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COFINITELY $\delta - H -$ SUPPLEMENTED MODULES

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ABSTRACT

Consider a module P over a ring S. We describe P as cofinitely $\delta - H - s$upplemented, in case there is a direct summand K of P with the property that the equality $P = A + X$ holds if and only if $P = K + X$ for any submodule X of P with singular P/X and for each cofinite submodule A of P. In this work, we demonstrate that P satisfies cofinitely $\delta - H - s$upplemented condition if and only if P has a direct summand K with the properties $(A + K)/A \ll_\delta P/A$ and $(A + K)/K \ll_\delta P/K$ for each cofinite submodule A of P. $\delta -$semiperfect rings are characterized by means of cofinitely $\delta - H -$supplemented modules, with the characterization expressed through a set of equivalent statements.

Keywords: Cofinitely $H -$supplemented module, $\delta -$Small submodule, Cofinite submodule, Cofinitely $\delta - H - s$upplemented module.

1 INTRODUCTION

In module theory, the distinct definitional approach, unconventional characteristics, and the fact that $H -$supplemented modules extend the concept of lifting modules have attracted significant attention, prompting further exploration beyond the foundational work by Mohamed and Müller [1]. A module P is defined as $H - supplemented$ if for its each submodule A, P has a direct summand K such that the equality $P = A + X$ holds if and only if the equality $P = K + X$ is satisfied for any submodule X of P [1]. Since its introduction, researchers have investigated the properties of $H -$supplemented modules, with significant contributions beginning with Koşan and Keskin Tütüncü's paper [2]. Their note explores some characteristics of H −supplemented modules, focusing in particular on their behavior under homomorphic images and direct summands. In [3], building on [2], the authors provided a characterization of H −supplemented modules in terms of small submodules. Several studies have been conducted on this class of modules, addressing their extended properties and presenting generalizations of the H −supplemented concept [4], [5], [6], [7].

The central objective of this study is to introduce the property of cofinitely $\delta - H - s$upplemented modules. This study defines the notion of cofinitely $\delta - H - s$upplemented modules and focus on their algebraic characteristics. Among various findings, it is proven a module P being cofinitely $\delta - H - s$upplemented is equivalent to P having a direct summand K with $(A+K)/A \ll_\delta P/A$ and $(A+K)/K \ll_\delta P/K$ for each cofinite submodule A of P. This study further shows that if a module P is either a singular module, or a module without a simple projective submodule, then the notions of cofinitely $\delta - H$ −supplemented and cofinitely H −supplemented coincide for the module P. Consequently, this study establishes that the modules which are cofinitely $\delta - H$ −supplemented coincide with cofinitely H −supplemented ones over the rings for which each simple module is singular. It is proven that a module P that is non−δ −cosingular is cofinitely δ −lifting if and only if it is cofinitely $\delta - H$ −supplemented. We demonstrate that for an indecomposable module P, being cofinitely $\delta - H$ −supplemented is equivalent to the condition that either all of its cofinite submodules are δ −small in P, or that P is simple. This study shows that whenever A is a projection invariant submodule of a cofinitely $\delta - H$ −supplemented module P, the quotient module P/A necessarily inherits the cofinitely $\delta - H$ −supplemented property. Furthermore, this study obtains a characterization of δ −semiperfect rings using cofinitely $\delta - H$ −supplemented modules, presented as a series of equivalent conditions. As a result of this characterization, this study provides an example of a module that is cofinitely $\delta - H - s$upplemented but not cofinitely $H - s$upplemented.

This study presents work on unitary rings. These rings are denoted by S and are considered together with the corresponding unitary left S −modules. Throughout, let P denote such a left S −module. The notation $A \leq P$ signifies that A is a submodule of P. $A \leq P$ is defined as *cofinite* if the quotient module P/A is finitely generated [8]. $A \leq P$ is said to be *small* in P, written as $A \ll P$, provided for each proper submodule B of P, the submodule $A + B$ does not equal P. Dually, $A \leq P$ is called *essential* in P, written $A \trianglelefteq P$, provided $A \cap K \neq 0$ for each nonzero $K \leq P$. P is called a *singular module* in case $P \cong B/A$ for some module B and for its essential submodule A. $A \leq P$ is defined as δ −*small* in P, indicated by

$A \ll_\delta P$, provided for each proper submodule B of P satisfying that P/B is singular, the submodule $A + B$ does not equal to P. Each small submodule and non-singular semisimple submodule of P satisfies the $\delta -$small condition. This study adopts the standard notation $\delta(P)$ for the sum of all $\delta -$small submodules of P (see [9], [10] for further information).

