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Research Article

Multiplicative set-valued equation related to certain set-valued mappings and its stability on Banach algebras

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ABSTRACT. Let $\mathcal{B}$ be a Banach algebra, and let $\mathcal{C}cb(\mathcal{B})$ denote the set of all closed convex bounded subsets of $\mathcal{B}$. Assume that $\succeq$ is a partial order defined on $\mathcal{C}cb(\mathcal{B})$, and define $\mathcal{D} := \{A \in \mathcal{C}_{cb}(\mathcal{B}) : A \succ \mathbf{0}\}$, where $\mathbf{0}$ denotes the zero element of $\mathcal{C}_{cb}(\mathcal{B})$. Furthermore, suppose that for every $A, B \in \mathcal{D}$, the set $A \otimes B$ also belongs to $\mathcal{D}$, where $A \otimes B$ means the closure of the product set AB. In this paper, general solutions $F : \mathcal{D} \to \mathcal{D}$ of the multiplicative set-valued functional equation

$$F(X \otimes Y) = F(X) \otimes F(Y)$$

for all $X, Y \in D$ are determined. These solutions are closely involved with some set-valued mappings. Moreover, its stability is also proved on Banach algebras. The results not only generalize classical findings in functional equations but also open avenues for further exploration in nonlinear analysis and set-valued operator theory.

Keywords: Banach algebras, Hausdorff metric, stability, multiplicative set-valued equations, set-valued mappings.

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1. INTRODUCTION

Consider the well-known functional equations, where $f_1, f_2, f_3, f_4, f_5 : \mathbb{R} \to \mathbb{R}$ are unknown functions, as follows:

$$f_1(x + y) = f_1(x) + f_1(y), \tag{1.1}$$

$$f_2(x + y) = f_2(x)f_2(y), \tag{1.2}$$

$$f_3(xy) = f_3(x)f_3(y), \tag{1.3}$$

$$2f_4\left(\frac{x + y}{2}\right) = f_4(x) + f_4(y), \tag{1.4}$$

$$f_5(x + y) + f_5(x - y) = 2f_5(x) + 2f_5(y) \tag{1.5}$$

for all $x, y \in \mathbb{R}$. The first three functional equations are known as *additive, exponential and multiplicative functional equations*, respectively. The function solutions $f_1, f_2, f_3 : \mathbb{R} \to \mathbb{R}$ satisfying (1.1), (1.2), and (1.3) are called *additive, exponential and multiplicative functions*, respectively. The equation (1.4) is called a *Jensen's functional equation*, while an equation (1.5) is called a *quadratic functional equation*. The mappings $f_4, f_5 : \mathbb{R} \to \mathbb{R}$ satisfying (1.4) and (1.5) are called *Jensen* and *quadratic functions*, respectively. These regarding functional equations have been solved by many mathematicians in many variants.

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In the article [4], Henney determined the general solutions $F : \mathbb{R} \to \mathcal{C}(E^n)$ of the following functional equation

$$F(x+y) + F(x-y) = 2F(x) + 2F(y) \tag{1.6}$$

for all $x, y \in \mathbb{R}$, where $\mathcal{C}(E^n)$ denotes the family of all non-empty compact subsets of an n-dimensional linear space E^n, and $A + B$ represents the sumset of $A, B \in \mathcal{C}(E^n)$, that is, the set of all vectors $a + b$ with $a \in A$ and $b \in B$. This equation is a set-valued version of the functional equation (1.5) so that it is called a *quadratic set-valued functional equation*. In another direction, the general solutions $F : \mathbb{R} \to \mathcal{CC}(X)$ of the functional equation (1.6) were determined in 1984 by Nikodem [9], where $\mathcal{CC}(X)$ denotes the family of all non-empty compact convex subsets of a real normed space X.

In 1987, Nikodem [10] determined all set-valued functions $F : [0, \infty) \to \mathcal{C}(V)$ satisfying the following set-valued functional equation

$$2F\left(\frac{x+y}{2}\right) = F(x) + F(y)$$

for all $x, y \in [0, \infty)$, where $\mathcal{C}(V)$ denotes the family of all non-empty compact subsets of a locally convex topological vector space V. This equation is a set-valued version of the functional equation (1.4). He also determined the general solutions $F, G, H : S \to Y$ of the set-valued functional equation

$$F(x+y) = G(x) + H(y)$$

for all $x, y \in S$, where S is an abelian semigroup and Y is a T_0 topological vector space, in 1988 [11]. This equation is a Pexiderized form of a so-called *additive set-valued functional equation*

$$F(x+y) = F(x) + F(y) \tag{1.7}$$

for all $x, y \in S$. Note here that the functional equation (1.7) is a set-valued version of the functional equation (1.1).

In 2017, by using fixed point method, an analysis of the stability of set-valued functional equation (1.6) was established by Lee et al. [7]. For an abelian group $(G, +)$ and a Hausdorff topological vector space Y over a field $\mathbb{R}$, the set-valued functional equation (1.6) was generalized to

$$F(x+y) + F(x-y) = aF(x) + bF(y) \tag{1.8}$$

for all $x, y \in G$, where $F : G \to \mathcal{P}_0(Y) := \{A \subseteq Y \ : \ A \neq \emptyset\}$ is an unknown function and a, b are nonnegative real numbers, in 2018, by Baias et al. [2]. They also determined all set-valued function solutions satisfying (1.8). Very recently, Park et al. [13] investigeted the stability of the set-valued functional equation (1.7) by using fixed point method as well. For other related problems, we refer to the works [3, 5, 6, 8], and [12].

