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Predicting bc̄ Meson Mass Spectra Using Neural Network with Combined Cornell and Quadratic Potentials

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Abstract: This research explores the mass spectra of the $b\bar{c}$ meson using the neural network (NN) method that combines the Cornell potential with the addition of another quadratic potential term. By training the NN model on theoretical results and comparing its predictions with existing theoretical and experimental data from the literature, we demonstrate that this mixed-potential model effectively describes the $b\bar{c}$ meson properties. Our treatment justifies the usefulness of merging machine learning methods with classical potential approaches for meson spectroscopy applications. In addition, the approach is generalized into an ideal four-dimensional space for its application in the study of the impact of more spatial dimensions on quarkonium systems. The results in this extended setting agree with expectations, confirming that the NN-type approach is strong and flexible with its application into more than three spatial dimensions for quantum systems. In all, our research proves the efficacy of neural network-facilitated models in promoting theoretical research in heavy meson physics and presents a door for further research into quantum systems in more dimensions.

Birleştirilmiş Cornell ve Kuadratik Potansiyeller ile Sinir Ağı Kullanarak bc̄ Mezon Kütle Spektrumlarının Tahmini

Keywords
Hadronlar,
Makine Öğrenmesi,
KRD

Öz: Bu araştırma, Cornell potansiyelini başka bir kuadratik potansiyel terimi ile birleştiren sinir ağı (NN) yöntemini kullanarak bc̄ mezonunun kütle spektrumlarını incelemektedir. NN modelini teorik sonuçlar üzerinde eğiterek ve tahminlerini literatürdeki mevcut teorik ve deneysel verilerle karşılaştırarak, bu karışık potansiyel modelinin bc̄ mezonunun özelliklerini etkili bir şekilde tanımladığını göstermekteyiz. Çalışmamız, makine öğrenimi yöntemlerinin klasik potansiyel yaklaşımlarla birleştirilmesinin mezon spektroskopisi uygulamaları için yararlı olduğunu doğrulamaktadır. Ayrıca, bu yaklaşım, daha fazla uzaysal boyutun kuarkonyum sistemleri üzerindeki etkisinin incelenmesinde uygulanabilmesi için ideal bir dört boyutlu uzaya genelleştirilmiştir. Bu genişletilmiş ortamdaki sonuçlar beklentilerle uyumludur ve NN tipi yaklaşımın kuantum sistemleri için üçten fazla uzaysal boyutta uygulanabilirliği açısından güçlü ve esnek olduğunu doğrulamaktadır. Sonuç olarak, araştırmamız ağır mezon fiziğinde teorik araştırmayı teşvik etmede sinir ağı destekli modellerin etkinliğini kanıtlamakta ve daha fazla boyutta kuantum sistemleri üzerine daha fazla araştırma için bir kapı açmaktadır.

1. Introduction

The examination of heavy quarkonium systems, including B_c ($b\bar{c}$) mesons, is essential to understanding the quantitative dimensions of Quantum Chromodynamics (QCD) and the Standard Model. Quarkonium systems offer a testing ground for various theoretical techniques including lattice QCD, QCD sum rules, and other methods that provide insights into the strong interactions [1-9]. The Schrödinger equation (SE) is one of the most important tools for investigating the properties of the constituent particles and the dynamics of their interactions in nuclear and subnuclear physics. Nevertheless, the Schrödinger equation is difficult to solve exactly, especially if the centrifugal potential is taken into account [10-12]. Consequently, several approximation methods have been developed, such as

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the asymptotic iteration method [10] and the Nikiforov-Uvarov method [13]. These methods are frequently used to derive analytical solutions for the Schrödinger equation with different potentials, including the Cornell potential and combinations of the Cornell potential with other potentials like the harmonic oscillator potential [12, 14-17].

The Cornell potential, distinguished by its Coulomb-like short-range and linear large-range terms, has been widely employed to describe quark-antiquark systems, which include mesons [18-24]. The Coulomb potential stems from one-gluon exchange, whereas the linear potential is conjectured to come from higher orders, though its first-principle derivation from the rules of QCD has been left as an open question [14]. Even with its relevance to the underlying physics, the exact solutions of the Schrödinger equation with the Cornell potential continue to pose an on-going challenge. In addition to the Cornell potential, we take into account quadratic term owing to its additional harmonic confinements.

