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Research Article

On the boundedness of Riesz potential operators: insights from net spaces

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ABSTRACT. This paper investigates the boundedness of Riesz potential operators on net spaces that are structured around special nets. We use the construction of net spaces and their intrinsic properties to establish conditions under which the considered operators are bounded. The methodology developed here provides a framework for establishing Hardy–Littlewood–Sobolev inequalities on net spaces, aiming at a deeper understanding of potential theory in non-standard settings.

Keywords: Riesz potential, Hardy–Littlewood–Sobolev inequality, net spaces, local nets.

2020 Mathematics Subject Classification: 46E30, 47B34.

1. INTRODUCTION

The Riesz potential operator has been actively studied due to its interlinked applications in harmonic analysis, Sobolev spaces, and the study of partial differential equations. Therefore, it has received significant attention from researchers over the last few decades. For a complete study of the Riesz potential operators in harmonic analysis and its applications, we refer to [23], [24] and the references therein. Formally, let $f \in L^1_{loc}(\mathbb{R}^n)$ and $0 < \gamma < n$, then the Riesz potential operator is defined as:

$$(I_\gamma^{(n)} f)(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}} dy. \tag{1.1}$$

The classical Hardy–Littlewood–Sobolev inequality states that if $1 < p < q < \infty$ and $\gamma = n\left(\frac{1}{p} - \frac{1}{q}\right)$, then $I_\gamma^{(n)}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. This result first appeared in the work of Hardy and Littlewood in 1928 for the one-dimensional case in [9] and later for the general case by Sobolev in [22]. The boundedness of the Riesz potential operator in Lorentz spaces was characterized, e.g., in [15] and [17]. In case of Morrey spaces, these types of results can be found e.g. see [1] and [20]. In case of non-standard function spaces, for variable Lebesgue spaces, Diening showed the boundedness of Riesz potential operators in the papers [7] and [8]. Such results for bounded domains were obtained by Samko in [21]. In a more general setting, for example, in grand Lebesgue spaces and non-standard function spaces, we refer to [12], [13] and [14].

Let M be a fixed family of finite measurable subsets of $\mathbb{R}^n$, which we refer to as a net of $\mathbb{R}^n$. For a function f which is integrable on each element of M, the average function $\bar{f}(t, M)$ is

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defined as:

$$\bar{f}(t,M)=\sup_{\substack{J\in M\\ |J|>t}}\frac{1}{|J|}\left|\int_J f(x)dx\right|,$$

and if $\sup\{|J|: J\in M\}=\alpha<\infty$ and $t>\alpha$, then we set $\bar{f}(t,M)=0$.

The net spaces were introduced by E. Nursultanov in [18]. For $0<p,q\leq\infty$, net space denoted by $N_{p,q}(M)$ is a collection of functions for which

$$\|f\|_{N_{p,q}(M)}=\begin{cases}\left(\int\limits_0^\infty (t^{1/p}\bar{f}(t,M))^q\frac{dt}{t}\right)^{1/q}, & q<\infty,\\ \sup\limits_{J\in M}\frac{1}{|J|^{1/p'}}\left|\int\limits_J f(x)dx\right|, & q=\infty,\end{cases}$$

is finite. The net spaces are quasi-normed spaces for $0<p,q\leq\infty$. In [18], it was shown that if M is a collection of all compact subsets of $\mathbb{R}^n$, then the net spaces coincide with the classical Lorentz spaces. Therefore, net spaces can be regarded as a generalization of Lorentz spaces. The following theorem about the inclusions of net spaces with respect to nets and exponents was proved in [18].

Theorem 1.1. *(a) If $M_1\subset M_2$, then $N_{p,q}(M_2)\hookrightarrow N_{p,q}(M_1)$.*

(b) For $1\leq q<q_1<\infty$, we have $N_{p,q}(M)\hookrightarrow N_{p,q_1}(M)$.

In [18], the following interpolation theorem was also proved.

Theorem 1.2. *Let $1\leq p_0<p_1<\infty$ and $1\leq q,q_0,q_1\leq\infty$ and $0<\theta<1$ and M be an arbitrary net of $\mathbb{R}^n$. Then*

$$(N_{p_0,q_0}(M),N_{p_1,q_1}(M))_{\theta,q}\hookrightarrow N_{p,q}(M),$$

where $\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{1}{p_1}$.