From the above historical background, it can be observed that most of the functional equations, namely (1.1), (1.4), and (1.5), have been studied and solved in the framework of set-valued equations. However, this is not the case for the functional equations (1.2) and (1.3). This gap naturally motivates the present research, which focuses on functional equations in the setting where the unknown function maps subsets of a Banach algebra $\mathcal{B}$ into elements of $\mathcal{C}cb(\mathcal{B})$, the family of all closed, convex, and bounded subsets of $\mathcal{B}$. Throughout this paper, we refer to such functions as set-valued functions, since their values lie in $\mathcal{C}cb(\mathcal{B})$. In addition, throughout the paper we let $\succeq$ denote a partial order defined on $\mathcal{C}cb(\mathcal{B})$, and we let $\mathbf{0}$ denote the zero element of $\mathcal{C}cb(\mathcal{B})$. Unless otherwise stated, these notations and conventions will be used throughout the paper. Define the *multiplication* between subsets of $\mathcal{B}$ by

$$UV := \{xy \ : \ x \in U, y \in V\}$$

for all $U, V \subseteq \mathcal{B}$. For all $A, B \in \mathcal{C}_{cb}(\mathcal{B})$, we define the operation $\otimes$ by

$$A \otimes B := \overline{AB},$$

which denotes the closure of the product AB. Moreover, throughout this paper, unless otherwise stated, we denote

$$\mathcal{D} := \{A \in \mathcal{C}_{cb}(\mathcal{B}) : A \succ \mathbf{0}\},$$

which satisfies the property that $A \otimes B \in \mathcal{D}$ for all $A, B \in \mathcal{D}$.

Our main objective in this paper is to investigate the general solutions of the functional equation

$$F(X \otimes Y) = F(X) \otimes F(Y) \tag{1.9}$$

for all $X, Y \in \mathcal{D}$, where $F : \mathcal{D} \to \mathcal{D}$ is an unknown function. This equation represents a generalized form of the functional equation (1.3). Our approach in this paper is twofold. First, we transform the main set-valued functional equation into a corresponding exponential-type form. Second, we address the problem of finding its general solution within this framework. In particular, we introduce the notions of exponential and logarithmic set-valued mappings, which play a crucial role in establishing the connection between the two functional equations and in deriving the complete set of solutions. Some interesting examples of such set-valued mappings are also demonstrated. Moreover, we prove the stability of the set-valued functional equation (1.9) on Banach algebras. This result requires a special property of the Hausdorff metric. Some examples of a such property are also given in this paper.

2. Preliminaries

In this section, we gather background knowledge needed for our results. Continuous with the recalling some notions in Section 1, for a Banach algebra $\mathcal{B}$, we denote generically, unless otherwise specified, by $2^{\mathcal{B}}, \mathcal{C}_b(\mathcal{B})$ and $\mathcal{C}_c(\mathcal{B})$ the set of all subsets of $\mathcal{B}$, the set of all closed bounded subsets of $\mathcal{B}$, and the set of all closed convex subsets of $\mathcal{B}$, respectively. Define the *addition*, *multiplication* and *scalar multiplication* for $A, B \subseteq \mathcal{B}$ and $\lambda \in \mathbb{R}$ as follows:

$$\begin{aligned} A + B &:= \{x + y \ : \ x \in A, y \in B\}, \\ AB &:= \{xy \ : \ x \in A, y \in B\}, \\ \lambda A &:= \{\lambda x \ : \ x \in A\}. \end{aligned} \tag{2.10}$$

Let Y be a Banach space. For all $A, B \in \mathcal{C}_{cb}(Y)$, define an operation $\oplus$ by

$$A \oplus B := \overline{A + B},$$

where $\overline{A + B}$ is the closure of the sumset $A + B$. It is easily checked ([14, Chapter II]) that

$$\lambda A + \lambda B = \lambda(A + B) \quad \text{and} \quad (\lambda + \mu)A \subseteq \lambda A + \mu A \tag{2.11}$$

for all $A, B \in 2^Y$ and all $\lambda, \mu \in \mathbb{R}$. Furthermore, we ([14, Chapter II]) have the following assertions:

(1) If $C \in 2^Y$ is convex, then

$$(\lambda + \mu)C = \lambda C + \mu C$$

for all $\lambda, \mu \in \mathbb{R}^+$.

(2) If $U, V, W \in \mathcal{C}_{cb}(Y)$, then

$$U \oplus W = V \oplus W \quad \text{implies} \quad U = V.$$

Let (X, d) be a metric space. For any pair of sets $A, B \in \mathcal{C}_b(Y)$, the *Hausdorff distance between A and B with respect to the metric d* is defined by

$$h(A,B) = \max\left\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\right\}. \tag{2.12}$$

In the case of a normed space, d denotes the metric induced by the norm. Since Y is a Banach space, it was proved that $(\mathcal{C}_{cb}(Y), \oplus, h)$ is a complete metric semigroup. Moreover, $(\mathcal{C}_{cb}(Y), \oplus, h)$ is isometrically embedded in a Banach space (see [15]).

Some results, which are directly obtained from the definition of the Hausdorff distance defined in (2.12), are revealed as follows.

Proposition 2.1 ([14]). *Let Y be a Banach space. For every $A, B, C, D \in \mathcal{C}_{cb}(Y)$ and $\lambda > 0$, we have*

(i) $h(A \oplus B, C \oplus D) \leq h(A, C) + h(B, D)$;
(ii) $h(\lambda A, \lambda B) = \lambda h(A, B)$;
(iii) $h(A, B) = h(A \oplus C, B \oplus C)$;
(iv) $h(A, B) = h(-A, -B)$.