2 MATERIAL AND METHOD

This study outlines key properties of $\delta -$small submodules in the following lemma, drawn from [10, Lemma 1.2 and 1.3].

Lemma 2.1. Suppose that P is a module.

1) For any submodule A of P, $A \ll_\delta P$ if and only if whenever $P = X + A$ there is a semisimple projective submodule A' of A with $X \oplus A' = P$.
2) If $A \ll_\delta P$ and $h : P \to W$ is a homomorphism, then $h(A) \ll_\delta W$. In particular, if $A \ll_\delta P \leq W$, then $A \ll_\delta W$.
3) If $A_1 \ll_\delta B_1 \leq P$ and $A_2 \ll_\delta B_2 \leq P$, then $A_1 + A_2 \ll_\delta B_1 + B_2$.
4) If $P = \bigoplus_{i \in I} P_i$, then $\delta(P) = \bigoplus_{i \in I} \delta(P_i)$.
5) If $A \leq B \leq P$, $A \ll_\delta P$ and B is a direct summand of P, then $A \ll_\delta B$.

Definition 2.2. A module P is called $\delta -$*supplemented* provided for each submodule A of P, there exists a submodule B of P such that $P = A + B$ and $A \cap B \ll_\delta B$. In this case, the submodule B is called a $\delta -$*supplement* of A in P [11].

Definition 2.3. A module P is defined as *cofinitely* $\delta -$*supplemented* provided each cofinite submodule possesses a $\delta -$supplement in P [12].

Definition 2.4. A module P is called $H -$*cofinitely supplemented* if for its each cofinite submodule A, the module P has a direct summand K such that the equality $P = A + X$ holds if and only if the equality $P = K + X$ is satisfied for any submodule X of P [13].

Definition 2.5. A module P is defined as *cofinitely* $\delta -$*lifting* if for each cofinite submodule A of P, P has a decomposition $P = P_1 \oplus P_2$ with $P_1 \leq A$ and $A \cap P_2 \ll_\delta P_2$ [12].

3 RESULTS AND DISCUSSION

Definition 3.1. Consider a module P. The module P is called *cofinitely* $\delta - H - supplemented$ provided for each cofinite submodule A of P, P has a direct summand K such that $P = A + B$ if and only if $P = K + B$ for each submodule B of P with singular P/B.

Clearly, each H −cofinitely supplemented module is cofinitely $\delta - H -$ supplemented. There are many examples of H −cofinitely supplemented modules in [5]. These examples also serve as examples of cofinitely $\delta - H -$ supplemented modules.

The subsequent result provides an alternative criterion for determining when a module possesses cofinitely $\delta - H - s$upplemented property. Throughout this study, the result of this lemma is used without further citation.

Lemma 3.2. A module P being cofinitely $\delta - H - s$upplemented is equivalent P having a direct summand K with the properties $(A + K)/A \ll_\delta P/A$ and $(A + K)/K \ll_\delta P/K$ for each cofinite submodule A of P.

Proof: ($\Longrightarrow$) If P is a cofinitely $\delta - H - s$upplemented module, then for each cofinite submodule A of P, P has a direct summand K satisfying the property that $P = A + B$ if and only if $P = K + B$ for any $B \leq P$ with singular P/B. Let A be a cofinite submodule of P. Assume now that for a submodule X of P containing A, where the quotient module P/X is singular, the relation $(A + K)/A + X/A = P/A$ is satisfied. We derive $P = K + X$. By assumption, $P = A + X$, and so $P = X$. On the other hand, assume that $(A + K)/K + Y/K = P/K$ for a submodule Y of P which contains K with singular P/Y. Thus $P = A + Y$. By assumption, $P = K + Y$, and hence $P = Y$.

($\Longleftarrow$) Assuming that for each cofinite submodule A of P, $P = A + B$ with singular P/B, we derive $P/K = (A + K)/K + (B + K)/K$ for a direct summand K of P. Here it should be noted that $P/(B + K)$ is a singular module as a quotient module of the singular module P/B. By assumption, $P = B + K$. Now assuming that $P = K + B$ with singular P/B for each direct summand K of P, we derive $(A + K)/A + (A + B)/A = P/A$ for each cofinite submodule A of P. Here $P/(A + B)$ is a singular module as a quotient module of the singular module P/B. By assumption, $P = A + B$.