There has been a recent interest in applying the use of Neural Networks (NNs) as a novel approach to solving ordinary and partial differential equations, including the Schrodinger equation [25]. There are various advantages of using NNs over conventional numerical methods, including the capability to approximate universal functions, generate continuum solutions over the entire domain, and remain computationally efficient irrespective of the sample points or dimensions of the problem [26-30]. Numerous studies have demonstrated that NNs can solve quantum mechanical problems effectively [25-30] and therefore are a potential method to calculate meson mass spectra.

In the present study, we seek to calculate the mass spectra of the $b\bar{c}$ mesons by solving the Schrödinger equation through an approach utilizing NNs with a combined Cornell and quadratic potential. The inclusion of the quadratic potential to the Cornell potential enables a more flexible and accurate description of the meson systems, with the possibility of better matching with the experimental results. This work extends the results of the quarkonium systems previously calculated and discusses the potential application of the method of neural networks to further advance the study of spectroscopy of the heavy mesons.

2. Materials and Methods

NNs are computational models consisting of interconnected units known as neurons. Each neuron receives multiple inputs and produces an output based on a nonlinear activation function, such as the sigmoid function. The sigmoid function is defined as:

$$\sigma(z) = \frac{1}{1+e^{-z}} \tag{1}$$

which is advantageous for its differentiability properties.

Neurons are usually structured in layers in a neural network. A single neuron accepts many inputs and has a single output in the most basic model. More advanced forms of the setup, such as the multilayer perceptron neural networks (MLPNs), contain neurons grouped in various numbers of layers, with a neuron in a layer linked to all the neurons in the next layer. This feed-forward structure is very versatile and can implement intricate functions and is thus widely used in a variety of applications [26, 28, 31].

In MLPNs (see Figure 1), the output of each neuron is computed as the activation function applied to the weighted sum of its inputs. Mathematically, this can be described for neurons in different layers using the following relations:

$$o_i = \sigma(n_i) \tag{2}$$

$$o_j = \sigma(n_j) \tag{3}$$

$$o_k = \sigma(n_k) \tag{4}$$

where i, j, and k denote the input, hidden, and output layers, respectively. The inputs to neurons in the hidden and output layers are given by:

$$n_j = \sum_{i=1}^{N_i} w_{ij} o_i + \theta_j, \tag{5}$$

$$n_j = \sum_{j=1}^{N_j} w_{jk} o_j + \theta_k. \tag{6}$$

Here, w_{ij} and w_{jk} are the weights connecting neurons between layers, and θ_j and θ_k are threshold parameters.

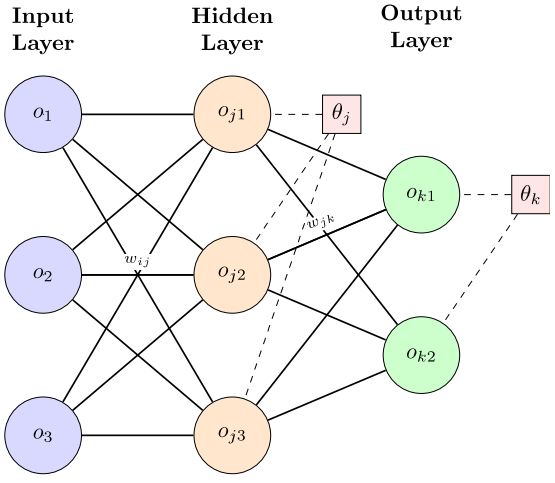

Figure 1. A representation of the DNN architecture, illustrating the input layer, hidden layers, and output layer.

The overall output o_k of the network can be expressed as a composite function of its inputs, weights, and thresholds:

$$o_k = \sum_{j=1}^{N_j} w_{jk}\sigma\left(\sum_{i=1}^{N_i} w_{ij}n_i + \theta_j\right) + \theta_k \tag{7}$$

This differentiable function allows the neural network to learn optimal parameters using training algorithms such as backpropagation.