The net spaces in the framework of the general measures were defined in [19]. The boundedness criterion for integral operators in these general net spaces was also studied in this article. For some other advancements on net spaces, we refer to [3], [4], [10], and [16].

The aim of this paper is to develop an analogue of the Hardy–Littlewood–Sobolev theorem on net spaces that are structured around special nets. We believe that these techniques can be useful for further development in this area. This paper has been divided into three sections, in Section 1 a brief introduction about the Riesz potential operator and net spaces is provided. In Section 2, our main results are presented, which is subdivided into several subsections in which proofs of the boundedness of Riesz potential operators on net spaces generated by local nets of intervals are given. We also prove the boundedness in higher dimensions for nets consisting of quasi-balls in $\mathbb{R}^n$. In the last section, the boundedness of the Riesz potential operators on Cesàro–type net spaces is established.

Throughout this paper, constants, often different within the same series of inequalities, will be denoted by c or C; by the symbol p' we denote the function $\frac{p}{p-1}$, $1<p<\infty$; the relation $a\asymp b$ indicates that there exist positive constants c_1 and c_2 such that $c_1a\leq b\leq c_2a$. Finally, the relation $a\lesssim b$ indicates that there exists positive constant C such that $a\leq Cb$.

2. Main Results

We begin this section with the definition of local nets. A family $G=\{G_t\}_{t>0}$ of subsets of $\mathbb{R}^n$ is said to be a local net if the following conditions are met:

(1) $G_t \subset G_s$ for $s > t$,
(2) $|G_t| = t$ for $t > 0$.

An example of local nets is the set of balls $\{Q_t(x)\}_{t>0}$ centered at x. The following interpolation theorem for local nets was proved in [11].

Theorem 2.3. *Let $0 < p_0 < p_1 < \infty$, $q_0, q_1, q \in (0, \infty]$ and $\theta \in (0, 1)$. If $G = \{G_t\}_{t>0}$ is a local net, then*

$$(N_{p_0,q_0}(G), N_{p_1,q_1}(G))_{\theta,q} = N_{p,q}(G),$$

where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

2.1. **Local nets of intervals.** In this subsection, we show analogous to the Hardy–Littlewood–Sobolev inequality on net spaces defined on $\mathbb{R}$ using closed intervals.

Let $x_0 \in \mathbb{R}$ and $\mathbb{Q}(x_0) := \{[x_0 - \delta, x_0 + \delta]\}_{\delta>0}$. Then $\mathbb{Q}(x_0)$ is a local net in $\mathbb{R}$. Our main result of this subsection is the following.

Theorem 2.4. *Let $1 < p < q < \infty$, $1 \le \tau \le \infty$ and $\gamma = \frac{1}{p} - \frac{1}{q}$. Then, the Riesz potential operator $I_\gamma^{(1)}$ is bounded from $N_{p,\tau}(\mathbb{Q}(x_0))$ to $N_{q,\tau}(\mathbb{Q}(x_0))$.*

Proof. We will prove quasi-weak type inequality first i.e.

$$\|I_\gamma^{(1)} f\|_{N_{q,\infty}(\mathbb{Q}(x_0))} \le c\|f\|_{N_{p,1}(\mathbb{Q}(x_0))}. \tag{2.2}$$

Let $\delta > 0$, $x_0 \in \mathbb{R}$ and $f \in N_{p,1}(\mathbb{Q}(x_0))$. Let $\Delta_\delta(x_0) = [x_0 - \delta, x_0 + \delta]$ and $\Delta_\delta = \Delta_\delta(0)$. Then the following estimates hold.

$$\left| \int\limits_{\Delta_\delta(x_0)} (I_\gamma^{(1)} f)(s) ds \right| = \left| \int\limits_{-\infty}^{\infty} f(t) \int\limits_{\Delta_\delta(x_0)} \frac{ds}{|s-t|^{1-\gamma}} dt \right| = \left| \int\limits_{-\infty}^{\infty} f(x_0 + t) \int\limits_{\Delta_\delta} \frac{ds}{|t-s|^{1-\gamma}} dt \right|.$$

Letting $\phi_\delta(t) = \int_{\Delta_\delta(t)} \frac{ds}{|s|^{1-\gamma}}$ and replacing $s \to t - s$, we have

$$\left| \int\limits_{\Delta_\delta(x_0)} (I_\gamma^{(1)} f)(s) ds \right| = \left| \int\limits_{-\infty}^{\infty} f(x_0 + t)\phi_\delta(t) dt \right| \le \left| \int\limits_{0}^{\infty} f(x_0 + t)\phi_\delta(t) dt \right| + \left| \int\limits_{-\infty}^{0} f(x_0 + t)\phi_\delta(t) dt \right| = I_1 + I_2.$$

