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Scalar characterization in Banach-Jordan algebras

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Abstract

Using a Diagonalization Theorem obtained when the spectrum is Lipschitzian, we extend a result of G. Braatvedt on scalar characterization in Banach algebras to Banach-Jordan algebras. We also establish that any element of a semisimple Banach-Jordan algebra with the property that all elements in some neighbourhood of the identity are spectrally invariant under multiplication by the quadratic U operator, has analogs with the identity.

1. Preliminaries

A unital Banach-Jordan algebra is a vector space with a binary product

$$(x,y) \mapsto x\cdot y$$

satisfying the identities:

$$x\cdot y = y\cdot x,\ (x\cdot y)^2 = x^2\cdot(y\cdot x^2) \quad \forall x,y\in A,$$

and endowed with a complete norm $\|\cdot\|$ such that, for all $x,y\in A$,

$$\| x\cdot y \| \leq \| x \| \| y \| .$$

N. Jacobson introduced the notion of invertibility in Jordan algebras, which generalizes the notion of invertibility in associative algebras. Given x in A we say that x is invertible in A if there exists y in A such that $x\cdot y = 1$ and $x^2\cdot y = x$. This element y is unique and is usually denoted by x^{-1}. It turns out that this notion of inverse is intimately related to the quadratic map $\mathrm{U}: A \mapsto \mathscr{BL}(A)$ defined by

$$\mathrm{U}_x y = 2x\cdot(x\cdot y) - x^2\cdot y$$

for any $x,y\in A$. Keeping in mind that the mapping $x\mapsto U_x$ from A into $\mathscr{BL}(A)$ is continuous, the invertible elements $\Omega = \{a\in A : \mathrm{U}_a \text{ is invertible}\}$ form an open subset of A; in particular, Ω is locally connected as shown by O. Loos in [7], so its connected components are open. Also, the space $\mathbb{C}[x]$ spanned by all powers of x is a commutative associative subalgebra with respect to the linear Jordan product. By continuity, the same holds for its closure $\mathscr{C}$ in A. We refer the reader to Chapter 4 of [6] for more details on spectral theory in Banach-Jordan algebras. For general theory of Jordan algebras see [8] and [10].

Theorem 1.1. *An element x of A is invertible if and only if U_x is invertible in $\mathscr{L}(A)$, the algebra of linear operators on A, in which case $\mathrm{U}_{x^{-1}} = \mathrm{U}_x^{-1}$. If $x,y\in A$, then they are both invertible if and only if $\mathrm{U}_x(y)$ is invertible in A. In particular, x is invertible if and only if x^n is invertible for every integer $n\geq 1$.*

This theorem implies that the set of invertible elements $\Omega(A)$ is invariant when taking powers, but unfortunately, it is not stable for the product. For $x\in A$ we denote respectively by $\mathrm{Sp}(x) = \{\lambda 1 - x \notin \Omega(A)\}$ and $\rho_A(x) = \sup\{|\lambda| : \lambda\in \mathrm{Sp}(x)\}$ the spectrum and spectral radius of x.
In what follows, an important tool will be the theory of subharmonic functions, based essentially on the celebrated result of Aupetit and Zraibi, which allow us to use analytic tools in Banach-Jordan algebras.

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Theorem 1.2 (Aupetit-Zraibi)**.** *Let $f : D \to A$ be a holomorphic function from a domain D of $\mathbb{C}$ into a Banach-Jordan algebra. Then the mapping $\lambda \to \mathrm{Sp}\,(f(\lambda))$ is an analytic multifunction. Consequently, $\lambda \longmapsto \rho(f(\lambda))$ and $\lambda \longmapsto \log\rho(f(\lambda))$ are subharmonic on D.*

We will require the following fundamental result from the theory of subharmonic functions [1].

Theorem 1.3 (Maximum Principle for Subharmonic Functions)**.** *Let f be a subharmonic function on a domain D of $\mathbb{C}$. If there exists $\lambda_0 \in D$ such that $f(\lambda) \leq f(\lambda_0)$ for all $\lambda \in D$, then $f(\lambda) = f(\lambda_0)$ for all λ in D.*

Another important ingredient is Aupetit's characterization of the McCrimmon radical $\mathrm{Rad}(A)$ of A (see [3]) and its corollaries.