Applying neural networks to quantum mechanical systems, particularly for solving differential equations like the Schrödinger equation, enables the approximation of solutions using a neural network. For a differential equation of the form

$$H\Psi(r) = f(r) \tag{8}$$

where H is a linear operator and $f(r)$ is a known function, a trial solution $\Psi_t(r)$ can be constructed using a neural network. This solution is parameterized by weights and biases represented by a vector $\boldsymbol{p}$ and is designed to satisfy boundary conditions.

To find an approximate solution, the problem is converted into a minimization problem where the objective is to minimize the error function:

$$E(\boldsymbol{p}) = \sum_i \frac{[H\Psi_t(r_i,\boldsymbol{p}) - f(r_i)]^2}{\int |\Psi_t|^2 d\boldsymbol{r}} \tag{9}$$

This transformation is achieved using the collocation method, where the differential equation is solved by adjusting the network parameters to minimize the discrepancy between the network output and the target function.

Specifically for the Schrödinger equation, this approach involves defining a trial wavefunction $\Psi_t(r)$ using a neural network and optimizing its parameters to minimize the energy eigenvalues. The trial wavefunction can be represented as:

$$\Psi_t(r) = B(\boldsymbol{r},\lambda)N(\boldsymbol{r},\boldsymbol{p}) \tag{10}$$

where $B(\boldsymbol{r},\lambda)$ ensures the boundary conditions are met, and $N(\boldsymbol{r},\boldsymbol{p})$ is the neural network output. By optimizing this network using backpropagation, we can find the parameters that best approximate the eigenvalues of the system under study.

Moreover, this method enables the calculation of multiple energy states by projecting out the already computed levels from the trial wavefunction. If we have computed orthonormal states $|\Psi_0\rangle$, $|\Psi_1\rangle, \dots$, $|\Psi_k\rangle$, a trial state $|\Psi_t\rangle$ orthogonal to all of them can be constructed by projecting out their components from a general function $|\tilde{\Psi}_t\rangle$ that satisfies the boundary conditions:

$$\begin{aligned}|\Psi_t\rangle = (1 - |\Psi_0\rangle\langle\Psi_0|)(1 - |\Psi_1\rangle\langle\Psi_1|)\dots \\ \times (1 - |\Psi_k\rangle\langle\Psi_k|)|\tilde{\Psi}_t\rangle\end{aligned} \tag{11}$$

This can also be expressed as:

$$|\Psi_t\rangle = (1 - |\Psi_0\rangle\langle\Psi_0| - \dots - |\Psi_k\rangle\langle\Psi_k|)|\tilde{\Psi}_t\rangle \tag{12}$$

This projection method ensures that the new trial wavefunction is orthogonal to all previously computed states, facilitating the systematic calculation of higher energy levels [26-28, 32].

Neural networks, particularly multilayer perceptrons, offer a powerful framework for modeling and solving complex problems in both classical and quantum domains due to their flexibility, differentiability, and capacity to approximate nonlinear functions [26-28, 31].

The Schrödinger equation (SE) for two particles interacting through a symmetric potential in an N-dimensional space can be formulated to describe the quantum states of systems like quarkonium. This equation, as described in Ref. [2], takes the following form:

$$\begin{aligned}\frac{d^2\Psi(r)}{dr^2} + \frac{N-1}{r}\frac{d\Psi(r)}{dr} - \frac{l(l+N-2)}{r^2}\Psi(r) \\ +2\mu\big(E - V(r)\big)\Psi(r) = 0\end{aligned} \tag{13}$$

where l is the angular momentum quantum number, N is the number of spatial dimensions, and μ is the reduced mass of the interacting particles, such as quarks in a quarkonium system.