For I_1, using integration by parts, we have

$$I_1 = \phi_\delta(t)G(t)\Big|_0^\infty - \int\limits_0^\infty G(t)\phi_\delta'(t) dt,$$

where $G(t) := \int\limits_0^t f(x_0 + s) ds$. The function $\phi_\delta(t)$ can be estimated as

$$\phi_\delta(t) = \begin{cases} \frac{(t+\delta)^\gamma - (t-\delta)^\gamma}{\gamma}, & |t| \ge \delta, \\ \frac{(t+\delta)^\gamma + (\delta-t)^\gamma}{\gamma}, & 0 \le |t| \le \delta. \end{cases} \tag{2.3}$$

Similarly for $\phi_{\delta}^{'}(t)$, we have

$$\phi_{\delta}^{'}(t)=\begin{cases}(t+\delta)^{\gamma-1}-(t-\delta)^{\gamma-1}, & |t|\geq\delta,\\ (t+\delta)^{\gamma-1}+(\delta-t)^{\gamma-1}, & 0\leq|t|\leq\delta.\end{cases}\tag{2.4}$$

For $|t|\geq\delta$, we have

$$\begin{aligned}0\leq|G(t)|\,\phi_\delta(t)&\leq ct^{1/p'}\frac{(t+\delta)^\gamma-(t-\delta)^\gamma}{\gamma}\|f\|_{N_{p,\infty}(\mathbb{Q}(x_0))}\\&\leq ct^{1/p'}\frac{(t+\delta)^\gamma-(t-\delta)^\gamma}{\gamma}\|f\|_{N_{p,1}(\mathbb{Q}(x_0))}.\end{aligned}$$

Therefore

$$\lim_{t\to\infty}t^{1/p'}((t+\delta)^\gamma)-(t-\delta)^\gamma)\sim\lim_{t\to\infty}t^{\frac{1}{p'}+\gamma-1}=\lim_{t\to\infty}t^{-1/q}=0.$$

Also,

$$\lim_{t\to0}|G(t)|\,\phi_\delta(t)=0.$$

On the other hand, for $0\leq|t|<\delta$,

$$|G(t)|\,\phi_\delta(t)=\left|\int\limits_0^t f(x_0+s)ds\right|\frac{(t+\delta)^\gamma+(\delta-t)^\gamma}{\gamma}\to0\quad\text{when}\quad t\to0.$$

Hence, I_1 is simplified as

$$I_1=\left|\int_0^\infty\int_0^t f(x_0+s)ds\right|\phi_{\delta}^{'}(t)dt.$$

Using the bounds for $\phi_{\delta}^{'}(t)$, we have

$$\begin{aligned}I_1&=\int_0^{2\delta}\left|\int_0^t f(x_0+s)ds\right|\phi_{\delta}^{'}(t)dt+\int_{2\delta}^\infty\left|\int_0^t f(x_0+s)ds\right|\phi_{\delta}^{'}(t)\\&\leq\int\limits_0^{2\delta}t^{\frac{1}{p'}}|\phi_{\delta}^{'}(t)|dt\|f\|_{N_{p,\infty}(\mathbb{Q}(x_0))}+\sup_{2\delta\leq t}t^{1+\frac{1}{p'}}|\phi_{\delta}^{'}(t)\|f\|_{N_{p,1}(\mathbb{Q}(x_0))}.\end{aligned}$$

Now, we compute

$$\begin{aligned}\int_0^{2\delta}t^{\frac{1}{p'}}|\phi_{\delta}^{'}(t)|\,dt&\leq\int_0^{2\delta}t^{\frac{1}{p'}}((t+\delta)^{\gamma-1}+|t-\delta|^{\gamma-1})\,dt\\&=\delta^{\frac{1}{q'}}\int_0^2 t^{\frac{1}{p'}}((t+1)^{\gamma-1}+|t-1|^{\gamma-1})\,dt=c_1\delta^{\frac{1}{q'}},\\ \sup_{2\delta\leq t}t^{1+\frac{1}{p'}}|\phi_{\delta}^{'}(t)|&=\sup_{2\delta\leq t}t^{1+\frac{1}{p'}}|((t+\delta)^{\gamma-1}-(t-\delta)^{\gamma-1}))|\\&=\delta^{\frac{1}{q'}}\sup_{2\leq t}t^{1+\frac{1}{p'}}|((t+1)^{\gamma-1}-(t-1)^{\gamma-1})|=c_2\delta^{\frac{1}{q'}}.\end{aligned}$$