Theorem 1.4 (Aupetit)**.** *Let a be an element of a Banach-Jordan algebra A. Then a is in the McCrimmon radical of A if and only if $\sup\{\rho(x+ta) : t \in \mathbb{C}\} < \infty$ for every x in A.*

Corollary 1.5. *An element a of a Banach-Jordan algebra A is in the McCrimmon radical of A if and only if $\sup\rho(U_x a) = 0$ for every x in A.*

Corollary 1.6. *An element a of a Banach- Jordan algebra A is in the McCrimmon radical of A if and only if there exists $C \geq 0$ such that $\rho(x) \leq C \parallel x - a \parallel$ for every x in a neighborhood of a.*

2. Some results under the condition of a Lipschitzian spectrum

The next lemma is a spectral characterization of the Jacobson radical in terms of the Lipshitzian behaviour of the spectrum. It was obtained by Aupetit in [3] for Banach algebras and we extend it here to Banach-Jordan algebras.

Lemma 2.1. *Let $q \in A$ be a quasi-nilpotent element. Suppose that there exists $r, C > 0$ such that $\rho(x) \leq C \parallel x - q \parallel$, for $\parallel x - q \parallel < r$, then $q \in \mathrm{Rad}(A)$.*

Proof. Let $y \in A$ be arbitrary. For $|\lambda| > \frac{\|y\|}{r}$, we have $\rho(q + \frac{y}{\lambda}) \leq C\frac{\|y\|}{|\lambda|}$, consequently $\rho(y + \lambda q) \leq C \parallel y \parallel$. Hence the upper semi-continuous function $\lambda \mapsto \rho(y + \lambda q)$ is bounded on the complex plane. Being subharmonic, it is constant by Liouville's Theorem for subharmonic functions. Thus $\rho(y+q) = \rho(y)$, for every $y \in A$ and by Aupetit's characterization of the radical [3], we obtain $q \in \mathrm{Rad}(A)$. □

We recall that the spectrum is said to be Lipschitzian at an element a of a Banach-Jordan algebra if there exists two positive constants r and C such that $\Delta(\mathrm{Sp}(x), \mathrm{Sp}(a)) \leq C \parallel x - a \parallel$ for all x satisfying $\parallel x - a \parallel < r$, where Δ represents the Hausdorff distance on compact sets of the complex plane defined by

$$\Delta(\sigma_1, \sigma_2) = \max\{\sup_{\lambda\in\sigma_2}\{\mathrm{dist}(\lambda, \sigma_1)\}, \sup_{\lambda\in\sigma_1}\{\mathrm{dist}(\lambda, \sigma_2)\}\}$$

where $\mathrm{dist}(\lambda, \sigma) = \inf\{|\lambda - \mu| : \mu \in \sigma\}$ is the distance of the point λ to the compact set σ (see [1]). Using the previous lemma, we obtained the following theorem in [9].

Theorem 2.2. *Let A be a semisimple complex Banach-Jordan algebra and let $a \in A$ have finite spectrum, $\mathrm{Sp}(a) = \{\alpha_1, \cdots, \alpha_n\}$. Suppose that the spectral mapping $x \mapsto \mathrm{Sp}_A(x)$ is Lipschitzian at a. Then there exist n nonzero orthogonal projections $p_1, \cdots, p_n$ whose sum is 1 and such that $a = \alpha_1 p_1 + \cdots + \alpha_n p_n$.*

The next theorem obtained in [5] for Banach algebras is in fact a particular case to our theorem quoted above. We extend here this theorem to Jordan algebras along with a new proof.

Theorem 2.3. *Let A be a semisimple complex Banach-Jordan algebra and let $a \in A$. If the spectrum is Lipschitzian at a and $\mathrm{Sp}(a) = \{\alpha\}$, then $a = \alpha 1$.*

Proof. Since the spectrum is Lipschitzian at a, it follows that there exists two positive constants r and C such that

$$\Delta(\mathrm{Sp}(x), \mathrm{Sp}(a)) \leq C \parallel x - a \parallel$$

for all x satisfying $\parallel x - a \parallel < r$. Clearly,

$$\Delta(\mathrm{Sp}(x), \mathrm{Sp}(a)) = \Delta(\mathrm{Sp}(x - \alpha), \mathrm{Sp}(a - \alpha)).$$

Since $\mathrm{Sp}(a - \alpha) = \{0\}$, taking $x - \alpha 1 = y$, we get from our assumption

$$\begin{aligned}
\rho(y) &= \Delta(\mathrm{Sp}(y), \{0\}) \\
&= \Delta(\mathrm{Sp}(y), \mathrm{Sp}(a - \alpha 1)) \\
&\leq C \parallel x - a \parallel \\
&= C \parallel y - (a - \alpha 1) \parallel
\end{aligned}$$

for all $\parallel y - (a - \alpha 1) \parallel < r$. It follows from Corollary 2 of Aupetit's characterization of the radical, that $a - \alpha 1 \in \mathrm{Rad}(A) = \{0\}$. Thus $a = \alpha 1$. □

3. Scalar characterization in a Banach-Jordan algebra

Another result obtained in [5] is the following multiplicative scalar characterization of elements in a Banach algebra.