To simplify the SE for radial solutions, we set the wave function $\Psi(r)$ in terms of a radial function $R(r)$ as $\Psi(r) = r^{\frac{1-N}{2}}R(r)$. Substituting this into Eq. (13) yields the radial Schrödinger equation:

$$\begin{aligned}\frac{d^2R(r)}{dr^2} = 2\mu(V(r) - E)R(r) \\ + \frac{\left(l + \frac{N-2}{2}\right)^2 - \frac{1}{4}}{2\mu r^2}R(r)\end{aligned} \tag{14}$$

To explore the properties of quarkonium, the potential energy function $V(r)$ often combines the Cornell potential term. On the other hand, the quadratic potential introduces additional effects. The potential $V(r)$ is given by:

$$V(r) = ar - \frac{b}{r} + cr^2 \tag{15}$$

where a, b, and c are constants that can be adjusted based on the specific properties of the system [5-7].

The Cornell potential, represented by the terms $ar - b/r$, encapsulates two fundamental characteristics of the strong force between quarks: confinement and asymptotic freedom. The linear term, ar, dominates at large distances, reflecting the confinement phenomenon where quarks are bound together. The inverse term, $-b/r$, is significant at short distances, illustrating asymptotic freedom, where quarks behave almost independently due to the decreasing interaction strength as they get closer [7, 33].

The quadratic potential, represented by the term cr^2 and where the parameter c is related to the mass m of the particle and the square frequency of the oscillation. This potential term is crucial in fine-tuning the model to reflect various quarkonium properties, as noted in Refs. [20, 34, 35].

Substituting the potential function $V(r)$ from Eq. (15) into the radial Schrödinger Eq. (14), we obtain:

$$\frac{d^2R(r)}{dr^2} = 2\mu\left(ar - \frac{b}{r} + cr^2 - E + \frac{(l + (N - 2/2))^2 - 1/4}{2\mu r^2}\right)R(r) \tag{16}$$

This modified Schrödinger equation allows for the exploration of the energy levels and wave functions of quarkonium systems under the influence of both the Cornell and quadratic potentials. The interplay of these potentials enables a detailed study of the quark-antiquark interaction, incorporating both the long-range confinement and short-range asymptotic freedom characteristic of strong interactions, along with additional harmonic confinement effects [34, 36, 37].

The radial wave function corresponding to this equation can be expressed in the form:

$$R_{nl}(r) = C_{nl} r^{-\frac{B}{\sqrt{2A}}-1} e^{\sqrt{2A}\,r}\left(r^2\frac{d}{dr}\right)^n \times \left(r^{-2n+\frac{2B}{\sqrt{2A}}} e^{-2\sqrt{2A}\,r}\right) \tag{17}$$

This solution is obtained by applying the Nikiforov-Uvarov method to Eq. (16) in [14]. The normalization constant C_{nl} is determined by ensuring $\int |R_{nl}(r)|^2\, dr = 1$. The parameters A and B in the radial Schrödinger equation are defined as:

$$A = -\mu\left(E - \frac{3a}{\delta} - \frac{6c}{\delta^2}\right), \quad B = \mu\left(\frac{3a}{\delta^3} + \frac{8c}{\delta^3} + b\right), \tag{18}$$

where $\delta = 1/r_0$, with r_0 being a characteristic radius of the meson [14, 17].

By analyzing solutions to this equation, one can derive important properties of quarkonium, such as energy eigenvalues and wavefunctions, which are essential for understanding the behavior of heavy quark pairs bound by the strong force [18, 33, 38].

In the comparison with Eq. (17), we parameterize the trial function as:

$$R_t(x) = x^\alpha\, e^{-\beta x^2} N(x, \boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}), \qquad \beta > 0, \alpha \in \mathbb{R}, \tag{19}$$

where N denotes a feed-forward artificial neural network with one hidden layer and m sigmoid hidden units, represented as:

$$N(x, \boldsymbol{u}, \boldsymbol{w}, \boldsymbol{v}) = \sum_{j=1}^{m} v_j \sigma\left(w_j x + u_j\right). \tag{20}$$

The minimization problem for this trial function is defined as:

$$\sum_i \frac{\left[HR_t(x_i - \epsilon R_t(x_i))\right]^2}{\int |R_t(x)|^2 dx} \tag{21}$$

This formulation leverages the pattern of the radial wave function to optimize the neural network parameters.