Note that the properties (i) and (iii) in the above proposition imply

$$h(A,C) \ \leq \ h(A,B) + h(B,C) \tag{2.13}$$

for all $A, B, C \in \mathcal{C}_{cb}(Y)$.

3. General solutions

In this section, firstly, we define the exponential and logarithmic set-valued mappings which play a crucial role in determining the solutions of the set-valued equation (1.9). Some examples of such set-valued mappings are presented here. The general solutions of the set-valued functional equation (1.9) are given later.

Definition 3.1. *Let $\succeq$ be a partial order on $\mathcal{C}_{cb}(\mathcal{B})$ and let $\mathbf{0}$ be the zero element in $\mathcal{C}_{cb}(\mathcal{B})$. Suppose that $\mathcal{R} \subseteq \mathcal{C}_{cb}(\mathcal{B})$ and $\mathcal{D} := \{A \in \mathcal{C}_{cb}(\mathcal{B}) : A \succ \mathbf{0}\}$ be such that $A \otimes B \in \mathcal{D}$ for all $A, B \in \mathcal{D}$.*

(i) A mapping $E : \mathcal{R} \to \mathcal{D}$ is called an exponential set-valued mapping if

$$E(A \oplus B) = E(A) \otimes E(B) \tag{3.14}$$

for all $A, B \in \mathcal{R}$.

(ii) A mapping $L : \mathcal{D} \to \mathcal{R}$ is called a logarithmic set-valued mapping if

$$L(A \otimes B) = L(A) \oplus L(B) \tag{3.15}$$

for all $A, B \in \mathcal{D}$.

Remark 3.1. *Note that the exponential and logarithmic set-valued mappings defined by* (3.14) *and* (3.15)*, respectively, in Definition 3.1 need not be bijective.*

Example 3.1. *Let $\mathcal{B} = \mathbb{R}$ be equipped with the usual norm, and multiplication of real numbers, making $\mathcal{B}$ a Banach algebra. For $A, B \in \mathcal{C}_{cb}(\mathcal{B})$, let $A \succeq B$ to mean that $a \geq b$ for all $a \in A$ and all $b \in B$. This defines a partial order $\mathcal{C}_{cb}(\mathcal{B})$. Let $\mathcal{R} := \{[u, v] \subseteq \mathbb{R} : 0 \leq u < v\}$ be a set of closed intervals on the real line $\mathbb{R}$, and let $\mathcal{D} := \{[0, e^w] \subseteq \mathbb{R} : w \in \mathbb{R}\}$. Define a mapping $E_1 : \mathcal{R} \to \mathcal{D}$ by*

$$[i, j] \mapsto [0, e^i] \ (\succ \mathbf{0}),$$

where $\mathbf{0}$ *represents a zero element* $[0,0] \in \mathcal{C}_{cb}(\mathcal{B})$*. Then, for all* $A, B \in \mathcal{R}$*, say* $A = [a,b], B = [u,v]$*, we have*

$$\begin{aligned} E_1(A \oplus B) &= E_1([a,b] \oplus [u,v]) \\ &= E_1(\overline{[a+u, b+v]}) \\ &= E_1([a+u, b+v]) \\ &= [0, e^{a+u}] \\ &= \overline{[0, e^{a+u}]} \\ &= \overline{[0, e^{a}] \cdot [0, e^{u}]} \\ &= \overline{E_1(A) \cdot E_1(B)} \\ &= E_1(A) \otimes E_1(B), \end{aligned}$$

showing that E_1 *is an exponential set-valued mapping.*

Example 3.2. *Let* $\mathcal{B} = \mathbb{R}$ *be equipped with the usual norm, and multiplication of real numbers, making* $\mathcal{B}$ *a Banach algebra. Taking the set* $\mathcal{R}$ *and the relation* $\succeq$ *as in Example 3.1, define a mapping* $E_2 : \mathcal{R} \to \mathcal{D} := \{\{e^k\} : k \in \mathbb{R}\}$ *by*

$$[i,j] \mapsto \{e^{i+j}\} \ (\succ \mathbf{0}).$$

Here, $\mathbf{0}$ *is a set of the usual zero element* $\{0\} \in \mathcal{C}_{cb}(\mathcal{B})$*. It is easily checked, by using the same arguments as in Example 3.1, that* E_2 *is an exponential set-valued mapping.*

Example 3.3. *Let* $\mathcal{B} = \mathbb{R}^2$ *equipped with the norm* $\|\cdot\|$ *defined by*

$$\|x\| = \max\{|\xi_1|, |\xi_2|\}$$

for all $x = (\xi_1, \xi_2) \in \mathcal{B}$*. Define multiplication on* $\mathcal{B}$ *by*

$$(\xi_1, \xi_2)(\eta_1, \eta_2) = (\xi_1\eta_1,\ \xi_2\eta_2)$$

for all $x = (\xi_1, \xi_2), y = (\eta_1, \eta_2) \in \mathcal{B}$*. Then* $\mathcal{B}$ *is a Banach algebra. For* $A, B \in \mathcal{C}_{cb}(\mathcal{B})$*, we write* $A \succeq B$ *to mean that* $a_1 \geq b_1$ *and* $a_2 \geq b_2$ *for all* $(a_1, a_2) \in A$ *and* $(b_1, b_2) \in B$*. This defines a partial order on* $\mathcal{C}_{cb}(\mathcal{B})$*. Let* $\mathcal{R} := \{R_1, R_1, R_3, \ldots\}$ *and* $\mathcal{D} := \{D_1, D_2, D_3, \ldots\}$*, where*