Theorem 3.1. *Let A be a semisimple complex Banach algebra and let $a \in A$. Then $a = 1$ if and only if* $\mathrm{Sp}(ax) = \mathrm{Sp}(x)$ *for all x in a neighborhood of* 1.

We extend this result to Banach-Jordan algebras with a slightly different conclusion. It is clear that if $a = 1$ then $\mathrm{Sp}(\mathrm{U}_a x) = \mathrm{Sp}(x)$ for all x in A. But the converse is not exactly as for Banach algebras. Indeed, instead of using the linear Jordan product as an analogue of multiplication in Banach algebras, we consider multiplication by the quadratic U operator in the situation of Banach-Jordan algebras. Precisely, we prove that if an element a in a semisimple Banach-Jordan algebra has the property that multiplication by U_a leaves all elements in some neighbourhood of the identity spectrally invariant, then clearly that element squares to the identity.

Theorem 3.2. *Let A be a complex semisimple Banach-Jordan algebra and a nonzero element a of A. If* $\mathrm{Sp}(\mathrm{U}_a x) = \mathrm{Sp}(x)$ *for all x in a neighborhood of* 1 *then $a^2 = 1$. In particular, a is invertible and $a^{-1} = a$.*

Proof. Note that if $a = 1$ then $\mathrm{Sp}(\mathrm{U}_a x) = \mathrm{Sp}(x)$. We are interested by the converse. Suppose that $\mathrm{Sp}(\mathrm{U}_a x) = \mathrm{Sp}(x)$ for all x in a neighborhood $\mathscr{V}(1)$ of 1. Let $x = 1$, then $\mathrm{Sp}(\mathrm{U}_a 1) = \mathrm{Sp}(a^2) = \mathrm{Sp}(1) = \{1\}$. So $\mathrm{Sp}(\mathrm{a}) \subseteq \{-1,1\}$, hence a is invertible. Let $y \in A$ arbitrary. Take λ sufficiently small, say $\lambda \in B(0,\varepsilon)$, such that

$$\mathrm{Sp}(\lambda y + a^2) = \mathrm{Sp}\left(\mathrm{U}_a(\lambda \mathrm{U}_{a^{-1}} y + 1)\right) = \mathrm{Sp}\left(\lambda \mathrm{U}_{a^{-1}} y + 1\right) = \mathrm{Sp}\left(\lambda \mathrm{U}_{a^{-1}} y\right) + 1.$$

So,

$$\mathrm{Sp}(\lambda y + a^2 - 1) = \mathrm{Sp}\left(\lambda \mathrm{U}_{a^{-1}} y\right)$$

and

$$\rho(y + \frac{1}{\lambda}(a^2 - 1)) = \rho\left(\mathrm{U}_{a^{-1}} y\right)$$

for all $0 \neq \lambda \in B(0,\varepsilon)$. Furthermore,

$$\begin{aligned} \rho(y + \frac{1}{\lambda}(a^2 - 1)) &\leq \| y + \frac{1}{\lambda}(a^2 - 1)) \| \\ &\leq \| y \| + |\frac{1}{\lambda}| \, \| a^2 - 1 \| \\ &\leq \| y \| + \frac{1}{\varepsilon} \| a^2 - 1 \| \end{aligned}$$

for all $\lambda \in \mathbb{C} \setminus B(0,\varepsilon)$.
Hence, there exists $M > 0$ such that

$$\rho(y + \frac{1}{\lambda}(a^2 - 1)) \leq M$$

for all $\lambda \in \mathbb{C} \setminus \{0\}$. Furthermore,

$$\limsup_{\lambda \to 0} \rho\left(y + \frac{1}{\lambda}(a^2 - 1)\right) \leq M.$$

Hence, taking $\mu = \frac{1}{\lambda}$ it follows that the subharmonic function

$$\phi : \mu \mapsto \rho\left(y + \mu(a^2 - 1)\right)$$

is bounded on $\mathbb{C}$, and

$$\limsup_{\mu \to \infty} \phi(\mu) \leq M.$$

By Liouville's theorem for subharmonic functions, ϕ is constant. Hence,

$$\rho\left(y + \mu(a^2 - 1)\right) = \rho(y)$$

for all $\mu \in \mathbb{C}$. By Aupetit's characterization of the radical, we obtain

$$a^2 - 1 \in \mathrm{Rad}(A) = \{0\},$$

that is $a^2 = 1$.

