GALERKIN METHOD FOR NUMERICAL SOLUTION OF TWO DIMENSIONAL HYPERBOLIC BOUNDARY VALUE PROBLEM WITH DIRICHLET CONDITIONS

Seda İĞRET ARAZ[1*], Hülya DURUR[2]

[1*]Siirt University, Faculty of Mathematic Education, 56100, Siirt, TURKEY

[2]Ardahan University, Faculty of Engineering, Computer Engineering, 75000, Ardahan, TURKEY

ABSTRACT

In this paper, our aim has been to deal with numerical solution of two dimensional hyperbolic boundary value problem. By applying Galerkin method for solution of this problem, numerical results are obtained and these results are compared with analytical solutions.

Keywords: Hyperbolic Boundary Value Problem, Numerical Approximation, Galerkin Method.

DIRICHLET KOŞULUNA SAHİP İKİ BOYUTLU HİPERBOLİK SINIR DEĞER PROBLEMİNİN NÜMERİK ÇÖZÜMÜ İÇİN GALERKİN METODU

ÖZET

Bu çalışmada, amacımız iki boyutlu hiperbolik sınır değer probleminin nümerik çözümü ile ilgilenmektir. Bu problemin çözümü için Galerkin metodu uygulanarak nümerik sonuçlar elde edilir ve bu sonuçlar analitik çözümler ile karşılaştırılır.

Anahtar Kelimeler: Hiperbolik Sınır Değer Problemi, Nümerik Yaklaşım, Galerkin Metodu.

**corresponding author: sedaaraz@siirt.edu.tr*

1. Introduction and Statement of the Problem

Hyperbolic boundary value problems have been appeared as mathematical modelling of physical phenomena like small vibration of a string, in the fields of science and engineering. In recent years, there has been much attention to studies related with these problems [1, 2, 3].

In this paper, on the domain $\Omega_t=\phi\times(0,T)$ where $\phi=(0,l_1)\times(0,l_2)$, we consider

$$u_{tt}-\Delta u+q(\mathrm{x})u=F(\mathrm{x},t),\ \ \mathrm{x}\in\phi,t\in(0,T) \tag{1}$$

$$u(\mathrm{x},0)=\varphi(\mathrm{x}),u_t(\mathrm{x},0)=\psi(\mathrm{x}),\ \mathrm{x}\in\phi \tag{2}$$

$$u(\mathrm{x},t)\big|_{\Gamma}=0,\ \ t\in(0,T) \tag{3}$$

hyperbolic initial-boundary value problem and this problem states wave equation in physical meaning. $\Delta=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ is the Laplace operator and Γ is the boundary of ϕ. Also the initial displacement $\varphi(\mathrm{x})$ and the initial velocity $\psi(\mathrm{x})$ are given.

Let us refer from some notations used in this paper.

The space $L_2(\Omega_t)$ is the set of all square integrable functions on the domain ϕ. The Hilbert space called $H^1(\Omega_t)$ is the set of functions taken from $L_2(\Omega_t)$ to the spatial variables and time variable of first order. The space $\overset{\circ}{H}{}^1(\Omega_t)$ which is subspace of the space $H^1(\Omega_t)$ consists of the functions vanishing at boundary of the domain ϕ.

The paper is organized as follows: In section 2, we obtain the generalized solution for hyperbolic problem. In section 3, we refer from the considered numerical method. In section 4, we give three examples and calculate their error norm on the space L_2.

2. Solvability of the Problem

Let us write a priori estimate utilizing energy method to demonstrate the existence, uniqueness and continuity of the solution according to input data[4].

Definition 1. The generalized solution of the problem (1)-(3) is the function $u\in\overset{\circ}{H}{}^1(\Omega_t)$ and it satisfies the following integral equality;

$$\int_{\Omega_t}\left(-u_t\mu_t+\nabla u.\nabla\mu+q(\mathrm{x})u\mu\right)d\mathrm{x}dt=\int_{\phi}\psi(\mathrm{x})\mu(\mathrm{x},0)d\mathrm{x}+\int_{\Omega_t}F(\mathrm{x},t)\mu(\mathrm{x},t)d\mathrm{x}dt \quad (4)$$

for all $\mu(\mathrm{x},t)\in \overset{o}{H}{}^1(\Omega_t),\ \mu(\mathrm{x},T)=0$ [5].