$$\begin{aligned} R_i &:= \{(\xi, \eta) \in [0,\infty)^2 : \xi + \eta \leq i\}, \\ D_i &:= \left\{\left(e^{\xi}, 0\right) \in [0,\infty)^2 : 0 \leq \xi \leq i\right\} \end{aligned}$$

for all $i \in \{1, 2, 3, \ldots\}$*. Define a mapping* $E_3 : \mathcal{R} \to \mathcal{D}$ *by*

$$R_t \mapsto D_t \qquad (t \text{ is fixed}).$$

Note here that $\mathbf{0}$ *is* $\{(0,0)\}$ *and that* $D_t \succ \mathbf{0}$ *for each* $t \in \mathbb{N}$*. For each* $R_i, R_j \in \mathcal{R}$*, we have*

$$\begin{aligned} E_3(R_i \oplus R_j) &= E_3(\{(\xi_1, \xi_2) \in [0,\infty)^2 : \xi_1 + \xi_2 \leq i\} \oplus \{(\eta_1, \eta_2) \in [0,\infty)^2 : \eta_1 + \eta_2 \leq j\}) \\ &= E_3(\overline{\{(\xi_1 + \eta_1, \xi_2 + \eta_2) \in [0,\infty)^2 : (\xi_1 + \eta_1) + (\xi_2 + \eta_2) \leq i + j\}}) \\ &= E_3(\{(\xi_1 + \eta_1, \xi_2 + \eta_2) \in [0,\infty)^2 : (\xi_1 + \eta_1) + (\xi_2 + \eta_2) \leq i + j\}) \\ &= \left\{\left(e^{\xi}, 0\right) : 0 \leq \xi \leq i + j\right\} \\ &= \overline{\{(e^{\xi}, 0) : 0 \leq \xi \leq i + j\}} \\ &= \overline{\{(e^{a}, 0) : 0 \leq a \leq i\} \cdot \{(e^{b}, 0) : 0 \leq b \leq j\}} \\ &= \overline{E_3(R_i) \cdot E_3(R_j)} \\ &= E_3(R_i) \otimes E_3(R_l). \end{aligned}$$

This shows that E_3 is an exponential set-valued mapping.

Example 3.4. *Let $\mathcal{B} = \mathbb{R}^2$ equipped with the norm $\|\cdot\|$ defined by*

$$\|x\| = \max\{|\xi_1|, |\xi_2|\}$$

for all $x = (\xi_1, \xi_2) \in \mathcal{B}$. Define multiplication on $\mathcal{B}$ by

$$(\xi_1, \xi_2)(\eta_1, \eta_2) = (\xi_1\eta_1,\ \xi_2\eta_2)$$

for all $x = (\xi_1, \xi_2), y = (\eta_1, \eta_2) \in \mathcal{B}$. Then $\mathcal{B}$ is a Banach algebra. For $A, B \in \mathcal{C}_{cb}(\mathcal{B})$, we write $A \succeq B$ to mean that $a_1 \geq b_1$ and $a_2 \geq b_2$ for all $(a_1, a_2) \in A$ and $(b_1, b_2) \in B$. This defines a partial order on $\mathcal{C}_{cb}(\mathcal{B})$. Let $\mathcal{R} := \{R_1, R_1, R_3, \ldots\}$ and $\mathcal{D} := \{D_1, D_2, D_3, \ldots\}$, where

$$D_i := \{(0, \eta) \in [0, \infty)^2 : 1 \leq \eta \leq i\},$$

$$R_i := \{(0, \ln \xi) \in [0, \infty)^2 : 1 \leq \xi \leq j\}$$

for all $i \in \{1, 2, 3, \ldots\}$. Define a mapping $L : \mathcal{D} \to \mathcal{R}$ by

$$D_t \mapsto R_t \qquad (t \geq 1 \text{ is fixed}).$$

For each $D_i, D_j \in \mathcal{D}$, we have

$$\begin{aligned}
L(D_i \otimes D_j) &= L(\{(\eta_1, 0) \in \mathbb{R}^2 : 1 \leq \eta_1 \leq i\} \otimes \{(\eta_2, 0) \in \mathbb{R}^2 : 1 \leq \eta_2 \leq j\}) \\
&= L(\overline{\{(\eta_1\eta_2, 0) \in \mathbb{R}^2 : 1 \leq \eta_1\eta_2 \leq ij\}}) \\
&= L(\{(\eta_1\eta_2, 0) \in \mathbb{R}^2 : 1 \leq \eta_1\eta_2 \leq ij\}) \\
&= \{(0, \ln \eta_1\eta_2) \in \mathbb{R}^2 : 1 \leq \eta_1\eta_2 \leq ij\} \\
&= \{(0, \ln \eta_1 + \ln \eta_2) \in \mathbb{R}^2 : 1 \leq \eta_1 \leq i, 1 \leq \eta_2 \leq j\} \\
&= \{(0, \ln \eta_1) + (0, \ln \eta_2) \in \mathbb{R}^2 : 1 \leq \eta_1 \leq i, 1 \leq \eta_2 \leq j\} \\
&= \overline{\{(0, \ln \eta_1) + (0, \ln \eta_2) \in \mathbb{R}^2 : 1 \leq \eta_1 \leq i, 1 \leq \eta_2 \leq j\}} \\
&= \overline{\{(0, \ln \eta_1) \in \mathbb{R}^2 : 1 \leq \eta_1 \leq i\} + \{(0, \ln \eta_2) \in \mathbb{R}^2 : 1 \leq \eta_2 \leq j\}} \\
&= \overline{L(A) + L(B)} \\
&= L(A) \oplus L(B).
\end{aligned}$$

This ensures that L is a logarithmic set-valued mapping.

