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Solvability for a nonlinear third-order three-point boundary value problem

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Abstract

In this article, the existence of positive solutions for a nonlinear third-order three-point boundary value problem with integral condition is investigated. By using Leray-Schauder fixed point theorem, sufficient conditions for the existence of at least one positive solution are obtained. Illustrative examples are also presented to show the applicability of our results.

1. Introduction

This paper is devoted to the existence of positive solutions for the following third-order nonlocal integral boundary value problem (BVP):

$$u'''(t) + a(t) f(t, u(t)) = 0, \quad 0 < t < T, \tag{1.1}$$

$$u(0) = u''(0) = 0, \; u(T) = \alpha \int_0^\eta u(s)\,ds, \tag{1.2}$$

where $0 < \eta < T$, $0 < \alpha < \frac{2T}{\eta^2}$ and

(H_1) $f([0,T] \times [0,+\infty), [0,+\infty))$;

(H_2) $a \in C([0,T], [0,+\infty))$ and there exists $t_0 \in [\eta, T]$ such that $a(t_0) > 0$.

Set

$$f_0 = \lim_{u \to 0^+} \frac{f(t,u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(t,u)}{u},$$

then $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case.
Third-order boundary-value problems for differential equations arise in variety of different areas of applied mathematics and physics. They have been many scholars' research object. For example, heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics can produce boundary-value problems with integral boundary conditions; see [3, 9, 11]. They include two, three, multipoint, and nonlocal boundary-value problems as special cases. By using the Krasnoselskii's fixed point theorem, Liu and Ma [19] studied the problem

$$u'''(t) + f(u(t)) = 0, \quad 0 < t < 1, \tag{1.3}$$

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subject to integral boundary condition of the form

$$u'(0)=0,\, u'(1)=0,\, u(0)=\int_0^1 k(s)u(s)ds. \tag{1.4}$$

Benaicha and Haddouchi [17] considered the fourth-order two-point boundary value problem

$$u''''(t)+f(u(t))=0, \quad t\in(0,1), \tag{1.5}$$

$$u'(0)=u'(1)=u''(0)=0,\, u(0)=\int_0^1 a(s)u(s)ds. \tag{1.6}$$

We quote also the reasearchs of [2, 4, 5, 6, 7, 8, 12, 13, 14, 15, 16, 18, 20] which concern the differential equations under various boundary conditions and by different approaches.
Motivated by the works mentioned above, we obtain the existence results for the problem (1.1)-(1.2) by using the Leray-Shauder fixed point theorem if $f_0=0$ (condition $f_\infty=\infty$ being unnecessary) , as well as, for $f_\infty=0$ (condition $f_0=\infty$ being unnecessary). In this way we remove the half of the assumptions to prove the existence of a solution when using Krasnoselskii's fixed point theorem.(See [10, 17, 19]).
Moreover, we establish our results for t in $[0,T]$.
Our main tool is the following Leray-Schauder fixed point theorem.

Theorem 1.1. *[1] Let Ω be the convex subset of Banach space E, $0\in\Omega$, $\Phi:\Omega\to\Omega$ be completely continuous operator. Then, either*
(i) Φ has at least one fixed point in Ω;
or
(ii) the set $\{x\in\Omega \mid u=\lambda\Phi u, 0<\lambda<1\}$ is unbounded.

2. Background

To prove the main existence results we will employ several straightforward lemmas.

Lemma 2.1. *Let $2T\neq\alpha\eta^2$. Then for $y\in C([0,T],[0,\infty))$, the problem*

$$u'''(t)+y(t)=0, \tag{2.1}$$

$$u(0)=u''(0)=0,\, u(T)=\alpha\int_0^\eta u(s)ds, \quad \eta\in(0,T), \quad \alpha>0, \tag{2.2}$$

has a unique solution given by

$$\begin{aligned} u(t) &= \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 y(s)ds - \frac{\alpha t}{3\left(2T-\alpha\eta^2\right)}\int_0^\eta (\eta-s)^3 y(s)ds \\ &\quad -\frac{1}{2}\int_0^t (t-s)^2 y(s)ds. \end{aligned}$$

