k} \left| \Delta^{(\widetilde{\alpha})}\left\{x_{\tau_k}(n)\right\} \right|^{p_n} + \sum_{n=m_k+1}^{\infty} \left| \frac{\Delta^{(\widetilde{\alpha})}\left\{x_{\tau_k}(n)\right\}}{k} \right|^{p_n} \\
& + \frac{\sigma_p(x_{\tau_1})+\sigma_p(x_{\tau_2})+\cdots+\sigma_p(x_{\tau_{k-1}})}{k} + \Omega(k-1) \\
\leq\; & \frac{1}{k} \sum_{n=1}^{m_k} \left| \Delta^{(\widetilde{\alpha})}\left\{x_{\tau_k}(n)\right\} \right|^{p_n} + \frac{1}{k^{\alpha}} \sum_{n=m_k+1}^{\infty} \left| \Delta^{(\widetilde{\alpha})}\left\{x_{\tau_k}(n)\right\} \right|^{p_n} \\
& + \frac{k-1}{k} + \Omega(k-1) \\
\leq\; & \frac{1}{k} - \frac{1}{k} \sum_{n=m_k+1}^{\infty} \left| \Delta^{(\widetilde{\alpha})}\left\{x_{\tau_k}(n)\right\} \right|^{p_n} + \frac{1}{k^{\alpha}} \sum_{n=m_k+1}^{\infty} \left| \Delta^{(\widetilde{\alpha})}\left\{x_{\tau_k}(n)\right\} \right|^{p_n} \\
& + \frac{k-1}{k} + \Omega(k-1) \\
\leq\; & 1 - \frac{\mu}{2} \left\{ \frac{1}{k} - \frac{1}{k^{\alpha}} \right\}.
\end{aligned}
$$

Therefore, (1) holds. This completes the proof, by Lemma 13. □

As an immediate consequence of Theorem 15 we have the following result.

Corollary 16. *The fractional Banach set* $\ell(\Delta^{(\widetilde{\alpha})}, p)$ *has* (β) *property.*

Lemma 17. *[15, Lemma 2.4] There exists* $\varphi = \varphi(\varepsilon) > 0$ *such that* $\|x\| \geq 1 + \varphi$ *implies* $\sigma_p(x) \geq 1 + \varepsilon$ *for any* $\varepsilon > 0$ *if* $\sigma_p \in \delta_2^s$.

Theorem 18. *Let* $\limsup p_k < \infty$*, then the fractional Banach set* $\ell(\Delta^{(\widetilde{\alpha})}, p)$ *has uniform Opial property.*

Proof. Let the sequence (x_n) be a weakly null sequence in $S[\ell(\Delta^{(\widetilde{\alpha})}, p)]$, where $x \in \ell(\Delta^{(\widetilde{\alpha})}, p)$ with $\|x\| \geq \varepsilon$ for $\varepsilon > 0$. There exists $\xi \in (0,1)$ independent of x such that $\sigma_p(x) > \xi$ by Lemma 14 since $\limsup p_k < \infty$, that is $\sigma_p(x) \in \delta_2^s$. There is also $\xi_1 = (0, \xi)$ such that

$$|\sigma_p(u+v) - \sigma_p(u)| < \xi/4 \tag{8}$$

whenever $\sigma_p(u) \leq 1$ and $\sigma_p(v) \leq \xi_1$ since $\sigma_p(x) \in \delta_2^s$. We now have the inequality

$$\sum_{k=a+1}^{\infty} \left|\Delta^{(\widetilde{\alpha})} x\left(k\right)\right|^{p_k} < \xi_1/4 \tag{9}$$

for a chosen number $a \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} \xi &< \sum_{k=1}^{a} \left|\Delta^{(\widetilde{\alpha})} x\left(k\right)\right|^{p_k} + \sum_{k=a+1}^{\infty} \left|\Delta^{(\widetilde{\alpha})} x\left(k\right)\right|^{p_k} \\ &\leq \frac{\xi_1}{4} + \sum_{k=1}^{a} \left|\Delta^{(\widetilde{\alpha})} x\left(k\right)\right|^{p_k} \end{aligned}$$

which implies

$$\sum_{k=1}^{a} \left|\Delta^{(\widetilde{\alpha})} x\left(k\right)\right|^{p_k} > \xi - \frac{\xi_1}{4} > \frac{3\xi}{4}. \tag{10}$$

