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Doubly Twisted Product Semi-Invariant Submanifolds of a Locally Product Riemannian Manifold

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Doubly Twisted Product, Doubly Warped Product, Semi-invariant Submanifold, Locally Product Riemannian Manifold.

Abstract: We define the notion of nearly doubly twisted product of type 1 and 2. We prove that there do not exist doubly twisted (respectively, doubly warped) product semi-invariant submanifolds in locally product Riemannian manifolds other than nearly doubly twisted product of type 2 (respectively, warped product) semi-invariant submanifolds.

1. Introduction

Warped product manifolds were first studied by R.L. Bishop and O'Neill [2] to construct new examples of negatively curved manifolds. In fact, warped product manifolds are natural generalization of usual product manifolds. The notion of warped product has been generalized in several ways such as twisted product, doubly warped product and doubly twisted product, see [5] and [6]. Also, these kinds of manifolds play very important roles in the theory of relativity.

The theory of warped product submanifolds has been becoming a very active research since Chen [3] studied the warped product CR-submanifolds in Kaehler manifolds. Lately, doubly warped product submanifols has been begun to study (see [7, 8, 10]). Most of studies related to theory of warped or doubly warped product submanifolds can be found Chen's book [4].

In this paper, we consider doubly twisted product semi-invariant submanifolds in locally product Riemannian manifolds. We check that the existence of this kind of submanifolds.

2. Preliminaries

In this section, we present the fundamental definitions and notions needed further study. In fact, in subsection 2.1, we will recall the definition of doubly twisted and doubly warped product submanifolds. In subsection 2.2, we will give the basic background for submanifolds of Riemannian manifolds. The definition of a locally product Riemannian manifold and semi-invariant submanifolds of a locally product Riemannian manifold are placed in subsection 2.3.

2.1. Doubly Twisted Product Manifolds

Let M_1 and M_2 be Riemannian manifolds endowed with metric tensors g_1 and g_2, respectively and let f_1 and f_2 are positive smooth functions defined on $M_1 \times M_2$. Then the *doubly twisted product manifold* ${}_{f_2}M_1 \times_{f_1} M_2$ is the product manifold $\bar{M} = M_1 \times M_2$ equipped with metric g given by

$$g = (f_2)^2 g_1 + (f_1)^2 g_2.$$

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Each function f_i is called a *twisting function* of the doubly twisted product $({}_{f_2}M_1 \times_{f_1} M_2, g)$.

If the twisting functions f_1 and f_2 only depend on the points of M_1 and M_2 respectively, then $({}_{f_2}M_1 \times_{f_1} M_2, g)$ is called *doubly warped product manifold*. In which case, f_1 and f_2 are called *warping functions* of doubly warped product.

Let $\bar{\nabla}$ be Levi-Civita conection of $({}_{f_2}M_1 \times_{f_1} M_2, g)$,. Then from Proposition 1 of [6], we have

$$\bar{\nabla}_V X = X(\ln f_2)V + V(\ln f_1)X \tag{1}$$

where V and X are arbitrary vector fields on (M_1, g_1) and (M_2, g_2) respectively.

Let $({}_{f_2}M_1 \times_{f_1} M_2, g)$ be a doubly twisted product manifold. If $f_2 = 1$, then we get *twisted product* [5] $M_1 \times_{f_1} M_2$ with twisting function f_1.

In addition, if the twisting function f_1 only depends on the point of M_1, then $M_1 \times_{f_1} M_2$ is called *warped product* [2] of (M_1, g_1) and (M_2, g_2) and the function f_1 is called *warping function*.

We now propose two new notions.

Definition 2.1. Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds and also let $f_1 : M_1 \to (0, \infty)$ (respectively, $f_1 : M_2 \to (0, \infty)$) and $f_2 : M_1 \times M_2 \to (0, \infty)$ be smooth functions. The *nearly doubly twisted product of type* 1 (respectively, *nearly doubly twisted product of type* 2) is the product manifold $M_1 \times M_2$ equipped with the metric tensor g defined by

$$g = (f_2)^2 \pi_1^*(g_1) + (f_1)^2 \pi_2^*(g_2), \tag{2}$$

where π_1 and π_2 are canonical projections of $M_1 \times M_2$ onto M_1 and M_2, respectively. For brevity in notation, we denote this Riemannian manifold (M, g) by ${}_{f_2}M_1 \times_{f_1} M_2$. In case of nearly doubly twisted of type 1, the function f_1 is called a *warping function* while in other case nearly doubly twisted of type 2, is called a *conformal factor*. In either case, the function f_2 is called a *twisting function*.

2.2. Submanifolds of Riemannian Manifolds

Let M be a Riemannian manifold isometrically immersed in a Riemannian manifold $(\bar{M}, g)$ and $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ with respect to the metric g. Also, let ∇ and $\nabla^\perp$ be the induced and induced normal connection on M, respectively. Then the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_V W = \nabla_V W + h(V, W) \tag{3}$$

$$\bar{\nabla}_V Z = -A_Z V + \nabla_V^\perp Z \tag{4}$$

where the vector fields V, W are tangent to M and Z is normal to M.