Theorem 1. Generalized solution of the problem (1)-(3) in the sense of (4) is exist, unique under conditions $q(\mathrm{x})\in L_\infty(\phi), F(\mathrm{x},t)\in L_2(\Omega_t), \varphi(\mathrm{x})\in L_2(\phi), \psi(\mathrm{x})\in H^1(\phi)$ and satisfies the following inequality;

$$\|u\|^2_{H^1(\Omega_t)}\le c_0\left(\|F\|^2_{L_2(\Omega_t)}+\|\varphi\|^2_{L_2(\phi)}+\|\psi\|^2_{H^1(\phi)}\right). \quad (5)$$

Proof: Multiplying the equation (3) with the function u_t and integrating on the domain ϕ and taking into account the condition (3), we write the following inequality;

$$\frac{1}{2}\int_{\phi}\left[\left(u_t(\mathrm{x},t)\right)^2+\left(\nabla u(\mathrm{x},t)\right)^2+q(\mathrm{x})\left(u(\mathrm{x},t)\right)^2\right]d\mathrm{x}=\frac{1}{2}\int_{\phi}\left[\left(u_t(\mathrm{x},0)\right)^2+\left(\nabla u(\mathrm{x},0)\right)^2+q(\mathrm{x})\left(u(\mathrm{x},0)\right)^2\right]d\mathrm{x}+\int_{\Omega}F(\mathrm{x},t)u_t d\mathrm{x}dt \quad (6)$$

Let us define the integral $J(t)$ called energy norm as follows;

$$J(t):=\|u\|_J:=\left\{\frac{1}{2}\int_{\phi}\left[\left(u_t(\mathrm{x},t)\right)^2+\left(\nabla u(\mathrm{x},t)\right)^2+q(\mathrm{x})\left(u(\mathrm{x},t)\right)^2\right]d\mathrm{x}\right\}^{1/2}$$

From energy norm, equality (6) is reorganized like this;

$$J^2(t)=J^2(0)+\int_{\Omega}F(\mathrm{x},t)u_t d\mathrm{x}dt \ \ t\in[0,T]. \quad (7)$$

Corollary 1. For the problem (1) – (3), the following inequalities are valid;

$$\|u\|_{L_2(\phi)}\le\sqrt{\frac{2}{q_1}}J(t),\ t\in[0,T] \quad (8)$$

$$\|\nabla u\|_{L_2(\phi)}\le\sqrt{2}J(t),\ t\in[0,T] \quad (9)$$

$$\|u_t\|_{L_2(\phi)}\le\sqrt{2}J(t),\ t\in[0,T]. \quad (10)$$

By taking derivative of the inequality (7) for $t>0$ and applying Cauchy-Bunyakovski inequality, it follows that

$$J(t) \le \frac{1}{\sqrt{2}}\left\{J(0)+\|F(\mathrm{x},t)\|_{L_2(\Omega_t)}\right\} \; t\in[0,T] \tag{11}$$

If we regulate the equality (11) and take as $c_2=\max\{1,q_2\}$, we write inequality

$$J(0) \le \sqrt{\frac{c_2}{2}}\left(\|u_t(\mathrm{x},0)\|_{L_2(\phi)}+\|\nabla u(\mathrm{x},0)\|_{L_2(\phi)}+\|u(\mathrm{x},0)\|_{L_2(\phi)}\right). \tag{12}$$