Proof. From equation (2.1) we have $u'''(t)=-y(t)$. Then, integrating from 0 to t we obtain

$$u''(t)=-\int_0^t y(s)ds.$$

For $t\in[0,T]$ we have, by integrating in t and using integration by parts,

$$\begin{aligned} u'(t) &= u'(0)-\int_0^t\left(\int_0^x y(s)ds\right)dx \\ &= u'(0)-\int_0^t (t-s)y(s)ds \end{aligned}$$

$$\begin{aligned} u(t) &= u'(0)t-\int_0^t\left(\int_0^x (x-s)y(s)ds\right)dx \\ &= u'(0)t-\frac{1}{2}\int_0^t (t-s)^2 y(s)ds. \end{aligned} \tag{2.3}$$

Thus, for $t=T$ we find

$$u(T)=u'(0)T-\frac{1}{2}\int_0^T (T-s)^2 y(s)ds. \tag{2.4}$$

Integrating again from 0 to η the expression (2.3), where $\eta\in(0,T)$, we obtain

$$\begin{aligned} \int_0^\eta u(s)ds &= \frac{1}{2}u'(0)\eta^2-\frac{1}{2}\int_0^\eta\left(\int_0^x (x-s)^2 y(s)ds\right)dx \\ &= \frac{1}{2}u'(0)\eta^2-\frac{1}{6}\int_0^\eta (\eta-s)^3 y(s)ds. \end{aligned} \tag{2.5}$$

From (2.2) and (2.4) we have

$$\int_0^\eta u(s)\,ds = \frac{1}{\alpha}u(T) = u'(0)\frac{T}{\alpha} - \frac{1}{2\alpha}\int_0^T (T-s)^2 y(s)\,ds.$$

Then, using (2.5) we see that

$$u'(0)\frac{T}{\alpha} - \frac{1}{2\alpha}\int_0^T (T-s)^2 y(s)\,ds = \frac{1}{2}u'(0)\eta^2 - \frac{1}{6}\int_0^\eta (\eta-s)^3 y(s)\,ds.$$

Thus,

$$u'(0)\left(\frac{2T-\alpha\eta^2}{2\alpha}\right) = \frac{1}{2\alpha}\int_0^T (T-s)^2 y(s)\,ds - \frac{1}{6}\int_0^\eta (\eta-s)^3 y(s)\,ds$$

or

$$u'(0) = \frac{1}{(2T-\alpha\eta^2)}\int_0^t (T-s)^2 y(s)\,ds - \frac{\alpha}{3(2T-\alpha\eta^2)}\int_0^\eta (\eta-s)^3 y(s)\,ds.$$

Therefore, the BVP (2.1)–(2.2) has a unique solution

$$u(t) = \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 y(s)\,ds - \frac{\alpha t}{3(2T-\alpha\eta^2)}\int_0^\eta (\eta-s)^3 y(s)\,ds$$
$$-\frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds.$$

□

The existence of positive solutions of the problem (2.1)–(2.2) is given in the next result.

Lemma 2.2. . *Let $0<\alpha<\frac{2T}{\eta^2}$. If $y\in C([0,T],[0,+\infty))$, then the unique solution of the problem (2.1)–(2.2) satisfies $u(t)\ge 0$ for $t\in[0,T]$.*

Proof. From $u'''(t) = -y(t)$, $t\in[0,T]$, we get that $u''(t)$ is decreasing on $[0,T]$. Then, the condition $u''(0)=0$ ensures that have $u''(t)\le 0$, $t\in[0,T]$, which implies $u(t)$ is concave. Observe also that if $u(T)\ge 0$, the concavity of u and the fact that $u(0)=0$ imply that $u(t)\ge 0$ for $t\in[0,T]$.
Since the graph of u is concave down $(0,T)$, we get

$$\int_0^\eta u(s)\,ds \ge \frac{1}{2}\eta u(\eta) \tag{2.6}$$

where $\frac{1}{2}\eta u(\eta)$ is the area of triangle under the curve $u(t)$ from $t=0$ to $t=\eta$ for $\eta\in(0,T)$.
If we assume that $u(T)<0$, then from 2.2 we have