Similarly, $I_2\leq c_p\delta^{1/q'}\|f\|_{N_{p,1}(\mathbb{Q}(x_0))}$. Hence,

$$\frac{1}{|\Delta_\delta(x_0)|^{1/q'}}\left|\int\limits_{\Delta_\delta(x_0)}(I_\gamma^{(1)}f)(s)ds\right|\leq c\|f\|_{N_{p,1}(\mathbb{Q}(x_0))}.$$

Since $\delta > 0$ is arbitrary, it implies the following desired inequality

$$\|I_\gamma^{(1)} f\|_{N_{q,\infty}(\mathbb{Q}(x_0))} \le c\|f\|_{N_{p,1}(\mathbb{Q}(x_0))}.$$

Thus, by using Theorem 2.3 together with the interpolation theorem [5, Theorem 1.12], we complete the proof, that is,

$$\|I_\gamma^{(1)} f\|_{N_{q,\tau}(\mathbb{Q}(x_0))} \le c\|f\|_{N_{p,\tau}(\mathbb{Q}(x_0))}.$$

□

2.2. **Net of balls with respect to metric** $\|\cdot\|_s$**.** In this subsection, we prove the Hardy–Littlewood–Sobolev inequality for the net M_s consisting of balls defined with respect to the metric

$$\|x\|_s = \left(\sum_{i=1}^{n} |x_i|^s\right)^{1/s}, \tag{2.5}$$

where $0 < s \le \infty$. Note that for $s = \infty$ it is the net of cubes, while for $s = 2$ it is the net of usual (Euclidean) balls in $\mathbb{R}^n$.

The following lemma is a variant of Hardy's inequality that will be used in the proof of the main results. This inequality can be obtained by choosing the appropriate function and exponents in [6, Theorem 6.12]. But here for the sake of completeness, we present a simple proof of the inequality.

Lemma 2.1. *Let f be non–negative measurable function on $[0,\infty)$. Let α, τ be real numbers such that $\alpha > 0$ and $\tau \ge 1$. Then the inequality*

$$\left(\int_0^\infty \left(r^\alpha \int_r^\infty f(t)\frac{dt}{t}\right)^\tau \frac{dr}{r}\right)^{\frac{1}{\tau}} \le \frac{1}{\alpha}\left(\int_0^\infty (r^\alpha f(r))^\tau \frac{dr}{r}\right)^{\frac{1}{\tau}}$$

holds.

Proof. By making the change of variable $t \to \frac{r}{s}$ and applying integral Minkowski's inequality and taking into account that $\alpha + \frac{1}{\tau'} > 0$, we have

$$\begin{aligned}
\left(\int_0^\infty \left(r^\alpha \int_r^\infty f(t)\frac{dt}{t}\right)^\tau \frac{dr}{r}\right)^{\frac{1}{\tau}} &= \left(\int_0^\infty \left(r^\alpha \int_0^1 f\left(\frac{r}{s}\right)\frac{ds}{s}\right)^\tau \frac{dr}{r}\right)^{\frac{1}{\tau}} \\
&\le \int_0^1 \left(\int_0^\infty \left(r^\alpha f\left(\frac{r}{s}\right)\frac{1}{s}\right)^\tau \frac{dr}{r}\right)^{\frac{1}{\tau}} ds \\
&= \int_0^1 s^{\alpha-\frac{1}{\tau}} ds \left(\int_0^\infty r^{\alpha\tau-1}(f(r))^\tau\right)^{\frac{1}{\tau}} \\
&\le \frac{1}{\alpha}\left(\int_0^\infty (r^\alpha f(r))^\tau \frac{dr}{r}\right)^{\frac{1}{\tau}}.
\end{aligned}$$

□

Our main result of this subsection is the following.

Theorem 2.5. *Let* $1 < p < q < \infty$, $1 \le \tau \le \infty$ *and* $\gamma = \frac{n}{p} - \frac{n}{q}$. *Then the Riesz potential operator defined by*

$$(I_{\gamma,s}f)(x) = \int\limits_{\mathbb{R}^n} \frac{f(x-y)}{\|y\|_s^{n-\gamma}} dy, \tag{2.6}$$

is bounded from $N_{p,\tau}(M_s)$ *to* $N_{q,\tau}(M_s)$.