Using the inequalities (8) and (11), (12), we obtain the following inequality;

$$\|u(\cdot,t)\|^2_{L_2(\phi)} \le \frac{3c_2}{2q_1}\left(\|u(\mathrm{x},0)\|^2_{L_2(\phi)}+\|u_t(\mathrm{x},0)\|^2_{L_2(\phi)}+\|\nabla u(\mathrm{x},0)\|^2_{L_2(\phi)}+2\|F(\mathrm{x},t)\|^2_{L_2(\Omega_t)}\right), \forall t\in[0,T] \tag{13}$$

and from the inequalities (9) and (11), (12), we write

$$\|\nabla u(\cdot,t)\|^2_{L_2(\phi)} \le 3c_2\left(\|u(\mathrm{x},0)\|^2_{L_2(\phi)}+\|u_t(\mathrm{x},0)\|^2_{L_2(\phi)}+\|\nabla u(\mathrm{x},0)\|^2_{L_2(\phi)}+2\|F(\mathrm{x},t)\|^2_{L_2(\Omega_t)}\right), \forall t\in[0,T]. \tag{14}$$

However, benefitting from the inequalities (10) and (11), (12) we formulate as follows;

$$\|u_t(\cdot,t)\|^2_{L_2(\phi)} \le 3c_2\left(\|u(\mathrm{x},0)\|^2_{L_2(\phi)}+\|u_t(\mathrm{x},0)\|^2_{L_2(\phi)}+\|\nabla u(\mathrm{x},0)\|^2_{L_2(\phi)}+2\|F(\mathrm{x},t)\|^2_{L_2(\Omega_t)}\right), \forall t\in[0,T]. \tag{15}$$

Summing the inequalities (13), (14) and (15) and after some manipulations, we get the inequality,

$$\|u(\cdot,t)\|^2_{H^1(\phi)} \le c_3\left(\|F\|^2_{L_2(\Omega)}+\|\varphi\|^2_{L_2(\phi)}+\|\psi\|^2_{L_2(\phi)}+\|\varphi'\|^2_{L_2(\phi)}\right), \forall t\in[0,T] \tag{16}$$

where $c_3=\max\left\{3c_2,\frac{3c_2}{2q_1},\frac{3c_2}{q_1},6c_2\right\}$ and

$$\|u(\cdot,t)\|^2_{H^1(\Omega)} \le c_4\left(\|F\|^2_{L_2(\Omega)}+\|\varphi\|^2_{H^1(\Omega)}+\|\psi\|^2_{L_2(\phi)}\right), c_4=c_3T \;. \tag{17}$$

3. Numerical Solution: Galerkin Method

In this section, we give information about Galerkin method. Galerkin methods have been utilized to solve the problems encountered structural mechanics, dynamics, fluid flow, heat and mass transfer, acoustics, microwave theory, neutron transport, etc. The Galerkin method can be thought of as the calculus of variations performed backwards. That is, the Galerkin method seeks to find the weak form expressed in terms of integrals, and solve that, instead of solving the strong form of the problem as a differential equation.

Galerkin method is an influential numerical method for solving different types of partial differential equations [6, 7, 8]. Subaşı *et al.* [9] have applied the Galerkin method to the problem of vibration of one dimensional system with free end conditions. Limaco et al. analyzed from the mathematical point of view a model for small vertical vibrations of an elastic string with fixed ends and the density of the material being not constant [10]. The basic Galerkin methods with piecewise linear basis functions and quadratic basis functions have been compared in [11].

In literature, there have been many researches about Galerkin method which is an important tool in for numerical solution of hyperbolic problems [12, 13, 14].

The main contribution of this study is to obtain a successful approximation for two dimensional hyperbolic boundary value problem with variable coefficient. The proposed method has been improved to solve equations with transverse elastic force which is the function $q(\mathrm{x})$ taken place in the coefficient of hyperbolic problem.

Now, we apply this method to the considered problem.