We are now ready to solve the set-valued functional equation (1.9).

Definition 3.2. *Let $\succeq$ be a partial order on $\mathcal{C}_{cb}(\mathcal{B})$ and let $\mathbf{0}$ be the zero element in $\mathcal{C}_{cb}(\mathcal{B})$. Suppose that $\mathcal{D} := \{A \in \mathcal{C}_{cb}(\mathcal{B}) : A \succ \mathbf{0}\}$ be such that $A \otimes B \in \mathcal{D}$ for all $A, B \in \mathcal{D}$ and $F : \mathcal{D} \to \mathcal{D}$ is a set-valued mapping. The multiplicative set-valued functional equation is defined by*

$$F(X \otimes Y) = F(X) \otimes F(Y) \tag{3.16}$$

for all $X, Y \in \mathcal{D}$. Every solution of (3.16) is called a multiplicative set-valued mapping.

Theorem 3.1. *Let $\succeq$ be a partial order on $\mathcal{C}_{cb}(\mathcal{B})$, $\mathcal{R} \subseteq \mathcal{C}_{cb}(\mathcal{B})$ and $\mathcal{D} := \{A \in \mathcal{C}_{cb}(\mathcal{B}) : A \succ \mathbf{0}\}$ be such that $A \otimes B \in \mathcal{D}$ for all $A, B \in \mathcal{D}$, where $\mathbf{0}$ is a zero element in $\mathcal{C}_{cb}(\mathcal{B})$. If a logarithmic set-valued mapping $L : \mathcal{D} \to \mathcal{R}$ and an onto exponential set-valued mapping $E : \mathcal{R} \to \mathcal{D}$, which are inverses of each other, exist, then the general solution $F : \mathcal{D} \to \mathcal{D}$ of the set-valued functional equation (3.16) is given by*

$$F(X) = E(G(L(X))) \tag{3.17}$$

for all $X \in \mathcal{D}$, *where* $G : \mathcal{R} \to \mathcal{R}$ *is an arbitrary additive set-valued mapping satisfying the set-valued functional equation*

$$G(X \oplus Y) = G(X) \oplus G(Y)$$

for all $X, Y \in \mathcal{R}$.

Proof. Define a set-valued mapping $H : \mathcal{R} \to \mathcal{D}$ by

$$H(X) = F(E(X)) \tag{3.18}$$

for all $X \in \mathcal{R}$. Then the equations (3.14) , (3.18), and the functional equation (3.16) imply

$$\begin{aligned} H(X \oplus Y) &= F(E(X \oplus Y)) \\ &= F(E(X) \otimes E(Y)) \\ &= F(E(X)) \otimes F(E(Y)) \\ &= H(X) \otimes H(Y) \end{aligned} \tag{3.19}$$

for all $X, Y \in \mathcal{R}$. Hence, solving the functional set-valued equation (3.16) is sufficient to treat the functional set-valued equation (3.19). Taking L of both sides of (3.19) and, with $G : \mathcal{R} \to \mathcal{R}$ defined by

$$G(X) = L(H(X)) \tag{3.20}$$

for all $X \in \mathcal{R}$, obtain

$$\begin{aligned} G(X \oplus Y) &= L(H(X \oplus Y)) \\ &= L(H(X) \otimes H(Y)) \\ &= L(H(X)) \oplus L(H(Y)) \\ &= G(X) \oplus G(Y) \end{aligned}$$

for all $X, Y \in \mathcal{R}$, by using (3.15) and (3.19), yielding that G is an additive set-valued mapping. The equation (3.20) now implies

$$H(X) = L^{-1}(G(X)) = E(G(X)) \tag{3.21}$$

for all $X \in \mathcal{R}$. Equating of (3.18) and (3.21), we arrive at

$$F(E(X)) = E(G(X))$$

for all $X \in \mathcal{R}$. The desired solutions (3.17) henceforth follows immediately. □

From the above theorem, we also obtain the general solution of the exponential set-valued functional equation as follows.

Corollary 3.1. *Let* $\succeq$ *be a partial order on* $\mathcal{C}_{cb}(\mathcal{B})$, $\mathcal{R} \subseteq \mathcal{C}_{cb}(\mathcal{B})$, *and* $\mathcal{D} := \{A \in \mathcal{C}_{cb}(\mathcal{B}) : A \succ \mathbf{0}\}$ *be such that* $A \otimes B \in \mathcal{D}$ *for all* $A, B \in \mathcal{D}$, *where* $\mathbf{0}$ *is a zero element in* $\mathcal{C}_{cb}(\mathcal{B})$. *If a logarithmic set-valued mapping* $L : \mathcal{D} \to \mathcal{R}$ *and an onto exponential set-valued mapping* $E : \mathcal{R} \to \mathcal{D}$, *which are inverses of each other, exist, then the general solution* $H : \mathcal{R} \to \mathcal{D}$ *of the exponential set-valued functional equation*

$$H(X \oplus Y) = H(X) \otimes H(Y) \tag{3.22}$$

for all $X, Y \in \mathcal{R}$, *is given by*

$$H(X) = E(G(X))$$

for all $X \in \mathcal{R}$, *where* $G : \mathcal{R} \to \mathcal{R}$ *is an arbitrary additive set-valued mapping satisfying the set-valued functional equation*

$$G(X \oplus Y) = G(X) \oplus G(Y)$$

for all $X, Y \in \mathcal{R}$.