Proof. Case I: $\tau = \infty$. Let $f \in N_{p,\infty}(M_s),\ r > 0$. Let Q_r be arbitrary ball of radius r. If the ball is centered at $x_0 \in \mathbb{R}^n$, it will be denoted by $Q_r(x_0)$. Then by Fubini's theorem, we have

$$\begin{aligned}
\frac{1}{|Q_r|}\left|\int\limits_{Q_r}(I_{\gamma,s}f)(x)dx\right| &= \left|\int\limits_{\mathbb{R}^n}\frac{1}{\|y\|_s^{n-\gamma}}\frac{1}{|Q_r|}\int\limits_{Q_r} f(x-y)dxdy\right| \\
&= (n-\gamma)\left|\int\limits_{\mathbb{R}^n}\left(\int\limits_{\|y\|_s}^{\infty}\frac{dt}{t^{n-\gamma+1}}\right)\frac{1}{|Q_r|}\int\limits_{Q_r} f(x-y)dxdy\right| \\
&= (n-\gamma)\left|\int\limits_{0}^{\infty}\frac{1}{t^{n-\gamma+1}}\int\limits_{\{y:\|y\|_s\le t\}}\frac{1}{|Q_r|}\int\limits_{Q_r} f(x-y)dxdydt\right| \\
&\le (n-\gamma)\int\limits_{0}^{\infty}\frac{1}{t^{n-\gamma+1}}\left|\int\limits_{Q_t(0)}\frac{1}{|Q_r|}\int\limits_{Q_r} f(x-y)dxdy\right|dt \\
&\asymp \int\limits_{0}^{\infty}\frac{1}{t^{1-\gamma}}\phi(t,r)dt, \qquad (2.7)
\end{aligned}$$

where $\phi(t,r) = \frac{1}{|Q_t||Q_r|}\left|\int\limits_{Q_t(0)}\int\limits_{Q_r} f(x-y)dxdy\right|$. For $r > t$, we have

$$\phi(t,r) \le \frac{1}{|Q_t|}\int\limits_{Q_t(0)}\sup_{Q_r\in M_s}\frac{1}{|Q_r|}\left|\int\limits_{Q_r} f(x-y)dx\right|dy \le \bar{f}(r^n, M_s). \tag{2.8}$$

While for $r \le t$, we the following inequality holds.

$$\phi(t,r) \le \sup_{z\in\mathbb{R}^n}\frac{1}{|Q_t|}\left|\int\limits_{Q_t(z)} f(x)dx\right| \lesssim \bar{f}(t^n, M_s). \tag{2.9}$$

Hence, we obtain

$$\frac{1}{|Q_r|}\left|\int_{Q_r}(I_{\gamma,s}f)(x)dx\right| \le \int_0^r t^{\gamma-1}\bar{f}(r^n,M_s)dt + \int_r^\infty t^{\gamma-1}\bar{f}(t^n,M_s)dt$$
$$\le \left(\frac{1}{r^{\frac{n}{p}}}\int_0^r t^{\gamma-1}dt + \int_r^\infty t^{\gamma-\frac{n}{p}-1}dt\right)\|f\|_{N_{p,\infty}(M_s)}$$
$$\asymp r^{-\frac{n}{q}}\|f\|_{N_{p,\infty}(M_s)}.$$

Thus,

$$\|I_{\gamma,s}f\|_{N_{q,\infty}(M_s)} \lesssim \|f\|_{N_{p,\infty}(M_s)}.$$

Case II: $0<\tau<\infty$. We have

$$\|I_{\gamma,s}(f)\|_{N_{q,\tau}(M_s)} = \left(\int_0^\infty \left(\xi^{\frac{1}{q}}\sup_{\substack{|Q_r|\ge\xi\\ Q_r\in M_s}}\frac{1}{|Q_r|}\left|\int_{Q_r}(I_{\gamma,s}f)(x)dx\right|\right)^\tau \frac{d\xi}{\xi}\right)^{\frac{1}{\tau}}.$$