By using Galerkin Method, the approximate solutions for the problem (1)-(3) are written as follows;

$$u^N(\mathrm{x},t)=\sum_{i=1}^{N}\sum_{j=1}^{N}c_{ij}^N(t)v_{ij}(\mathrm{x})$$

where the coefficients $c_{ij}^N(t)$ are the functions such that

$$c_{ij}^N(t)=\left\langle u^N(\mathrm{x},t),v_{ij}(\mathrm{x})\right\rangle_{L_2(\phi)}$$

for $i,j=1,2,...,N$ and the functions $v_{ij}(\mathrm{x})$ are basis functions for which $\left\langle v_{ij}(\mathrm{x}),v_{kl}(\mathrm{x})\right\rangle_{L_2(\phi)}=\delta_{kl}^{ij}$ is valid for $k,l=1,2,...,N$. Here δ_{kl}^{ij} is Kronecker delta. Let us write the equation (1) for the approximations $u^N(\mathrm{x},t)$ so that we obtain the coefficients $c_{ij}^N(t)$

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{d^2c_{ij}^N(t)}{dt^2}v_{ij}(\mathrm{x})-\sum_{i=1}^{N}\sum_{j=1}^{N}c_{ij}^N(t)\Delta v_{ij}(\mathrm{x})+q(x)\sum_{i=1}^{N}\sum_{j=1}^{N}c_{ij}^N(t)v_{ij}(\mathrm{x})=F(\mathrm{x},t) \quad (19)$$

and then integrate on the domain ϕ after multiplying both side of equality (19) with the function $v_{kl}(\mathrm{x})$. Hence we obtain the equality such as,

$$\int_{\phi}\left[\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{d^2c_{ij}^N(t)}{dt^2}v_{ij}(\mathrm{x})-\sum_{i=1}^{N}\sum_{j=1}^{N}c_{ij}^N(t)\Delta v_{ij}(\mathrm{x})+q(x)\sum_{i=1}^{N}\sum_{j=1}^{N}c_{ij}^N(t)v_{ij}(\mathrm{x})\right]v_{kl}(\mathrm{x})d\mathrm{x}=\int_{\phi}F(\mathrm{x},t)v_{kl}(\mathrm{x})d\mathrm{x} \quad (20)$$

where $\left\langle v_{ij}(\mathrm{x}),v_{kl}(\mathrm{x})\right\rangle_{L_2(\phi)}=0$ and $\left\langle \Delta v_{ij}(\mathrm{x}),v_{kl}(\mathrm{x})\right\rangle_{L_2(\phi)}=0$ as $\{i,j\}\neq\{k,l\}$.

Let us we write this system in the matrix form of

$$\begin{aligned}&\frac{d^2}{dt^2}C^N(t)+V(t)C^N(t)=F(t)\\&C^N(0)=D,\ \frac{d}{dt}C^N(0)=E\end{aligned} \quad (21)$$

where $C^N(t)$ is the matrix of searched functions that is,

$$C^N(t)=\begin{bmatrix}c_{11}^N(t) & c_{12}^N(t) & \cdots & c_{1N}^N(t)\\ c_{21}^N(t) & c_{22}^N(t) & \cdots & c_{2N}^N(t)\\ \vdots & \vdots & \ddots & \vdots\\ c_{N1}^N(t) & c_{N2}^N(t) & \cdots & c_{NN}^N(t)\end{bmatrix} \quad (22)$$

and the coefficient matrix $V(t)$ defined by

$$V(t)=\begin{bmatrix}\left\langle -\Delta v_{11}(\mathrm{x},t)+q(x)v_{11}(\mathrm{x}),v_{11}(\mathrm{x})\right\rangle_{L_2(\phi)} & \cdots & \left\langle -\Delta v_{1N}(\mathrm{x},t)+q(x)v_{1N}(\mathrm{x}),v_{1N}(\mathrm{x})\right\rangle_{L_2(\phi)}\\ \vdots & \ddots & \vdots\\ \left\langle -\Delta v_{N1}(\mathrm{x},t)+q(x)v_{N1}(\mathrm{x}),v_{N1}(\mathrm{x})\right\rangle_{L_2(\phi)} & \cdots & \left\langle -\Delta v_{NN}(\mathrm{x},t)+q(x)v_{NN}(\mathrm{x}),v_{NN}(\mathrm{x})\right\rangle_{L_2(\phi)}\end{bmatrix}. \quad (23)$$