From the above theorem, we now give some examples of the two set-valued mappings which are inverse of each other satisfying the hypothesis of the theorem. We start with a simplest example as follows.

Example 3.5. *Let $\mathcal{B} = \mathbb{R}$ be equipped with the usual norm, and multiplication of real numbers, making $\mathcal{B}$ a Banach algebra. Take $\mathcal{D} := \{\{e^x\} : x \in \mathbb{R}\}$ and $\mathcal{R} := \{\{x\} : x \in \mathbb{R}\}$. Define a set-valued mapping $E : \mathcal{R} \to \mathcal{D}$ by*

$$\{x\} \longmapsto \{e^x\}, \tag{3.23}$$

and also define a set-valued mapping $L : \mathcal{D} \to \mathcal{R}$ by

$$\{x\} \longmapsto \{\ln x\}. \tag{3.24}$$

Then it is easily verified that these two mappings, respectively, satisfy the equations (3.14) and (3.15); i.e, they are exponential and logarithmic set-valued mappings, respectively. Here, note from (3.23) and (3.24) that

$$L(E(\{x\})) = L(\{e^x\}) = \{\ln e^x\} = \{x\}$$

for all $\{x\} \in \mathcal{R}$ and that

$$E(L(\{x\})) = E(\{\ln x\}) = \{e^{\ln x}\} = \{x\}$$

for all $\{x\} \in \mathcal{D}$. This shows that these two set-valued mappings defined by (3.23) and (3.24) are inverse of each other.

Example 3.6. *Let $\mathcal{B} = \mathbb{R}$ be equipped with the usual norm, and multiplication of real numbers, making $\mathcal{B}$ a Banach algebra. Take $\mathcal{B} = \mathbb{R}, \mathcal{D} := \{[e^a, e^b] \subseteq \mathbb{R} : a, b \in \mathbb{R},\ a \leq b\}$ and $\mathcal{R} := \{[a, b] \subseteq \mathbb{R} : a, b \in \mathbb{R},\ a \leq b\}$. Define a set-valued mapping $E : \mathcal{R} \to \mathcal{D}$ by*

$$[a, b] \longmapsto [e^a, e^b],$$

and also define a set-valued mapping $L : \mathcal{D} \to \mathcal{R}$ by

$$[a, b] \longmapsto [\ln a, \ln b].$$

Then it is easy to check that these two mappings indeed satisfy (3.14) and (3.15), respectively, and that they are inverse of each other, by using the same manner as in the previous example.

By using the set-valued mappings defined in the previous two examples, we here obtain some particular solutions of our set-valued functional equation (3.16) as follows.

Example 3.7. *By using the exponential and logarithmic set-valued mappings defined in Example 3.5, and picking an additive set-valued mapping $G : \mathcal{R} \to \mathcal{R}$ defined by*

$$\{x\} \longmapsto \{2x\},$$

by Theorem 3.1, a particular solution $F : \mathcal{D} \to \mathcal{D}$ of the equation (3.16) is of the form

$$F(\{e^x\}) = \{e^{2x}\}$$

for all $\{e^x\} \in \mathcal{D}$.

Example 3.8. *By using the exponential and logarithmic set-valued mappings defined in Example 3.6, and picking an additive set-valued mapping $G : \mathcal{R} \to \mathcal{R}$ defined by*

$$[x, y] \longmapsto [x, 2y],$$

by Theorem 3.1, a particular solution $F : \mathcal{D} \to \mathcal{D}$ of the equation (3.16) is of the form

$$F([e^x, e^y]) = [e^x, e^{2y}]$$

for all $[e^x, e^y] \in \mathcal{D}$.

The function solutions so obtained in each example are easily checked to satisfy the set-valued functional equation (3.16).

4. Stability analysis

It was proved, in 1980, by Baker [1] that for $\delta > 0$ and a complex-valued function f defined on a semigroup S satisfying

$$|f(xy) - f(x)f(y)| \leq \delta$$

for all $x, y \in S$, either $|f(x)| \leq (1 + \sqrt{1+4\delta})/2$ for all $x \in S$, or $f(xy) = f(x)f(y)$ for all $x \in S$. This leads us, in this section, to prove the stability of the set-valued functional equation (3.16) on Banach algebras. The obtained result here requires some special property of the Hausdorff distance h induced by the metric d on the Banach algebra $\mathcal{B}$. Such a property here, indeed, is extended from the basic property of real numbers as follows: for every $x, y, z \in \mathbb{R}$,

$$|xy - xz| = |x|\,|y - z|\,.$$

The latter property may not hold in general for the set of all closed convex bounded subsets of the Banach algebra $\mathcal{B}$. It takes us to consider the following examples.