Proposition 3.3. Let P be a module. If either

1) P is a singular module, or
2) P does not have any simple projective submodule, then P is a cofinitely $\delta-H-s$upplemented module if and only if P is a cofinitely $H-s$upplemented module.

Proof: Suppose that P is a singular module. It should be noted that $\delta-$small submodules of a singular module are small submodules. As such, using [5, Theorem 2.10] it can be easily checked that a singular module P is a cofinitely $\delta-H-s$upplemented module if and only if P is a cofinitely $H-s$upplemented module.

Furthermore, assume that P is a cofinitely $\delta-H-s$upplemented module which does not have any simple projective module and $A \leq P$ is a cofinite submodule. Then by assumption, P has a direct summand K such that $(A+K)/A \ll_{\delta} P/A$ and $(A+K)/K \ll_{\delta} P/K$. Let $(A+K)/A + T/A = P/A$ for a submodule T/A of P/A. By Lemma 2.1 $(A+K)/A$ includes a direct summand X/A of P/A which is semisimple projective such that $X/A \oplus T/A = P/A$. So that X has a submodule A' such that $X = A \oplus A'$ as X/A is projective. This yields that A' includes a submodule which is simple projective as A' is semisimple. Accordingly, we infer $X = A$, and so $T/A = P/A$. For this reason, $(A+K)/A \ll P/A$. In like manner, it can be observed that $(A+K)/K \ll P/K$. As a result, P is a cofinitely $H-s$upplemented module by [5, Theorem 2.10].

Corollary 3.4. Assume that S is a ring such that every simple left $S-$module is singular**.** In this case, an $S-$module P is a cofinitely $H-$supplemented module if and only if P is a cofinitely $\delta-H-s$upplemented module. Specifically, a $\mathbb{Z}-$module P is a cofinitely $H-$supplemented module if and only if P is a cofinitely $\delta-H-s$upplemented module.

Recall from [14] that a module P is said to be *non- δ-cosingular* if $\bar{Z}_{\delta}(P) = \bigcap\{Ker g \mid g : P \to W$ where there exists another module Y such that $W \ll_{\delta} Y\} = P$.

Proposition 3.5. A module P which is non- δ-cosingular is a cofinitely $\delta-$lifting module if and only if P is a cofinitely $\delta-H-s$upplemented module.

Proof: $(\Longrightarrow)$ Assume that A is a cofinite submodule of P. Since P is a cofinitely $\delta-$lifting module, then P has a decomposition $P = P_1 \oplus P_2$ with $P_1 \leq A$ and $A \cap P_2 \ll_{\delta} P_2$. Here we derive $A = P_1 + (P_2 \cap A)$. Let $P = A + X$ with singular P/X. Then $P = \big(P_1 + (P_2 \cap A)\big) + X$, and so $P = P_1 + X$ by Lemma 2.1.

($\Leftarrow$) Suppose that P is a cofinitely $\delta-H-s$upplemented module and A is a cofinite submodule of P. Then there exists a direct summand K of P such that $(A+K)/A \ll_\delta P/A$ and $(A+K)/K \ll_\delta P/K$. Since P is non- δ-cosingular, then K is so by [14, Proposition 2.4]. It should be noted that $(A+K)/A$ is $\delta-$small and also by [14, Proposition 2.4] it is non- δ-cosingular module as a factor module of the non- δ-cosingular module K. This yields that $A = A+K$ by [14, Proposition 2.4]. Accordingly, A/K is a submodule which is $\delta-$small in P/K, so that P is a cofinitely $\delta-$lifting module.

Proposition 3.6. A module P which is indecomposable is cofinitely $\delta-H-s$upplemented if and only if either P is a simple module, or each cofinite submodule of P is $\delta-$small in P.

Proof: ($\Rightarrow$) Suppose that A is a cofinite submodule of P. Then there exists a direct summand K of P with the properties that $(A+K)/A \ll_\delta P/A$ and $(A+K)/K \ll_\delta P/K$. Firstly, let $K = 0$. Then $A \ll_\delta P$. In the other case, if $K = P$, then $P/A \ll_\delta P/A$. By Lemma 2.1 P/A is a semisimple projective module. This yields that A is a direct summand of P. Since P is indecomposable, $A = 0$. As a result, P is a simple module.