The right hand side of matrix is in the form of

$$F(t)=\begin{bmatrix}\left\langle F(\mathrm{x},t),v_{11}(\mathrm{x})\right\rangle_{L_2(\phi)} & \left\langle F(\mathrm{x},t),v_{12}(\mathrm{x})\right\rangle_{L_2(\phi)} & \cdots & \left\langle F(\mathrm{x},t),v_{1N}(\mathrm{x})\right\rangle_{L_2(\phi)}\\ \left\langle F(\mathrm{x},t),v_{21}(\mathrm{x})\right\rangle_{L_2(\phi)} & \left\langle F(\mathrm{x},t),v_{22}(\mathrm{x})\right\rangle_{L_2(\phi)} & \cdots & \left\langle F(\mathrm{x},t),v_{2N}(\mathrm{x})\right\rangle_{L_2(\phi)}\\ \vdots & \vdots & \ddots & \vdots\\ \left\langle F(\mathrm{x},t),v_{N1}(\mathrm{x})\right\rangle_{L_2(\phi)} & \left\langle F(\mathrm{x},t),v_{N2}(\mathrm{x})\right\rangle_{L_2(\phi)} & \cdots & \left\langle F(\mathrm{x},t),v_{NN}(\mathrm{x})\right\rangle_{L_2(\phi)}\end{bmatrix}. \quad (24)$$

However, the matrices of initial conditions are given by

$$D=\begin{bmatrix} c_{11}^N(0) & c_{12}^N(0) & \cdots & c_{1N}^N(0) \\ c_{21}^N(0) & c_{22}^N(0) & \cdots & c_{2N}^N(0) \\ \vdots & \vdots & \ddots & \vdots \\ c_{N1}^N(0) & c_{N2}^N(0) & \cdots & c_{NN}^N(0) \end{bmatrix}, \quad E=\begin{bmatrix} \frac{d}{dt}c_{11}^N(0) & \frac{d}{dt}c_{12}^N(0) & \cdots & \frac{d}{dt}c_{1N}^N(0) \\ \frac{d}{dt}c_{21}^N(0) & \frac{d}{dt}c_{22}^N(0) & \cdots & \frac{d}{dt}c_{2N}^N(0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d}{dt}c_{N1}^N(0) & \frac{d}{dt}c_{N2}^N(0) & \cdots & \frac{d}{dt}c_{NN}^N(0) \end{bmatrix} \tag{25}$$

subject to $c_{ij}^N(0)=\left\langle \varphi(\mathrm{x}),v_{ij}(\mathrm{x})\right\rangle_{L_2(\phi)}$ and $\frac{d}{dt}c_{ij}^N(0)=\left\langle \psi(\mathrm{x}),v_{ij}(\mathrm{x})\right\rangle_{L_2(\phi)}$.

By calculating the coefficients $c_{ij}^N(t)$, we obtain the approximate solution $u^N(t)$. The equation (21) states a system of second order ODE and the solution of this system is uniquely solvable.

4. Numerical Illustrations

In this section, we present test problems after giving theoretical information mentioned in the previous sections. Before giving illustrations, we refer from its properties and basis functions which we will use for solving examples. In this article, we take the following fundamental set as basis functions

$$\left\{v_{ij}(\mathrm{x})\right\}=\left\{\frac{2}{\sqrt{l_1l_2}}\sin\left(\frac{\pi x}{l_1}\right)\sin\left(\frac{\pi y}{l_2}\right),\dots,\frac{2}{\sqrt{l_1l_2}}\sin\left(\frac{i\pi x}{l_1}\right)\sin\left(\frac{j\pi y}{l_2}\right),\dots,\frac{2}{\sqrt{l_1l_2}}\sin\left(\frac{N\pi x}{l_1}\right)\sin\left(\frac{N\pi y}{l_2}\right)\right\}$$

for which the following statements are hold;