From (2.7), it follows that

$$\|I_{\gamma,s}f\|_{N_{q,\tau}(M_s)} \le \left(\int_0^\infty\left(\xi^{\frac{1}{q}}\sup_{r^n\ge\xi}\int_0^\infty t^{\gamma-1}\phi(t,r)\right)^\tau\frac{d\xi}{\xi}\right)^{\frac{1}{\tau}}$$

$$\lesssim \left(\int_0^\infty\left(\xi^{\frac{1}{q}}\sup_{r^n\ge\xi}\left(\int_0^{\xi^{\frac{1}{n}}} t^{\gamma-1}\phi(t,r)dt + \int_{\xi^{\frac{1}{n}}}^\infty t^{\gamma-1}\phi(t,r)dt\right)^\tau\frac{d\xi}{\xi}\right)\right)^{\frac{1}{\tau}} \le I_1+I_2.$$

Using the estimates as in (2.8), we have

$$I_1 \le \left(\int_0^\infty\left(\xi^{\frac{1}{q}}\sup_{r^n\ge\xi}\int_0^{\xi^{\frac{1}{n}}} t^{\gamma-1}\bar{f}(r^n,M_s)dt\right)^\tau\frac{d\xi}{\xi}\right)^{\frac{1}{\tau}}$$
$$\le \frac{1}{\gamma}\left(\int_0^\infty(\xi^{\frac{1}{p}}\bar{f}(\xi,M_s))^\tau\frac{d\xi}{\xi}\right)^{\frac{1}{\tau}} \asymp \|f\|_{N_{p,\tau}(M_s)}.$$

Now by using (2.9) and Lemma 2.1, we obtain

$$I_2 \le \left(\int_0^\infty\left(\xi^{\frac{1}{q}}\sup_{r^n\ge\xi}\int_{\xi^{\frac{1}{n}}}^\infty t^\gamma\bar{f}(t^n,M_s)\frac{dt}{t}\right)^\tau\frac{d\xi}{\xi}\right)^{\frac{1}{\tau}}$$
$$\lesssim \left(\int_0^\infty\left(\xi^{\frac{1}{q}}\int_\xi^\infty t^{\frac{\gamma}{n}}\bar{f}(t,M_s)\frac{dt}{t}\right)^\tau\frac{d\xi}{\xi}\right)^{\frac{1}{\tau}}$$

$$\lesssim \left(\int_0^\infty \left(\xi^{\frac{1}{p}} \bar{f}(\xi, M_s) \right)^\tau \frac{d\xi}{\xi} \right)^{\frac{1}{\tau}} = \|f\|_{N_{p,\tau}(M_s)}.$$

□

3. Boundedness on Cesàro–type net spaces

In this section, we prove the boundedness of Riesz potential operators on Cesàro–type net spaces. Let $G = \{G_t\}_{t>0}$ be local net in $\mathbb{R}^n$. Then the collection

$$M = \{G_t + y; t > 0, y \in \mathbb{R}^n\}$$

is called the net generated by the local net G. Now we define a variant of net spaces, denoted by $\mathcal{N}_{p,q}(M)$, $0 < p, q \leq \infty$, as follows,

$$\|f\|_{\mathcal{N}_{p,q}(M)} = \begin{cases} \left(\int_0^\infty (t^{1/p} \tilde{f}(t, M))^q \frac{dt}{t} \right)^{1/q}, & q < \infty, \\ \sup_{t>0} t^{\frac{1}{p}} \tilde{f}(t, M), & q = \infty, \end{cases}$$

where

$$\tilde{f}(t, M) = \sup_{y \in \mathbb{R}^n} \frac{1}{|G_t|} \left| \int_{G_t + y} f(x) dx \right|.$$

We call $\mathcal{N}_{p,q}(M)$ a Cesàro-type net space. These types of spaces are particularly important due to their structural similarity to classical Cesàro spaces, offering generalizations and flexible settings for studying integral operators. We refer to [2] and the references cited therein for a comprehensive review of Cesàro–type function spaces.

Let M_Q denote the net generated by the local net of balls in $\mathbb{R}^n$ with respect to the norm $\|\cdot\|_s$. The following result gives the boundedness of the Riesz potential operator on Cesàro–type net space.

Theorem 3.6. *Let $1 < p < q < \infty$, $\gamma = \frac{n}{p} - \frac{n}{q}$ and $1 \leq \tau \leq \infty$. Then the Riesz potential is defined by*

$$(I_{\gamma,s} f)(x) = \int_{\mathbb{R}^n} \frac{f(x-y)}{\|y\|_s^{n-\gamma}} dy, \tag{3.10}$$

is bounded from $\mathcal{N}_{p,\tau}(M_Q)$ to $\mathcal{N}_{q,\tau}(M_Q)$.