Example 4.9. *Let $\mathcal{B} = \mathbb{R}$ be equipped with the usual norm and multiplication of real numbers, making $\mathcal{B}$ a Banach algebra. Define $A := [-2,-1]$, $B := [2,3]$, and $C := \{0\}$. By the definitions of multiplication* (2.10) *and* (2.12) *of h induced by the usual metric d, it is easily see that*

$$h(A \otimes B, A \otimes C) = h([-6,-2], \{0\}) = 6,$$

and that

$$h(A, \mathbf{0}) = h([-2,-1], \{0\}) = 2, \quad \text{and} \quad h(B, C) = h([2,3], \{0\}) = 3.$$

The result so obtained here is that

$$h(A \otimes B, A \otimes C) \;=\; h(A, \mathbf{0})h(B, C).$$

Example 4.10. *Let $\mathcal{B} = \mathbb{R}$ be equipped with the usual normed, and multiplication of real numbers, making $\mathcal{B}$ a Banach algebra. Define $A := [-2,-1]$, $B := [2,3]$, and $C := \{-2,0,3\}$. Then, by the definitions of multiplication* (2.10) *and* (2.12) *of h induced by the usual metric d, it is easily see that*

$$h(A \otimes B, A \otimes C) = h([-6,-2], [-6,-3] \cup \{0\} \cup [2,4]) = 6,$$

and that

$$h(A, \mathbf{0}) = h([-2,-1], \{0\}) = 2, \quad \text{and} \quad h(B, C) = h([2,3], \{-2,0,3\}) = 4.$$

In this case, we see that

$$h(A \otimes B, A \otimes C) \;\neq\; h(A, \mathbf{0})h(B, C).$$

Example 4.11. *Let $\mathcal{B} = \mathbb{R}^2$ be equipped with the Euclidean norm and the usual coordinatewise multiplication of ordered pairs of real numbers, making $\mathcal{B}$ a Banach algebra. Define*

$$A := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\},$$
$$B := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\},$$

and

$$C := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}.$$

Note that $A \otimes B = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\} = B$ and $A \otimes C = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\} = C$. The definitions of multiplication (2.10) *and* (2.12) *of h induced by the Euclidean metric d now yields that*

$$h(A \otimes B, A \otimes C) = h(B, C) = 1.$$

Note here that $h(A,\mathbf{0}) = h(A,\{(0,0)\}) = 1 = h(B,C)$, *and so we get*

$$h(A\otimes B, A\otimes C) \;=\; h(A,\mathbf{0})h(B,C).$$

To obtain the stability result for the multiplicative set-valued functional equation (3.16), we use the Hausdorff distance h on $\mathcal{C}_{cb}(\mathcal{B})$ to measure the distance between two sets. For this purpose, a special condition on the Hausdorff metric h is required, and the resulting theorem is as follows:

Theorem 4.2. *Let* $\succeq$ *be a partial order on* $\mathcal{C}_{cb}(\mathcal{B})$, *and* $\mathcal{D} := \{A \in \mathcal{C}_{cb}(\mathcal{B}) : A \succ \mathbf{0}\}$ *be such that* $A\otimes B \in \mathcal{D}$ *for all* $A, B \in \mathcal{D}$, *where* $\mathbf{0}$ *is a zero element in* $\mathcal{C}_{cb}(\mathcal{B})$. *Assume that* $F : \mathcal{D} \to \mathcal{D}$ *satisfies*

$$h\left(F(X\otimes Y), F(X)\otimes F(Y)\right) \;\le\; \delta \tag{4.25}$$

for all $X, Y \in \mathcal{D}$, *where* $\delta > 0$ *and* h *is a Hausdorff distance fulfilling the property*

$$h(A\otimes B, A\otimes C) \;=\; h(A,\mathbf{0})h(B,C) \tag{4.26}$$

for all $A, B, C \in \mathcal{C}_{cb}(\mathcal{B})$. *Then either*

$$h\left(F(X),\mathbf{0}\right) \;\le\; \frac{1+\sqrt{1+4\delta}}{2}$$

for all $X \in \mathcal{D}$, *or* F *is a multiplicative set-valued mapping.*

Proof. Letting $\varepsilon := \dfrac{1+\sqrt{1+4\delta}}{2}$, we see that

$$\begin{aligned}
\varepsilon^2-\varepsilon &= \left(\frac{1+\sqrt{1+4\delta}}{2}\right)^2 - \left(\frac{1+\sqrt{1+4\delta}}{2}\right)\\
&= \frac{1}{4}\left(1+\sqrt{1+4\delta}\right)\left(\sqrt{1+4\delta}-1\right)\\
&= \frac{1}{4}\left(1+4\delta-1\right)\\
&= \delta,
\end{aligned} \tag{4.27}$$

and, obviously, that $\varepsilon > 1$. Assume that there exists $A \in \mathcal{D}$ such that $h(F(A),\mathbf{0}) > \varepsilon$. Then $h(F(A),\mathbf{0}) = \varepsilon + k$ for a suitable positive real number k. By the property (2.13), we get

$$h\left(F(A)\otimes F(A),\mathbf{0}\right) \le h\left(F(A)\otimes F(A), F(A^2)\right) + h\left(F(A^2),\mathbf{0}\right), \tag{4.28}$$

where A^n denotes $\overbrace{A\otimes A\otimes\cdots\otimes A}^{\text{n-terms}}$, and then using (4.25), (4.26), (4.28), and triangle inequality, we have

$$\begin{aligned}
h\left(F(A^2),\mathbf{0}\right) &\ge h\left(F(A)\otimes F(A),\mathbf{0}\right) - h\left(F(A)\otimes F(A), F(A^2)\right)\\
&\ge h\left(F(A)\otimes F(A), F(A)\otimes \mathbf{0}\right) - \delta\\
&= h\left(F(A),\mathbf{0}\right) h\left(F(A),\mathbf{0}\right) - \delta\\
&= h\left(F(A),\mathbf{0}\right)^2 - \delta\\
&= (\varepsilon+k)^2 - \delta\\
&= (\varepsilon^2-\delta) + 2\varepsilon k + k^2\\
&\overset{(4.27)}{=} \varepsilon + 2\varepsilon k + k^2\\
&> \varepsilon + 2k.
\end{aligned}$$