$$\int_0^\eta u(s)\,ds < 0. \tag{2.7}$$

By concavity of u and $\int_0^\eta u(s)\,ds<0$, it implies that $u(\eta)<0$.
Hence

$$u(T) = \alpha\int_0^\eta u(s)\,ds \ge \frac{2T}{\eta^2}\times\frac{1}{2}\eta u(\eta) = \frac{T}{\eta}u(\eta),$$

which contradicts the concavity of u. □

Lemma 2.3. *Let $\alpha>\frac{2T}{\eta^2}$. If $y\in C([0,T],[0,+\infty))$, then the problem (2.1)-(2.2) has no positive solution.*

Proof. Suppose that the problem (2.1)-(2.2) has a positive solution u.
If $u(T)>0$, then $\int_0^\eta u(s)\,ds>0$. It implies that $u(\eta)>0$ and

$$\frac{u(T)}{T} = \frac{\alpha}{T}\int_0^\eta u(s)\,ds > \frac{2}{\eta^2}\left(\frac{1}{2}\eta u(\eta)\right) = \frac{u(\eta)}{\eta}$$

This contradicts the concavity of u.
If $u(T)=0$, then $\int_0^\eta u(s)\,ds=0$, this is $u(t)\equiv 0$ for all $t\in[0,\eta]$. If there exists $t_0\in(\eta,T)$ such that $u(t_0)>0$, then $u(0)=u(\eta)<u(t_0)$, which contradicts the concavity of u. Therefore, no positive solutions exist. □

Lemma 2.4. . *Let $0<\alpha<\frac{2T}{\eta^2}$. If $y\in C([0,T],[0,+\infty))$, then the unique solution of the problem (2.1)–(2.2) satisfies*

$$\min_{t\in[\eta,T]} u(t) \ge \gamma\|u\|,\ \|u\| = \max_{t\in[0,T]}|u(t)|, \tag{2.8}$$

where

$$\gamma := min\left\{\frac{\eta}{T}, \frac{\alpha\eta^2}{2T}, \frac{\alpha\eta(T-\eta)}{2T-\alpha\eta^2}\right\}. \tag{2.9}$$

Proof. Set $u(\tau)=\|u\|$. We consider three cases.
Case 1. If $\eta \le \tau \le T$ and $\min_{t\in[\eta,T]} u(t)=u(\eta)$, then the concavity of u implies that

$$\frac{u(\eta)}{\eta} \ge \frac{u(\tau)}{\tau} \ge \frac{u(\tau)}{T}$$

Thus,

$$\min_{t\in[\eta,T]} u(t) \ge \frac{\eta}{T}\|u\|.$$

Case 2. If $\eta \le \tau \le T$ and $\min_{t\in[\eta,T]} u(t)=u(T)$, then (2.2)-(2.6) and the concavity of u implies

$$u(T)=\alpha\int_0^{\eta} u(s)\,ds \ge \alpha\frac{\eta^2}{2}\left[\frac{u(\eta)}{\eta}\right] \ge \alpha\frac{\eta^2}{2}\left[\frac{u(\tau)}{\tau}\right]) \ge \frac{\alpha\eta^2}{2T}u(\tau).$$

Therefore,

$$\min_{t\in[\eta,T]} u(t) \ge \frac{\alpha\eta^2}{2T}\|u\|.$$

Case 3. If $\tau \le \eta \le T$, then $\min_{t\in[\eta,T]} u(t)=u(T)$. Using the concavity of u and (2.2)-(2.6), we have

$$\begin{aligned} \frac{u(\tau)-u(T)}{\tau-T} &\ge \frac{u(T)-u(\eta)}{T-\eta} \\ u(\tau) &\le u(T)+\frac{u(T)-u(\eta)}{T-\eta}(\tau-T) \\ u(\tau) &\le u(T)+\frac{u(T)-u(\eta)}{T-\eta}(0-T) \\ &\le u(T)\left[1-T\frac{1-\frac{2}{\alpha\eta}}{T-\eta}\right] \\ &= u(T)\left[\frac{2T-\alpha\eta^2}{\alpha\eta(T-\eta)}\right]. \end{aligned} \tag{2.10}$$

This implies that

$$\min_{t\in[\eta,T]} u(t) \ge \frac{\alpha\eta(T-\eta)}{2T-\alpha\eta^2}\|u\|.$$

This completes the proof. □

3. Main results

In this section, we establish the existence of positive solution for the (BVP) (1.1)-(1.2).