□

Our last result concerns bounded elements in a finite dimensional Banach-Jordan algebra. Exactly as for Banach algebras, we get the following theorem which extends another result of G. Braatvedt from associative Banach algebras to Non-associative Banach algebras. The proof follows the same arguments as the associative one. Recall that an element a of a Banach-Jordan algebra is said to be power bounded if there exists a positive constant M such that $\| a^n \| \leq M$ for all $n \in \mathbb{N}$ (more details on powers of elements in Banach-Jordan algebras can be found in [5]).

Theorem 3.3. *Let A be a finite-dimensional Banach-Jordan algebra and $a \in A$. If $\mathrm{Sp}(a) = \{1\}$ and a is power bounded, then $a = 1$.*

Proof. Note that since $\mathrm{Sp}(a) = \{1\}$, then $\mathrm{Sp}(a-1) = \{0\}$. Hence, $(a-1)$ is quasi-nilpotent and therefore nilpotent since A is finite-dimensional. Thus $(a-1)^N = 0$ for some $N \in \mathbb{N}$. Hence for all $n \geq N$, we get

$$a^n = ((a-1)+1)^n = \sum_{k=0}^{n} \binom{k}{n} (a-1)^k = \sum_{k=0}^{N-1} \binom{k}{n})(a-1)^k.$$

Since a is power bounded, for some $M > 0$ and all $n \in \mathbb{N}$ we have $\| a^n \| \leq M$. Hence for all $n \in \mathbb{N}$,

$$\| \sum_{k=0}^{N-1} \binom{k}{n} (a-1)^k \| \leq M \Longrightarrow \| \sum_{k=0}^{N-2} \binom{k}{n} (a-1)^k + \binom{n}{N-1} (a-1)^{N-1} \| \leq M$$

Dividing both sides by $\binom{n}{N-1}$ gives

$$\| \sum_{k=0}^{N-2} \frac{(N-1)!}{(n-k)(n-(k+1))\cdots(n-(N-2))k!} (a-1)^k + (a-1)^{N-1} \| \leq \frac{M}{\binom{n}{N-1}}$$

(because $k \leq N-2 < N-1$). Now considering the limit as $n \to \infty$ gives $0 \leq \| (a-1)^{N-1} \| \leq 0$, and so $(a-1)^{N-1} = 0$. It follows by induction that $(a-1) = 0$ that is $a = 1$.

□

Remark 3.4. *The previous theorem is valid for Banach-Jordan algebras because the proof takes place in a subalgebra generated by* 1 *and* a, *that is a full subalgebra of a Banach-Jordan algebra. In that case everything works as in classical Banach algebras as described and well explained in Chapter 4 of [6].*

References

[1] Aupetit Bernard, A Primer on spectral theory, Universitext, Springer, New York, 1991.
[2] Aupetit Bernard, "Spectral characterisation of the radical in Banach and Jordan-Banach algebras", *Math. Proc. Camb. Philos. Soc*. **114** (1993), 31-35.
[3] Aupetit Bernard, "Spectrum-preserving linear mappings between Banach algebras and Jordan-Banach algebras", *J. London Math. Soc*. 2**62** (2000), 917-924.
[4] Aupetit Bernard and Abdelaziz Maouche, "Trace and determinant in Jordan-Banach algebras", *Publ. Mat*. **46 (2002)**, 3-16.
[5] Braatvedt , G., Brits, R., and Raubeuheimer, H., "Spectral characterization of scalars in a Banach algebra", *Bull. London Math. Soc*. **41 (2009)**, 1095-1104.
[6] Miguel Cabrera Garcia and Angel Rodriguez Palacios, Non-Associative Normed Algebras, Encyclopedia of Mathematics and its Applications 154, Cambridge, 1994.
[7] O. Loos, "On the set of invertible elements in Banach-Jordan algebras", *Results in Math*. **29 (1996)**, no. 1-2, 111-114.
[8] Kevin McCrimmon, A Taste of Jordan algebras, Universitext, Springer, New York, 2004.
[9] Abdelaziz Maouche, "Diagonalization in Jordan-Banach algebras", *Extracta Mathematicae*. **Vol. 19**, Num. 2 **(2004)**, 257-260.
[10] K.A. Zhevlakov, A.M. Slin'ko, I.P. Shestakov and A. I. Shirshov, Rings that are nearly associative, Academic Press, New York, 1982.