We now proceed by induction on $n \in \mathbb{N}$ to show that

$$h\left(F\left(A^{2^n}\right), \mathbf{0}\right) > \varepsilon + (n+1)k \tag{4.29}$$

for all $n \in \mathbb{N}$. The asserted inequality being clearly true when $n = 1$. Assume, as the induction hypothesis, that $n > 1$ and that the result holds for every integers less than n. Then (4.25), (4.26), (4.28), and induction hypothesis ensure that

$$\begin{aligned}
h\left(F\left(A^{2^n}\right), \mathbf{0}\right) &= h\left(F\left(A^{2^{n-1}\cdot 2}\right), \mathbf{0}\right) \\
&\geq h\left(F\left(A^{2^{n-1}}\right) \otimes F\left(A^{2^{n-1}}\right), \mathbf{0}\right) - h\left(F\left(A^{2^{n-1}}\right) \otimes F\left(A^{2^{n-1}}\right), F\left(A^{2^{n-1}\cdot 2}\right)\right) \\
&\geq h\left(F\left(A^{2^{n-1}}\right), \mathbf{0}\right) h\left(F\left(A^{2^{n-1}}\right), \mathbf{0}\right) - \delta \\
&= h\left(F\left(A^{2^{n-1}}\right), \mathbf{0}\right)^2 - \delta \\
&> (\varepsilon + nk)^2 - \varepsilon^2 + \varepsilon \\
&= (nk)^2 + 2n\varepsilon k + \varepsilon \\
&> \varepsilon + 2nk \\
&\geq \varepsilon + (n+1)k,
\end{aligned}$$

completing the induction step, and the argument. Note that the inequality (4.25) gives

$$h\left(F(X \otimes Y) \otimes F(Z), F(X \otimes Y \otimes Z)\right) \leq \delta \text{ and } h\left(F(X \otimes Y \otimes Z), F(X) \otimes F(Y \otimes Z)\right) \leq \delta \tag{4.30}$$

for all $X, Y, Z \in \mathcal{D}$. Using Proposition 2.1 (iii), (2.13), and (4.30), we see that

$$\begin{aligned}
&h\left(F(X \otimes Y) \otimes F(Z), F(X) \otimes F(Y \otimes Z)\right) \\
=&h\left(F(X \otimes Y \otimes Z) \oplus F(X \otimes Y) \otimes F(Z), F(X) \otimes F(Y \otimes Z) \oplus F(X \otimes Y \otimes Z)\right) \\
\leq&h\left(F(X \otimes Y \otimes Z), F(X) \otimes F(Y \otimes Z)\right) + h\left(F(X \otimes Y) \otimes F(Z)), F(X \otimes Y \otimes Z)\right) \\
\leq&\delta + \delta \\
=&2\delta
\end{aligned}$$

for all $X, Y, Z \in \mathcal{D}$. By using (2.13), (4.25), and (4.26), we have

$$\begin{aligned}
&h\left(F(X \otimes Y) \otimes F(Z), F(X) \otimes F(Y) \otimes F(Z)\right) \\
\leq&h\left(F(X \otimes Y) \otimes F(Z), F(X) \otimes F(Y \otimes Z)\right) \\
+&h\left(F(X) \otimes F(Y \otimes Z), F(X) \otimes F(Y) \otimes F(Z)\right) \\
\leq&2\delta + h\left(F(X), \mathbf{0}\right) h\left(F(Y \otimes Z), F(Y) \otimes F(Z)\right) \\
\leq&2\delta + h\left(F(X), \mathbf{0}\right) \delta
\end{aligned}$$

for all $X, Y, Z \in \mathcal{D}$. This is equivalent to

$$h\left(F(X \otimes Y), F(X) \otimes F(Y)\right) h(F(Z), \mathbf{0}) \leq 2\delta + h\left(F(X), \mathbf{0}\right) \delta \tag{4.31}$$

for all $X, Y, Z \in \mathcal{D}$. Note that the inequality (4.29) ensures that $h(f(A^{2n}), \mathbf{0}) \neq 0$ for all $n \in \mathbb{N}$. By putting $Z = A^{2^n}$ with $n \in \mathbb{N}$ in (4.31), this now forces that

$$h\left(F(X \otimes Y), F(X) \otimes F(Y)\right) \leq \frac{2\delta + h\left(F(X), \mathbf{0}\right) \delta}{h(F(A^{2n}), \mathbf{0})} \to 0 \text{ as } n \to \infty,$$

showing that F is a multiplictive set-valued mapping. The proof is now completed. □

5. Conclusion

It was shown that the solutions of the set-valued functional equations (1.9) and (3.22) are of the forms of exponential and logarithmic set-valued mappings, which are presented in Section 3. These two set-valued mappings must be inverses of each other. Moreover, we have also proved the stability result for the multiplicative set-valued equation (1.9), which appears in Section 4, on Banach algebras. This result requires a crucial property of the Hausdorff metric.

Recently, many mathematicians have been interested in the stability problems of functional equations; in particular, of set-valued functional equations, in many variant spaces. They investigate the stability of set-valued functional equations related to additive operations; in particular, the additive set-valued mappings, but not the multiplicative ones. The exponential and logarithmic set-valued mappings defined in this paper will help solve set-valued functional equations involving multiplicative mappings. Furthermore, for stability problems, a special property of the Hausdorff distance will also help prove the stability of set-valued functional equations that are associated with the multiplicative operation on the variables.

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Author contributions

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Conflict of Interest

The authors declare no conflicts of interest.

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