There exists $a \in \mathbb{N}$ such that

$$\sum_{k=1}^{a} \left|\Delta^{(\widetilde{\alpha})} \left(x_n\left(k\right) + x\left(k\right)\right)\right|^{p_k} > \frac{3\xi}{4} \tag{11}$$

since the weak convergence implies the coordinate-wise convergence, the fractional operator $\Delta^{(\widetilde{\alpha})}$ is linear and (10) holds. Since (x_n) is a weakly null sequence, there exists $a_1 > a$ for all $n > a_1$ such that $\|x\| < 1 - (1 - ((4-\xi)/4))^{1/M}$, where $p_n \leq M \in \mathbb{N}$ for each $n \in \mathbb{N}$. Then, we have $\|x\| > ((4-\xi)/4)^{1/M}$ since the norm satisfies the triangle inequality. Taking into account the definition of the norm, we have

$$\begin{aligned} 1 &\leq \sum_{k=a+1}^{\infty} \left(\frac{\left|\Delta^{(\widetilde{\alpha})} x_n\left(k\right)\right|}{((4-\xi)/4)^{1/M}} \right)^{p_k} \\ &\leq \left(\frac{1}{[(4-\xi)/4]^{1/M}} \right)^{M} \sum_{k=a+1}^{\infty} \left|\Delta^{(\widetilde{\alpha})} x_n\left(k\right)\right|^{p_k} \\ &= \left(\frac{4}{4-\xi} \right) \sum_{k=a+1}^{\infty} \left|\Delta^{(\widetilde{\alpha})} x_n\left(k\right)\right|^{p_k}. \end{aligned}$$

Taking into account this result and the relations (8), (9), (10) and (11), we have for any $n > a_1$

$$\begin{aligned}\sigma_p\left(x_n + x\right) &= \sum_{k=1}^{a}\left|\Delta^{(\widetilde{\alpha})}\left(x_n\left(k\right) + x\left(k\right)\right)\right|^{p_k} + \sum_{k=a+1}^{\infty}\left|\Delta^{(\widetilde{\alpha})}\left(x_n\left(k\right) + x\left(k\right)\right)\right|^{p_k} \\ &\geq \frac{\xi}{2} + \sum_{k=a+1}^{\infty}\left|\Delta^{(\widetilde{\alpha})}\left(x_n\left(k\right)\right)\right|^{p_k} \geq 1 + \frac{\xi}{4}\end{aligned}$$

since the operator is linear and the weakly convergence implies the coordinate-wise convergence. By means of Lemma 17, there exists φ only depending on ξ such that $\|x_n + x\| \geq 1+\varphi$ since $\sigma_p \in \delta_2^s$. It means $\liminf \|x_n + x\| \geq 1+\varphi$, which completes the proof. □

Remark 19. *Let X be a Banach space, then a non-empty set $C \subset X$ is said to be approximatively compact if for any sequence $(x_n)_{n=1}^{\infty}$ in C and any $y \in X$ such that $\|x_n - y\| \longrightarrow d(y, C)$, it follows that $(x_n)_{n=1}^{\infty}$ has a Cauchy subsequence. The set X is called approximatively compact if any non-empty closed and convex set in X is approximatively compact [14]. It is worth mentioning that approximatively compactness is strongly related to the approximation theory for Banach spaces. It is also known that a Banach space X is approximatively compact if and only if X is reflexive and X has property (H) [19, Theorem 3]. Note that, the notion of approximative compactness is connected with reflexivity and property (H) in Banach spaces; that is the drop property and approximative compactness coincide.*

3. Conclusion

While the Hausdorff measure of non-compactness is established to study modulus of noncompact convexity which is important in the geometry of Banach and Hilbert spaces it is also used to find necessary and sufficient conditions for a matrix operator on a given sequence space to be a compact operator (see [24, 25, 32]). Topological properties of certain sets of sequence spaces are investigated in the papers [1–3, 7–10, 13, 16, 21–23, 31, 33, 34] and [40]. Geometric properties of certain sets were also studied in [11, 12, 17, 29]. A comprehensive study about the theory of FK spaces and their applications can be found in the monograph [7]. Note that, the graphical representations of neighborhoods in the norms of certain FK spaces and in their duals were illustrated in [26–28, 39]. The main aim of this study is to consider the fractional Banach set $\ell(\Delta^{(\widetilde{\alpha})}, p)$, which is established by gamma function. The modular structure of this set is investigated and the geometric characterizations of the given set are determined. It is found that the fractional Banach set $\ell(\Delta^{(\widetilde{\alpha})}, p)$ is strictly convex (rotund) and k nearly uniformly convex by Theorem 10 and Theorem 15. It is also proved that this set has (H) and (β) properties by Theorem 12 and Corollary 16. Taking into account these facts we conclude that the set has drop property and it is convex and reflexive. Since defined fractional Banach set is reflexive and it has property (H), that is it has drop property, it is also approximatively compact. The modulus of noncompact convexity $\Delta(\varepsilon)$ for the defined Banach set is 1 as $\varepsilon \to 1^-$. It means the set has (L) property. The set has weak fixed point property since it has uniform Opial and (L) properties by Theorem 18. Note that, this study extends the scope of the fractional calculus and it is related with fixed point and approximation theories.

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