Let

$$E=C[0,T],\ \beta=\int_0^T (T-s)^2 a(s)\,ds$$

Theorem 3.1. *Assume (H1) and (H2) hold and $0<\alpha<\frac{2T}{\eta^2}$. If $f_0=0$, then the problem (1.1)-(1.2) has at least one positive solution.*

Proof. From Lemma 2.1, u is a solution to the boundary value problem (1.1)-(1.2) if and only if u is a fixed point of operator A, where A is defined by

$$\begin{aligned} Au(t) &= \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) f(s,u(s))\,ds \\ &\quad -\frac{\alpha t}{3\left(2T-\alpha\eta^2\right)}\int_0^{\eta} (\eta-s)^3 a(s) f(s,u(s))\,ds-\frac{1}{2}\int_0^t (t-s)^2 a(s) f(s,u(s))\,ds. \end{aligned} \tag{3.1}$$

Denote that

$$\Omega=\left\{u \mid u\in C([0,T],\mathbb{R}),\ u\ge 0,\ \min_{t\in[\eta,T]} u(t)\ge\gamma\|u\|\right\},$$

where γ is defined in (2.9). Then Ω is the convex subset of E.
We choose $\varepsilon>0$ and $\varepsilon \le \frac{2T-\alpha\eta^2}{T\beta}$. By $f_0=0$, it there exists constant $M>0$, such that $f(u)<\varepsilon u$ for $0<u<M$. For $u\in\Omega$, from Lemma 2.2 and Lemma 2.4, we have $Au(t)\ge 0$ and $\min_{t\in[\eta,T]} Au(t)\ge\gamma\|Au\|$.

On the other hand,

$$
\begin{aligned}
Au(t) &\leq \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) f(u(s))\, ds \\
&\leq \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\varepsilon u(s)\, ds \\
&\leq \|u\| \frac{T\varepsilon}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\, ds \\
&\leq \|u\| \leq M.
\end{aligned}
$$

Thus $\|Au\| \leq \|u\|$. $u \in K \cap \partial\Omega_1$. Hence $A\Omega \subset \Omega$. It easy to check that $A : \Omega \to \Omega$ is completely continuous. For $u \in \Omega$ and $0 < \lambda < 1$, we have $u(t) = \lambda Au(t) < Au(t) \leq M$, which implies $\|u\| \leq M$. So $\{u \in \Omega \mid u = \lambda Au,\ 0 < \lambda < 1\}$ is bounded. By Theorem 1.1 the operator A has at least one fixed point in Ω. Thus the problem (1.1)-(1.2) has at least one positive solution. The proof is complete. □

Theorem 3.2. *Assume (H1) and (H2) hold, and $0 < \alpha < \frac{2T}{\eta^2}$. If $f_\infty = 0$, then the problem (1.1)-(1.2) has at least one positive solution.*

Proof. Choose $\varepsilon < \frac{2T-\alpha\eta^2}{2T\beta}$. By $f_\infty = 0$, we know there exists Constant N, such that $f(u) < \varepsilon u$ for $u > N$.
Select

$$
M \geq N + 1 + \frac{2T\beta}{2T-\alpha\eta^2}\max_{0\leq u\leq N} f(u)
$$

Let

$$
\Omega = \left\{ u \mid u \in C[0,T],\ u \geq 0,\ \|u\| \leq M,\ \min_{t\in[\eta,T]} u(t) \geq \gamma\|u\| \right\},
$$

then Ω is the convex subset of E. For $u \in \Omega$, by Lemma 2.2 and Lemma 2.4 we know $Au(t) \geq 0$ and $\min_{t\in[\eta,T]} Au(t() \geq \gamma\|Au\|)$.
On the other hand,