- $\left\langle v_{ij}(\mathrm{x}),v_{kl}(\mathrm{x})\right\rangle_{L_2(\phi)}=\begin{cases}0, & \{i,j\}\neq\{k,l\} \\ 1, & \{i,j\}=\{k,l\}\end{cases}$
- $\left\langle \Delta v_{ij}(\mathrm{x}),v_{kl}(\mathrm{x})\right\rangle_{L_2(\phi)}=\begin{cases}0, & \{i,j\}\neq\{k,l\} \\ \left(\frac{i\pi}{l_1}\right)^2+\left(\frac{j\pi}{l_2}\right)^2, & \{i,j\}=\{k,l\}\end{cases}.$

Example 1. Let us consider problem (1)-(3) on the domain $(x,y)\in[0,2]\times[0,2]$ and $t\in[0,t]$ where $\varphi(x,y)=0,\ \psi(x,y)=0$, $q(\mathrm{x})=\begin{cases}-1 & 0\le x \text{ and } x<1,\, y\in[0,2] \\ \frac{1}{x-3} & 1\le x \text{ and } x\le 2,\, y\in[0,2]\end{cases}$

and

$$F(x,t)=\begin{cases}-\dfrac{1}{2}\sin\left(\dfrac{\pi x}{2}\right)\sin\left(\dfrac{\pi y}{2}\right)\left(\pi^2\cos t-\pi^2-4\cos t+2\right) & 0<x<1, y\in[0,2], t\in[0,t]\\ -\dfrac{1}{2}\dfrac{\sin\left(\dfrac{\pi x}{2}\right)\sin\left(\dfrac{\pi y}{2}\right)\left(\pi^2 x\cos t-3\pi^2\cos t-\pi^2 x+3\pi^2-2x\cos t+8\cos t-2\right)}{x-3} & x\le 2, y\in[0,2], t\in[0,t]\end{cases}$$

In this case, by solving the system (21), the exact solution is equal to approximate solution which is $u(\mathrm{x},t)=u^N(\mathrm{x},t)=(1-\cos t)\sin\left(\dfrac{\pi x}{2}\right)\sin\left(\dfrac{\pi x}{2}\right)$ for $N=10$.

The graphs of these solutions are given in Figure 1.

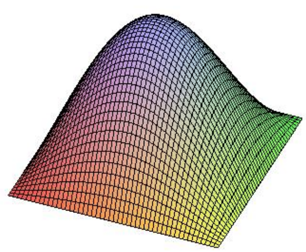
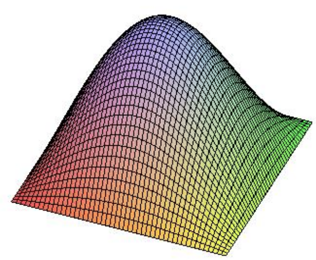
Approximate Solution	Exact Solution	Errors for $\left\Vert u^N-u\right\Vert_{L_2(\phi)}$	
		$t=0$	0.0001
		$t=2$	0.0002
		$t=4$	0.0001
		$t=6$	0.0005
		$t=8$	0.00006
		$t=10$	0.0004

Figure 1: The graph of the approximate solution and exact solution for $N=10,\ t=1$.

Example 2. Let us analyze problem (1)-(3) on the domain $(x,y)\in[0,1]\times[0,2]$ and $t\in[0,t]$ where $\psi(x,y)=0,\ q(\mathrm{x})=x$

$$\varphi(x,y)=\begin{cases}\dfrac{1}{2}x(2x-1)\sin\left(\dfrac{3\pi y}{2}\right) & 0< x<\dfrac{1}{2}, y\in[0,2]\\ -\dfrac{1}{2}(x-1)(2x-1)\sin\left(\dfrac{3\pi y}{2}\right) & x\le 1, y\in[0,2]\end{cases}$$

and

$$F(x,t)=\begin{cases}\dfrac{1}{8}\cos t\sin\left(\dfrac{3\pi y}{2}\right)\left(18\pi^2x^2-9\pi^2x+8x^3-12x^2+4x-16\right) & x<\dfrac{1}{2}, y\in[0,2], t\in[0,t]\\ -\dfrac{1}{8}\cos t\sin\left(\dfrac{3\pi y}{2}\right)\left(18\pi^2x^2-27\pi^2x+8x^3+9\pi^2-20x^2+16x-20\right) & x<1, y\in[0,2], t\in[0,t]\end{cases}$$