($\Leftarrow$) Straightforward.

Proposition 3.7. If P is a cofinitely $\delta-H-s$upplemented module with $\delta(P) = 0$, then each cofinite submodule of P is a direct summand.

Proof: Assuming that A is a cofinite submodule of P, we derive that there exists a direct summand K of P with the properties that $(A+K)/A \ll_\delta P/A$ and $(A+K)/K \ll_\delta P/K$. Since K is a direct summand of P and $\delta(P) = 0$, then $\delta(P/K) = 0$. Accordingly, we infer $K = A + K$ meaning that $K/A \ll_\delta P/A$. Here it should be noted that the sum $K/A + (K'+A)/A = P/A$ is direct for any submodule $(K'+A)/A$ of P/A, because K is a direct summand of P. So that K/A is a semisimple projective module by Lemma 2.1. This yields that A is a direct summand of K. Hence A is a direct summand of P.

Recall that for a submodule A of a module P, A is called *fully invariant (projection invariant)* if for each endomorphism (idempotent endomorphism) h of P, the image of A under h remains within A; that is, $h(A) \subseteq A$. Based on fundamental definitions, it is readily inferable that any fully invariant submodule necessarily satisfies projection invariance. Moreover, the module P is called *(weak) duo* in case each submodule (direct summand) of P is fully invariant.

Proposition 3.8. If P is a cofinitely $\delta - H - s$upplemented module and A is a projection invariant submodule of P, then its quotient module P/A is a cofinitely $\delta - H - s$upplemented module.

Proof: Assuming that T/A is a cofinite submodule of P/A, we conclude that T is a cofinite submodule of P. Accordingly, P has a direct summand K with the property that the sum $P = T + X$ holds if and only if the sum $P = K + X$ holds for each submodule X of P with singular P/X. Let $P = K \oplus K'$. Since A is a projection invariant submodule of P, then $A = (A \cap K) \oplus (A \cap K')$ by [15, Proposition 3.1]. Therefore, $(K + A) \cap (K' + A) = [K \oplus (K' \cap A)] \cap [(K \cap A) \oplus K'] = (K \cap A) \oplus (K' \cap A) = A$. This immediately implies that, $P/A = ((A + K)/A) \oplus ((A + K')/A)$. Now assuming that $P/A = T/A + Y/A$ for a submodule Y/A of P/A with singular P/Y, we deduce that $P = T + Y$, and hence $P = K + Y$ by assumption. So $P/A = (A + K)/A + Y/A$. In order to show the rest of the proof, let $P/A = (A + K)/A + B/A$ with singular P/B. Then $P = K + B$, and so $P = T + B$ by assumption. As a result, we infer $P/A = T/A + B/A$.

Recall that a module P is said to be *distributive* provided its lattice of submodules satisfies distributivity; in particular, for any submodules A, B, C of P the following conditions are met: $(A \cap B) + (A \cap C) = A \cap (B + C)$ and $A + (B \cap C) = (A + B) \cap (A + C)$.

The next result can be observed with the fact that for a module $P = \bigoplus_{i \in I} P_i$ and for its fully invariant submodule A, $A = \bigoplus_{i \in I} (A \cap P_i)$ given in [16].

Corollary 3.9.

1) Each homomorphic image of a cofinitely $\delta - H - s$upplemented module which is distributive is cofinitely $\delta - H - s$upplemented.

2) Each direct summand of a cofinitely $\delta - H - s$upplemented module which is weak duo is cofinitely $\delta - H - s$upplemented.

Theorem 3.10. Suppose that $P = P_1 \oplus P_2$ is a distributive module. Then P is a cofinitely $\delta - H - s$upplemented module if and only if P_1 and P_2 are cofinitely $\delta - H - s$upplemented modules.

Proof: $(\Longrightarrow)$ The necessity follows from Corollary 3.9.