Optimization of the trial wavefunction parameters is done through the Broyden– Fletcher– Goldfarb–Shanno (BFGS) algorithm, a quasi-Newton method with the advantage of being efficient when dealing with problems described by a nonlinear function. We aim to minimize the total error function E_{total} by optimizing the trial wavefunction's parameters [28, 29].

Optimization for each quantum state begins with a group of initial random parameters, and the BFGS algorithm iteratively updates the parameters to minimize the total error. This optimization is run for a specified number of trials to guarantee convergence to the global minimum [26].

The optimum parameters provide a trial wavefunction that most accurately represents the actual wavefunction of the quantum state in question. These optimized parameters are used to calculate the energy, and the results are compared to experimental results to verify the validity of the model.

This method enables the systematic calculation of energy levels and wavefunctions and offers useful information regarding the $b\bar{c}$ meson system [21, 22, 28].

3. Results

In this section, we present the mass spectra calculations for the $b\bar{c}$ states using the neural network approach with the combined Cornell and quadratic potentials. The results are displayed in Tables 1 and 2 for different quantum states, and we compare them with those from other theoretical studies and available experimental data. The constants used, as presented in [14], are:

$$\begin{aligned} m_b &= 4.823\ GeV, \\ m_c &= 1.209\ GeV, \\ a &= 0.2\ GeV^2, \\ b &= 1.2, \\ c &= 0.04\ GeV^3. \end{aligned} \tag{22}$$

We trained the network using 200 evenly spaced points with $m = 10$ over the interval $-10 \leq r \leq 10$ and solved Eq. (16) (see Table 1). The loss function curve resulting from the training process, as illustrated in Figure 2, offers valuable insights into the convergence of the neural network. Furthermore, the learning curve of the plot of the error function versus BFGS iteration (Figure 3) also supports and plots the optimizer convergence path. At first, the error function falls very rapidly, dropping sharply over the initial epochs, suggesting that the neural network converges to an optimal value close to the global minimum very rapidly. Following around 50 epochs, the loss converges to a consistently low error plateau. Such consistency validates the effectiveness and stability of the optimization process utilized, showcasing the applicability of having modeled the combined Cornell and quadratic potentials using neural networks. Our method reaches consistent convergence before epoch 50; however, when considering the all $b\bar{c}$ states—which follow the same pattern of convergence—it is generally necessary to extend training up to 300 epochs to ensure stability across all cases. Such stability provides assurance over calculated- B_c meson mass spectra, supporting the robustness and precision of the methodology under consideration.

Table 1. NN Architecture

Hidden Layer	Hidden Units	Epoch	BFGS Iteration
1	10	300	100

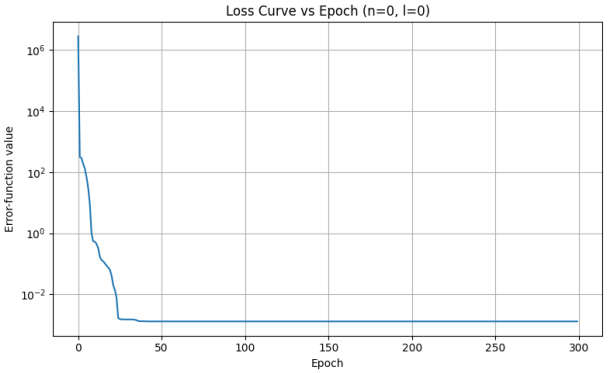

Figure 2. Loss curve illustrating convergence of the neural network training over epochs for the $n = 0, l = 0$ state.