In this case, the exact solution is

$$u(\mathrm{x},t)=\begin{cases}\frac{1}{2}x(2x-1)\cos t\,\sin\left(\frac{3\pi y}{2}\right) & x<\frac{1}{2}, y\in[0,2], t\in[0,t]\\ -\frac{1}{2}(x-1)(2x-1)\cos t\,\sin\left(\frac{3\pi y}{2}\right) & x\le 1, y\in[0,2], t\in[0,t]\end{cases}.$$

The graphs of these solutions are given in Figure 2.

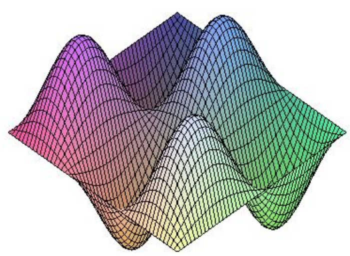
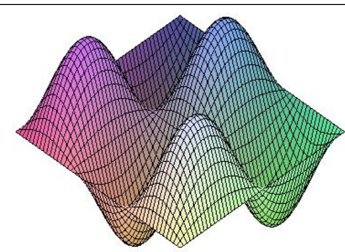
Approximate Solution	Exact Solution	Errors for $\left\Vert u^N-u\right\Vert_{L_2(\phi)}$	
		$t=0$	0.31622×10^{-6}
		$t=2$	0.02
		$t=4$	0.02
		$t=6$	0.0005
		$t=8$	0.01
		$t=10$	0.02

Figure 2: The Graph of the Approximate Solution and Exact Solution for $N=10,\ t=1$

Example 3. Let us solve problem (1)-(3) on the domain $(x,y)\in[0,1]\times[0,1]$ and $t\in[0,t]$. For this example, we shall take as $q(\mathrm{x})=x+y$, $\varphi(x,y)=\sin\pi x\sin\pi y$, $\psi(x,y)=-\sin\pi x\sin\pi y$,

$$F(\mathrm{x},t)=e^{-t}\sin\pi x\sin\pi y\left(2\pi^2+x+y+1\right)$$

In this case, it can be seen that the exact solution is $u(\mathrm{x},t)=e^{-t}\sin\pi x\sin\pi y$.

The graphs of these solutions are given in Figure 3.

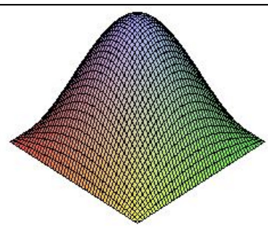
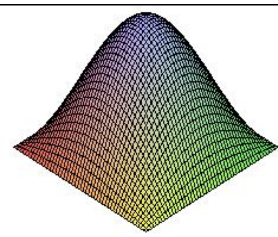
<table>
<tr><th>Approximate Solution</th><th>Exact Solution</th><th colspan="2">Errors for $\left\|u^N - u\right\|_{L_2(\phi)}$</th></tr>
<tr><td rowspan="6"></td><td rowspan="6"></td><td>$t=0$</td><td>0</td></tr>
<tr><td>$t=2$</td><td>0.0008</td></tr>
<tr><td>$t=4$</td><td>0.002</td></tr>
<tr><td>$t=6$</td><td>0.0007</td></tr>
<tr><td>$t=8$</td><td>0.002</td></tr>
<tr><td>$t=10$</td><td>0.001</td></tr>
</table>

Figure 3: The Graph of the Approximate Solution and Exact Solution for $N=10,\ t=1$

5. Conclusion

In this paper, Galerkin method with basis functions has been proposed to solve two dimensional hyperbolic boundary value problems. The existence, uniqueness and dependence continuously of generalized solution for problem have been demonstrated by Theorem 1. It is considered three illustrations for the hyperbolic boundary value problem in this article.

Speaking for the examples presented, it can be seen that the method is successful. In other words, the numerical results confirm the validity of the technique. It has been utilized from Maple in numerical calculations.

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