$(\Longleftarrow)$ To prove the sufficiency, let P_1 and P_2 be cofinitely $\delta - H - s$upplemented modules and A be a cofinite submodule of P. For brevity in notation, say $A_1 = A \cap P_1$ and $A_2 = A \cap P_2$. Then $A = A_1 + A_2$. By assumption, P_i has a direct summand K_i for $i = 1,2$ with the properties

$(A_i + K_i)/A_i \ll_\delta P_i/A_i$ and $(A_i + K_i)/K_i \ll_\delta P_i/K_i$. We claim that $(A+K)/A \ll_\delta P/A$ and $(A+K)/K \ll_\delta P/K$ where $K = K_1 \oplus K_2$ is a direct summand of P. For this, assuming that $P/A = (A+K)/A + X/A$ for a submodule X of P with singular P/X, we derive that $P = K + X$. Accordingly, we obtain that $K_1 + (X \cap P_1) = P_1$. For this reason, $(A_1 + K_1)/A_1 + (X \cap P_1)/A_1 = P_1/A_1$ and $P_1/(X \cap P_1) \cong (X + P_1)/X \leq P/X$ is a singular module. Since $(A_1 + K_1)/K_1 \ll_\delta P_1/K_1$, we conclude the equality $X \cap P_1 = P_1$ meaning that X contains P_1. The equality $P = K + X$ yields that $K_2 + (X \cap P_2) = P_2$. As $(A_2 + K_2)/A_2 + (X \cap P_2)/A_2 = P_2/A_2$ and $(A_2 + K_2)/A_2 \ll_\delta P_2/A_2$, and furthermore $P_2/(X \cap P_2) \cong (X + P_2)/X \leq P/X$ is singular, so that we infer $P_2 = X \cap P_2$. So that $P = X$. In addition, assuming that $(A+K)/K + T/K = P/K$ for a submodule T of P with singular P/T, we derive that $P = A + T$. As such, $P_1 = A_1 + (T \cap P_1)$. Since $(A_1 + K_1)/K_1 \ll_\delta P_1/K_1$ and $P_1/(T \cap P_1)$ is singular, we derive $P_1 = T \cap P_1$ meaning that $P_1 \leq T$. By the same arguments, we can provide that T contains P_2. Therefore, $P = T$. As a result, P is a cofinitely $\delta - H - s$upplemented module.

Recall that an epimorphism $h: P \to W$ is called a δ −*cover* of W if the kernel of h is δ −small in P. If P is furthermore projective, such a morphism is specifically called a *projective* δ −*cover*. Following [10], a ring S is said to be δ −*semiperfect* provided each simple S −module possesses a projective δ −cover.

A module P is called *cofinitely* δ −*semiperfect,* if each quotient module of P by a cofinite submodule possesses a projective δ −cover. A module P is said to be $\oplus$ −*cofinitely* δ −*supplemented* provided that each cofinite submodule of P possesses a δ −supplement which forms a direct summand in P [12].

Theorem 3.11. For a ring S, the statements listed below are all equivalent:

1) S is a δ −semiperfect ring.
2) Each S −module is cofinitely δ −semiperfect.
3) ${}_SS$ is a cofinitely δ −semiperfect module.
4) ${}_SS$ is a cofinitely $\delta - H - s$upplemented module.
5) ${}_SS$ is a $\oplus$ −cofinitely $\delta - s$upplemented module.

Proof: $(1) \Leftrightarrow (2) \Leftrightarrow (5)$ By [12, Theorem 3.9].

The implications $(2) \Rightarrow (3)$ and $(4) \Rightarrow (5)$ are straightforward.

(3) $\Longrightarrow$ (4) A cofinitely $\delta-$semiperfect module ${}_SS$ is cofinitely $\delta-$lifting by [12, Theorem 3.6]. Hence it can be seen that ${}_SS$ is a cofinitely $\delta - H -$ supplemented module by proving in a similar way to the necessity proof of Proposition 3.5.

Example 3.12. (see [12, Example 3.10]) Let F be a field, $I = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ and $S = \{(x_1, x_2, \dots, x_n, x, x, \dots) | n \in \mathbb{N}, x_i \in M_2(F), x \in I\}$ be a ring with component-wise operations. By [12, Example 3.10], S is a $\delta-$semiperfect ring. As another result of [12, Example 3.10], we derive that the module ${}_SS$ is not a $H-$cofinitely supplemented module. However, the module ${}_SS$ is a cofinitely $\delta - H -$ supplemented module according to Theorem 3.11.