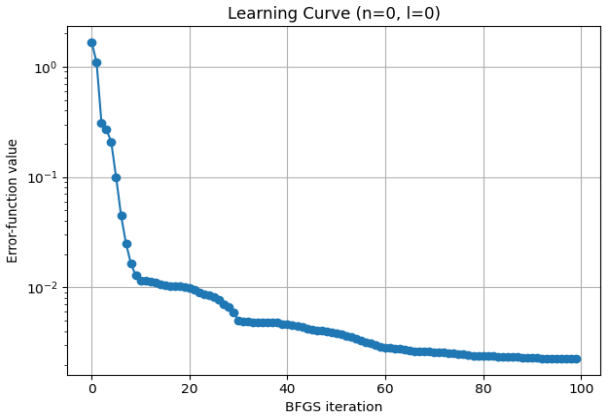

Figure 3. Learning curve illustrating convergence of the neural network training over BFGS iterations for the $n = 0$, $l = 0$ state.

Table 2 shows the calculated mass spectra of the $b\bar{c}$ meson for $N = 3$ spatial dimensions, which corresponds to our three-dimensional space. The results from our present work are compared with those from other studies, including the NNs used in [25], as well as theoretical models in [10, 14, 19, 39]. Where available, comparisons with experimental data from [12] are also included.

In the first excited state ($2S$), we calculated the mass to be $7.043\ GeV$, which is consistent with other theoretical calculations but somewhat greater than the experimental value of $6.842\ GeV$ [14, 18, 19, 25].

Table 2. Mass spectra of B_c meson (in GeV) for $N = 3$.

State	Present Work	[25]	[10]	[14]	[18]	[19]	Exp. [12]
1S	6.276	6.274	6.362	6.277	6.264	6.270	6.275
1P	6.669	6.743	6.792	6.666	6.700	6.699	-
1D	7.052	7.046	7.051	-	-	-	-
2S	7.043	6.839	6.911	7.042	6.856	6.835	6.842
2P	7.212	*7.187*	7.178	7.207	7.108	7.091	-
3S	7.287	7.245	7.284	7.384	7.244	7.193	-

3P	7.471	7.467	7.494	-	-	-	-
4S	7.527	7.522	7.593	-	-	-	-

This might be a result of the additional quadratic potential term in our approach, which might raise the calculated masses for the excited states.

Theoretical mass calculated for the $1P$ state comes out to be 6.669 GeV, consistent with the values from the other models, assuring the strength of our method for the calculation of orbital excitations. We find the same consistency in the calculations for the $2P$ and $3P$ states even though we do not possess experimental results for these states for direct comparison.

Table 3 presents the calculated mass spectra for the $b\bar{c}$ meson in a hypothetical four-dimensional space ($N = 4$). These results are compared with those in [14], which uses the same potential model.

The ground state mass, calculated as $1S$, in $N = 4$ dimensions comes out to be 6.350 GeV, which is somewhat larger than the result in the $N = 3$ dimensional case. This can be anticipated because the increased spatial dimension changes the dynamics in the quark-antiquark system. Excited state masses are typically higher than in the $N = 3$ dimensional case.

Specifically, for the $2S$ state, our predicted mass of 6.879 GeV agrees almost exactly with that given in [14], demonstrating the consistency of the neural network method with conventional analytical calculations even at larger dimensions. The calculated masses for the remaining states, including $3S$ and $3P$, agree likewise reasonably well with past results.

Table 3. Mass spectra of B_c meson (in GeV) for $N = 4$.

State	Present Work	[14]
1S	6.350	6.355
1P	6.862	6.883
1D	7.052	-
2S	6.879	6.878
2P	7.165	7.161
3S	7.516	8.035
3P	7.282	-
4S	7.246	-

Results for $N = 4$ dimensions are noteworthy because they provide useful theoretical insight on the variation of the quark-antiquark interaction for systems of larger numbers of dimensions. An increase in spatial dimensions modifies the overall dynamics of a system through a variation of the combined potential, leading to computable observable changes of calculated meson masses. These results are of particular interest for studies of beyond-Standard Model scenarios, e.g., of string models or models of extra spatial dimensions. Such models are, of course, speculative without experimental confirmation but are a useful starting point for investigating the strength and versatility of theoretical approaches of meson spectroscopy.

4. Discussion and Conclusion

When compared to the results using the literature, it becomes apparent that the method using the neural network is a valid and effective technique to obtain the masses of the spectra of the heavy meson system of $b\bar{c}$. Adding the quadratic potential to the Cornell potential makes the model flexible, especially in the prediction of the excited states [10, 14, 25].