$$
\begin{aligned}
Au(t) &\leq \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) f(u(s))\, ds \\
&\leq \frac{T}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\varepsilon u(s)\, ds \\
&= \frac{T}{2T-\alpha\eta^2}\int_{I_1=\{s\in[0,T],\,u(s)>N\}} (T-s)^2 a(s) f(u(s))\, ds + \frac{T}{2T-\alpha\eta^2}\int_{I_2=\{s\in[0,T],\,u(s)\leq N\}} (T-s)^2 a(s) f(u(s))\, ds \\
&\leq \frac{T}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\varepsilon u(s)\, ds + \frac{T}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\, ds. \max_{0\leq u\leq N} f(u) \\
&\leq \frac{T\varepsilon}{2T-\alpha\eta^2}\|u\|\int_0^T (T-s)^2 a(s)\, ds + \frac{T}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\, ds. \max_{0\leq u\leq N} f(u) \\
&\leq \frac{T\varepsilon}{2T-\alpha\eta^2} M \int (T-s)^2 a(s)\, ds + \frac{T}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s)\, ds. \max_{0\leq u\leq N} f(u) \\
&\leq \frac{T\varepsilon}{2T-\alpha\eta^2} M\beta + \frac{T}{2T-\alpha\eta^2}\beta \max_{0\leq u\leq N} f(u) \\
&\leq \frac{1}{2}M + \frac{1}{2}M = M.
\end{aligned}
$$

Thus $\|Au\| \leq M$. Hence, $A\Omega \subset \Omega$. IT easy to check that $A : \Omega \to \Omega$ is completely continuous.
For $u \in \Omega$ and $u = \lambda Au$, $0 < \lambda < 1$, we have $u(t) = \lambda Au(t) < Au(t) \leq M$, which implies $\|u\| \leq M$. So, $\{u \in \Omega : u = \lambda Au,\ 0 < \lambda < 1\}$ is bounded. By Theorem 1.1, we know the operator A has at least one fixed point in Ω. Thus the problem (1.1)-(1.2) has at least one positive solution. The proof is complete. □

Theorem 3.3. *Assume (H1) and (H2) hold, and $0 < \alpha < \frac{2T}{\eta^2}$. If there exists constant $\rho_1 > 0$, such that $f(u) \leq \frac{(2T-\alpha\eta^2)\rho_1}{T\beta} T\beta$ for $0 < u < \rho_1$, then the problem (1.1)-(1.2) has at least one positive solution.*

Proof. Let $\Omega = \left\{ u \mid u \in C[0,1],\ u \geq 0,\ \|u\| \leq \rho_1,\ \min_{t\in[\eta,T]} u(t) \geq \gamma\|u\| \right\}$, then Ω is the convex subset of E.
For $u \in \Omega$, by Lemma 2.2 and Lemma 2.4, we have

$$
Au(t) \geq 0 \text{ and } \min_{t\in[\eta,T]} Au(t) \geq \gamma\|Au\|. \tag{3.2}
$$

On the other hand

$$
\begin{aligned}
Au(t) &\leq \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 a(s) f(u(s)) \\
&\leq \frac{t}{2T-\alpha\eta^2}\int_0^T (T-s)^2 \frac{(2T-\alpha\eta^2)\rho_1}{T\beta}\, ds = \rho_1.
\end{aligned}
$$

Then $\|Au\| \leq \rho_1$. Hence, $A\Omega \subset \Omega$. It easy to check yhat $A : \Omega \to \Omega$ is completely continuous.
For $u \in \Omega$ and $u = \lambda Au$, $0 < \lambda < 1$, we have $u(t) = \lambda Au(t) < Au(t) \leq \rho_1$, which implies $\|u\| \leq d$. So $\{u \in \Omega : u = \lambda Au,\ 0 < \lambda < 1\}$ is bounded. By Theorem 1.1, we know the operator A has at least one fixed point in Ω. Thus the problem (1.1)-(1.2) has at least one positive solution. The proof is complete. □