4 CONCLUSION AND SUGGESTIONS

This study introduces a novel generalization of $H-$supplemented modules, which themselves extend the concept of lifting modules. When $H-$supplemented modules are analyzed from the perspective of singularity, the notion of modules that are $\delta - H-$supplemented naturally arises. This study introduces a new class of modules derived from the cofinite submodules of $\delta - H-$supplemented modules. This study defines this new class as cofinitely $\delta - H-$supplemented modules and investigate their algebraic properties in detail. This study proves that the homomorphic image of any cofinitely $\delta - H-$supplemented module that satisfies the distributive condition also retains this property. Furthermore, this study demonstrates that if a cofinitely $\delta - H-$supplemented module is weak duo, then each of its direct summands is likewise cofinitely $\delta - H-$supplemented. Finally, this study provides a new characterization of cofinitely $\delta - H-$supplemented modules by providing equivalent conditions in the context of S being a $\delta-$semiperfect ring. The definitions and results presented in this study can also be considered for co-coatomic submodules, which constitute a proper generalization of cofinite submodules and were introduced in [17], and $\delta - H-$supplemented modules may be approached via co-coatomic submodules in the same manner as carried out in this paper.

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Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

Artificial Intelligence (AI) Contribution Statement

This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools. All content, including text, data analysis, and figures, was solely generated by the author.

REFERENCES

[1] S. H. Mohamed, and B. J. Müller, *London mathematical society lecture note series: Continuous and discrete modules series number 147*. Cambridge, England: Cambridge University Press, 1990.

[2] M. T. Koşan, and D. Keskin Tütüncü, "H −supplemented duo modules," *J. Algebra Appl.*, vol. 06, no. 06, pp. 965–971, 2007.

[3] D. Keskin Tütüncü, M. J. Nematollahi, and Y. Talebi, "On H −supplemented modules", *Algebra Colloq*., 18 Spec., pp. 915–924, 2011.

[4] Y. Talebi, A.R. Moniri Hamzekolaee, and D. Keskin Tütüncü, "H −supplemented modules with respect to a preradical," *Algebra Discrete Math.*, vol. 12, no. 1, pp. 116–131, 2011.

[5] Y. Talebi, R. Tribak, and A. R. Moniri Hamzekolaee, "On H −cofinitely supplemented modules," *Bull. Iranian Math. Soc.*, vol. 30, no. 2, pp. 325–346, 2013.

[6] A. R. Moniri Hamzekolaee, A. Harmancı, Y. Talebi, and B. Üngör, "A new approach to H −supplemented modules via homomorphisms" *Turk. J. Math.*, vol. 42, pp. 1941–1955, 2018.

[7] A. R. Moniri Hamzekolaee, "H^* −condition on the set of submodules of a module," *J. Algebraic Struc. Appl.*, vol. 6, no. 2, pp. 13–20, 2019.

[8] R. Alizade, G. Bilhan, and P. F. Smith, "Modules whose maximal submodules have supplements," *Comm. Algebra*, vol. 29, no. 6, pp. 2389–2405, 2001.

[9] R. Wisbauer, *Foundations of module and ring theory*. London, England: Routledge, 2018.

[10] Y. Zhou, "Generalizations of perfect, semiperfect and semiregular rings," *Algebra Colloq.*, vol. 7, no. 3, pp. 305–318, 2000.

[11] M. T. Koşan, "δ −lifting and δ −supplemented modules," *Algebra Colloq*., vol. 14, no. 01, pp. 53–60, 2007.

[12] A. Takhman, "Cofinitely δ −supplemented and cofinitely δ −semiperfect modules," *Int. J. Algebra*, vol. 1, no. 12, pp. 601–613, 2007.

[13] M. T. Koşan, "H−Cofinitely supplemented modules," *Vietnam J. Math*., vol. 35, no. 2, pp. 215–222, 2007.

[14] A. Ç. Özcan, "The torsion theory cogenerated by δ−M−small modules and GCO-modules," *Commun. Algebra*, vol. 35, pp. 623–633, 2007.

[15] C. Abdioğlu, M. T. Koşan, and S. Şahinkaya, "On modules for which all submodules are projection invariant and the lifting condition," *Southeast Asian Bull. Math*., vol. 34, pp. 807-818, 2010.

[16] A. Ç. Özcan, P. F. Smith, and A. Harmancı, "Duo modules," *Glasgow Math. J.*, vol. 48, no. 1, pp. 533–545, 2006.

[17] R. Alizade, and S. Güngör, "Co-coatomically supplemented modules," *Ukr. Math. J.*, vol. 69, no. 7, pp. 867-876, 2017.