In addition, h is the second fundamental form of M and A_Z is the Weingarten endomorphism associated with Z. The second fundamental form h and the shape operator A are related by

$$g(h(V, W), Z) = g(A_Z V, W). \tag{5}$$

For the more details on the theory of submanifolds, we refer to [11].

2.3. Semi-invariant Submanifolds of Locally Product Riemannian Manifolds

Let $\bar{M}$ be any manifold equipped with a tensor field of type $(1,1)$ such that

$$F^2 = I, \qquad (F \neq \mp I) \tag{6}$$

where I is the identity endomorphism on the tangent bundle $T\bar{M}$ of $\bar{M}$. Then we say that $(\bar{M}, F)$ is an almost product manifold with almost product structure F. If the almost product manifold $(\bar{M}, F)$ admits a metric tensor g such that

$$g(F\bar{V}, F\bar{W}) = g(\bar{V}, \bar{W}) \tag{7}$$

for all $\bar{V},\bar{W} \in \Gamma(T\bar{M})$, then $(\bar{M},F,g)$ is called an almost product Riemannian manifold. Let $\bar{\nabla}$ be the Levi-Civita connection of $(\bar{M},F,g)$, then we say that $(\bar{M},F,g)$ is a *locally product Riemannian manifold*, (briefly, l.p.R. manifold) if F is parallel with respect to $\bar{\nabla}$, i.e.

$$\bar{\nabla}_{\bar{V}}F \equiv 0 \tag{8}$$

for all $\bar{V} \in \Gamma(T\bar{M})$.

Now we recall the definition of a semi-invariant submanifold of a l.p.R. manifold.

Definition 2.2. ([1]) Let M be a isometrically immersed submanifold of a l.p.R. manifold $(\bar{M},F,g)$. Then M is called a semi-invariant submanifold of $\bar{M}$, if there exist two orthogonal complementary distribution $\mathscr{D}$ and $\mathscr{D}^{\perp}$ such that

(a) TM admits the orthogonal direct decomposition $TM = \mathscr{D} \oplus \mathscr{D}^{\perp}$

(b) The distribution $\mathscr{D}$ is F-invariant, i.e. $F\mathscr{D} \subseteq \mathscr{D}$

(c) The distribution $\mathscr{D}^{\perp}$ is F-anti-invariant, i.e. $F\mathscr{D}^{\perp} \subseteq T^{\perp}M$,

where TM is the tangent bundle of M and $T^{\perp}M$ is the normal bundle of M in $\bar{M}$.

For the general properties of semi-invariant submanifolds of a l.p.R. manifold, see [1] and [9].

3. Doubly Twisted and Doubly Warped Product Semi-invariant Submanifolds

Theorem 3.1. *Let ${}_fM_T \times_b M_{\perp}$ be a doubly twisted semi-invariant submanifold of a locally product Riemannian manifold $(\bar{M},F,g)$. Then M is nearly doubly twisted product of type 2 such that M_T is an invariant submanifold and $M_{\perp}$ is an anti invariant submanifold of $(\bar{M},F,g)$.*

Proof. Assume that $M =_f M_T \times_b M_{\perp}$ be a doubly twisted product semi-invariant submanifold of a l.p.R. $(\bar{M},F,g)$. Let $X \in \Gamma(TM_T)$ and $U,V \in \Gamma(TM_{\perp})$. Then using (1), we have

$$g(\nabla_U X,V) = X(\ln b)g(U,V),$$

hence, we obtain

$$g(\nabla_U V,X) = -X(\ln b)g(U,V).$$

Using (3), (7) and (8), we get

$$g(\bar{\nabla}_U FV,FX) = -X(\ln b)g(U,V).$$

Again, using (3) and (7), we arrive at

$$g(h(U,FX),FV) = X(\ln b)g(U,V). \tag{9}$$

On the other hand, using (1), (3) and (7), we have

$$g(h(U,FX),FV) = g(\bar{\nabla}_{FX}U,FV) = g(\bar{\nabla}_{FX}FU,V).$$

Hence, we obtain

$$g(h(U,FX),FV) = -g(FU,\bar{\nabla}_{FX}V),$$

since $g(FU,V) = 0$. Again, using (3) , we get

$$g(h(U,FX),FV) = -g(h(V,FX),FU). \tag{10}$$

From (9) and (10), it follows that

$$X(lnb)g(U,V) = 0.$$

From the last equation, we deduce that $X(\ln b) = 0$. Which means that the warping function b only depends on the points of $M_{\perp}$. Thus, M is a nearly doubly twisted product of type 2 semi-invariant submanifold with conformal factor b and twisting function f. □

We have immediately the following result from Theorem 3.1.

Corollary 3.2. *There do not exist doubly warped product semi-invariant submanifolds ${}_fM_T \times_b M_{\perp}$ of a l.p.R. manifold $(\bar{M},F,g)$, such that M_T is an invariant and $M_{\perp}$ is an anti-invariant submanifold of $(\bar{M},F,g)$.*

CONCLUSION: There are no doubly twisted (resp. doubly warped) product submanifolds in locally product Riemannian manifolds other than nearly doubly twisted product of type 2 (resp. warped product) semi-invariant submanifolds.

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