Proof. Let $f \in \mathcal{N}_{p,\tau}(M_Q)$. Let Q_r be arbitrary ball with radius $r(|Q_r|) = r$. As in the proof of Theorem 2.5, we have

$$\frac{1}{|Q_r|} \left| \int_{Q_r + y} (I_{\gamma,s} f)(x) dx \right| \leq c \int_0^\infty \frac{\phi(t, r)}{t^{1-\frac{\gamma}{n}}} dt \leq c \int_0^\infty \frac{\tilde{f}(\max(t, r), M_Q)}{t^{1-\frac{\gamma}{n}}} dt.$$

Therefore, we get

$$\|I_{\gamma,s} f\|_{\mathcal{N}_{q,\tau}(M_Q)} = \left(\int_0^\infty \left(r^{\frac{1}{q}} \sup_{y \in \mathbb{R}^n} \frac{1}{|Q_r|} \left| \int_{Q_r + y} (I_{\gamma,s} f)(x) dx \right| \right)^\tau \frac{dr}{r} \right)^{\frac{1}{\tau}}$$

$$
\begin{aligned}
&\leq \left(\int_0^\infty \left(r^{\frac{1}{q}} \int_0^\infty \frac{\tilde{f}(\max(t,r),M_Q)}{t^{1-\frac{\gamma}{n}}}dt\right)^\tau \frac{dr}{r}\right)^{\frac{1}{\tau}} \\
&\asymp \left(\int_0^\infty \left(r^{\frac{1}{q}} \left(\int_0^r \frac{\tilde{f}(r,M_Q)}{t^{1-\frac{\gamma}{n}}}dt + \int_r^\infty \frac{\tilde{f}(t,M_Q)}{t^{1-\frac{\gamma}{n}}}dt\right)\right)^\tau \frac{dr}{r}\right)^{\frac{1}{\tau}} \\
&\asymp \left(\int_0^\infty \left(r^{\frac{1}{q}} \left(r^{\frac{1}{p}-\frac{1}{q}} \tilde{f}(r,M_Q) + \int_r^\infty \frac{\tilde{f}(t,M_Q)}{t^{1-\frac{\gamma}{n}}}dt\right)\right)^\tau \frac{dr}{r}\right)^{\frac{1}{\tau}}.
\end{aligned}
$$

Furthermore, Minkowski's inequality implies

$$
\begin{aligned}
\|I_{\gamma,s}f\|_{\mathcal{N}_{q,\tau}(M_Q)} &\leq c\left(\int_0^\infty \left(r^{\frac{1}{p}}\tilde{f}(r,M_Q)\right)^\tau \frac{dr}{r}\right)^{\frac{1}{\tau}} + \left(\int_0^\infty \left(r^{\frac{1}{q}} \int_r^\infty \frac{\tilde{f}(t,M_Q)}{t^{1-\frac{\gamma}{n}}}dt\right)^\tau \frac{dr}{r}\right)^{\frac{1}{\tau}} \\
&\leq \|f\|_{\mathcal{N}_{p,\tau}(M_Q)} + A.
\end{aligned}
$$

Now the estimate for A follows from Lemma 2.1, that is,

$$
\begin{aligned}
A &\leq \left(\int_0^\infty \left(r^{\frac{1}{q}} \int_r^\infty t^{\frac{\gamma}{n}} \tilde{f}(t,M_Q)\frac{dt}{t}\right)^\tau \frac{dr}{r}\right)^{\frac{1}{\tau}} \\
&\leq c\left(\int_0^\infty (r^{\frac{1}{q}+\frac{\gamma}{n}}\tilde{f}(r,M_Q))^\tau \frac{dr}{r}\right)^{\frac{1}{\tau}} \\
&= \left(\int_0^\infty (r^{\frac{1}{p}}\tilde{f}(r,M_Q))^\tau \frac{dr}{r}\right)^{\frac{1}{\tau}} = \|f\|_{\mathcal{N}_{p,\tau}(M_Q)}.
\end{aligned}
$$

Hence, we arrive at

$$
\|I_{\gamma,s}f\|_{\mathcal{N}_{q,\tau}(M_Q)} \leq c\|f\|_{\mathcal{N}_{p,\tau}(M_Q)}.
$$

□

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