Yet, the minor variations seen in the excited states hint at the possibility of further tuning of the potential parameters or the incorporation of additional physical phenomena, e.g., relativistic correction or spin-spin interaction, to enhance the precision of predictions [7, 23, 40-42].

Overall, the combined Cornell and quadratic potential used in the neural network method presents a wide range of potential for the study of heavy meson spectroscopy, giving results competitive to, and sometimes better than, the conventional theory.

References

[1] Kuchin, S., Maksimenko, N. 2013. Theoretical estimations of the spin-averaged mass spectra of heavy quarkonia and Bc mesons. Universal Journal of Physics and Application, 7(3), 295-298.

[2] Kumar, R., Chand, F. 2013. Asymptotic study to the N-dimensional radial Schrödinger equation for the quark-antiquark system. Communications in Theoretical Physics, 59(5), 528.

[3] Dib, C. O., Neill, N. A. 2012. χ b (3 P) splitting predictions in potential models. Physical Review D—Particles, Fields, Gravitation, and Cosmology, 86(9), 094011.

[4] Evans, P., Allton, C., Skullerud, J. I. 2014. Ab initio calculation of finite-temperature charmonium potentials. Physical Review D, 89(7), 071502.

[5] Hall, R. L. 1984. Simple eigenvalue formula for the Coulomb-plus-linear potential. Physical Review D, 30(2), 433.

[6] Thacker, B. A., Lepage, G. P. 1991. Heavy-quark bound states in lattice QCD. Physical Review D, 43(1), 196.

[7] Godfrey, S., Isgur, N. 1985. Mesons in a relativized quark model with chromodynamics. Physical Review D, 32(1), 189.

[8] Lepage, G. P., Magnea, L., Nakhleh, C., Magnea, U., Hornbostel, K. 1992. Improved nonrelativistic

QCD for heavy-quark physics. Physical Review D, 46(9), 4052.

[9] Soto, J. 2007. Overview of non-relativistic QCD. The European Physical Journal A, 31, 705-710.

[10] Ciftci, H., Hall, R. L., Saad, N. 2003. Asymptotic iteration method for eigenvalue problems. Journal of Physics A: Mathematical and General, 36(47), 11807.

[11] Gönül, B., Köksal, K., Bakir, E. 2006. An alternative treatment for Yukawa-type potentials. Physica Scripta, 73(3), 279.

[12] Patrignani, C., Agashe, K., Aielli, G., Amsler, C., Antonelli, M., Asner, D., Baer, H., Banerjee, S., Barnett, R., Basaglia, T. 2016. Review of particle physics.

[13] Nikiforov, A.F., Uvarov, V.B. 1988. Special functions of mathematical physics (Vol. 205). Springer.

[14] Abu-Shady, M. 2016. Heavy quarkonia and Bc-mesons in the cornell potential with harmonic oscillator potential in the N-dimensional Schrodinger equation. International Journal of Applied Mathematics Physics, 2(2), 16-20.

[15] Vigo-Aguiar, J., Simos, T. 2005. Review of multistep methods for the numerical solution of the radial Schrödinger equation. International journal of quantum chemistry, 103(3), 278-290.

[16] Gupta, P., Mehrotra, I. 2012. Study of heavy quarkonium with energy dependent potential. Journal of Modern Physics, 3(10), 1530.

[17] Kumar, R., Chand, F. 2012. Series solutions to the N-dimensional radial Schrödinger equation for the quark–antiquark interaction potential. Physica Scripta, 85(5), 055008.

[18] Eichten, E.J., Quigg, C. 1994. Mesons with beauty and charm: Spectroscopy. Physical Review D, 49(11), 5845.

[19] Ebert, D., Faustov, R., Galkin, V. 2003. Properties of heavy quarkonia and B c mesons in the relativistic quark model. Physical Review D, 67(1), 014027.

[20] Abou-Salem, L. 2005. A systematic study on nonrelativistic quark–antiquark interactions. International Journal of Modern Physics A, 20(17), 4113-4122.

[21] Eichten, E., Gottfried, K., Kinoshita, T., Kogut, J., Lane, K., Yan, T.-M. 1976. Spectrum of Charmed Quark-Antiquark Bound States. Physical Review Letters, 36(21), 1276.

[22] Quigg, C., Rosner, J.L. 1979. Quantum mechanics with applications to quarkonium. Physics Reports, 56(4), 167-235.

[23] Alford, J., Strickland, M. 2013. Charmonia and bottomonia in a magnetic field. Physical Review D—Particles, Fields, Gravitation, and Cosmology, 88(10), 105017.

[24] Barnes, T., Godfrey, S., Swanson, E. 2005. Higher charmonia. Physical Review D—Particles, Fields, Gravitation, and Cosmology, 72(5), 054026.

[25] Mutuk, H. 2019. Cornell potential: a neural network approach. Advances in High Energy Physics, 2019(1), 3105373.

[26] Lagaris, I.E., Likas, A., Fotiadis, D.I. 1997. Artificial neural network methods in quantum mechanics. Computer Physics Communications, 104(1-3), 1-14.

[27] Haykin, S. 1994. Neural networks: a comprehensive foundation. Prentice Hall PTR.

[28] Yadav, N., Yadav, A., Kumar, M. 2015. An introduction to neural network methods for differential equations. SpringerBriefs in Applied Sciences and Technology.

[29] Spining, M., Darsey, J., Sumpter, B., Nold, D. 1994. Opening up the black box of artificial neural networks. Journal of chemical education, 71(5), 406.

[30] Darsey, J., Noid, D., Upadhyaya, B. 1991. Application of neural network computing to the solution for the ground-state eigenenergy of two-dimensional harmonic oscillators. Chemical physics letters, 177(2), 189-194.

[31] Barron, A. R. 1993. Universal approximation bounds for superpositions of a sigmoidal function. IEEE Transactions on Information theory, 39(3), 930-945.

[32] Karakoc, M., Boztosun, I. 2006. Accurate iterative and perturbative solutions of the Yukawa potential. International Journal of Modern Physics E, 15(06), 1253-1262.

[33] Gönül, B., Cançelik, Y. 2017. Remarks on the treatments of nonsolvable potentials. Turkish Journal of Physics, 41(4), 326-336.

[34] Chen, J. K. 2013. Spectral method for the Cornell and screened Cornell potentials in momentum space. Physical Review D—Particles, Fields, Gravitation, and Cosmology, 88(7), 076006.

[35] Chung, H.S., Lee, J., Kang, D. 2008. Cornell potential parameters for S-wave heavy quarkonia. arXiv preprint arXiv:0803.3116.

[36] Eichten, E., Gottfried, K., Kinoshita, T., Lane, K., Yan, T.-M. 1980. Charmonium: comparison with experiment. Physical review D, 21(1), 203.

[37] Fulcher, L.P. 1994. Matrix representation of the nonlocal kinetic energy operator, the spinless Salpeter equation and the Cornell potential. Physical Review D, 50(1), 447.

[38] Vega, A., Flores, J. 2016. Heavy quarkonium properties from Cornell potential using variational method and supersymmetric quantum mechanics. Pramana, 87, 1-7.

[39] Brambilla, N., Eidelman, S., Heltsley, B., Vogt, R., Bodwin, G., Eichten, E., Frawley, A., Meyer, A., Mitchell, R., Papadimitriou, V. 2011. Heavy quarkonium: progress, puzzles, and opportunities. The European Physical Journal C, 71, 1-178.

[40] Lichtenberg, D. 1987. Energy levels of quarkonia in potential models. International Journal of Modern Physics A, 2(06), 1669-1705.

[41] Akan, T. 2024. Exploring machine learning models for predicting meson mass and width. Physica Scripta, 99(2), 125305.

[42] Assi, B., Wagman, M. L. 2023. Baryons, multihadron sytstems, and composite dark matter in nonrelativistic QCD. Physical Review D, 108(9), 096004.