Theorem 3.4. *Assume (H1) and (H2) hold, and* $0<\alpha<\frac{2T}{\eta^2}$. *If there exists constant* $\rho_2>0$, *such that* $f(u)\leq\frac{(2T-\alpha\eta^2)\rho_2}{T\beta}$ *for* $0<u<\rho_1$, *then the problem (1.1)-(1.2) has at least one positive solution.*

Proof. Choose

$$d>1+\rho_2+\frac{T\beta}{2T-\alpha\eta^2}\cdot\max_{0\leq u\leq\rho_2}f(u).$$

Let

$$\Omega=\left\{u\mid u\in C[0,T],\,u\geq 0,\,\|u\|\leq d,\,\min_{t\in[\eta,T]}u(t)\geq\gamma\|u\|\right\},$$

then Ω is the convex subset of E.
For $u\in\Omega$, by Lemma 2.2 and Lemma 2.4, we know $Au(t)\geq 0$ and $\min_{t\in[\eta,T]}Au(t)\geq\gamma\|Au\|$.
On the other hand,

$$\begin{aligned}Au(t)&\leq\frac{t}{2T-\alpha\eta^2}\int_0^T(T-s)^2a(s)f(u(s))ds\\&\leq\frac{T}{2T-\alpha\eta^2}\int_0^T(T-s)^2a(s)f(u(s))ds\\&=\frac{T}{2T-\alpha\eta^2}\int_{I_1=\{s\in[0,T],\,u(s)>\rho_2\}}(T-s)^2a(s)f(u(s))ds+\frac{T}{2T-\alpha\eta^2}\int_{I_2=\{s\in[0,T],\,u(s)\leq\rho_2\}}(T-s)^2a(s)f(u(s))ds\\&\leq\frac{T}{2T-\alpha\eta^2}\int_0^T(T-s)^2a(s)\frac{(2T-\alpha\eta^2)\rho_2}{T\beta}ds+\frac{T}{2T-\alpha\eta^2}\int_0^T(T-s)^2a(s)\cdot\max_{0\leq u\leq\rho_2}f(u)ds\\&\leq\rho_2+\frac{T\beta}{2T-\alpha\eta^2}\cdot\max_{0\leq u\leq\rho_2}f(u)<d.\end{aligned}$$

Thus $\|Au\|\leq d$. Hence $A\Omega\subset\Omega$. It easy to check that the operator A is completely continuous. For $u\in\Omega$ and $u=\lambda Au$, $0<\lambda<1$, we have $u(t)=\lambda Au(t)<Au(t)\leq d$, which implies $\|u\|\leq d$. So $\{u\in\Omega:\,u=\lambda Au,\,0<\lambda<1\}$ is bounded. By Theorem 1.1, we know the operator A has at least one fixed point in Ω. Thus the problem (1.1)-(1.2) has at least one positive solution. The proof is complete. □

4. Examples

Example 4.1. *Consider the boundary value problem*

$$u'''(t)+\frac{t^2u}{t+e^u}=0,\quad 0<t<\frac{5}{4},\tag{4.1}$$

$$u(0)=0,\,u''(0)=0,\,u\left(\frac{5}{4}\right)=35\int_0^{\frac{1}{4}}u(s)ds,\tag{4.2}$$

where $\alpha=35$, $\eta=\frac{1}{4}$, $T=\frac{5}{4}$, $0<\alpha=35<40=\frac{2T}{\eta^2}$, $f(t,u)=\frac{u}{t+e^u}\in C([0,T]\times[0,\infty),[0,\infty))$ *and* $a(t)=t^2>0$ *for* $t\in\left[\frac{1}{4},\frac{5}{4}\right]$. *Since* $f_\infty=0$ *and from Theorem 3.2, we can get that the (4.1)- (4.2) has at least one positive solution. Consequently, we cannot apply the Krasnoselskii's fixed point theorem like in [10, 17, 19]*

Example 4.2. *Consider the boundary value problem*

$$u'''(t)+e^t\left(u-\frac{u}{\sqrt{1+u}}\right)=0,\quad 0<t<\frac{3}{4},\tag{4.3}$$

$$u(0)=0,\,u''(0)=0,\,u\left(\frac{3}{4}\right)=15\int_0^{0,2}u(s)ds,\tag{4.4}$$

where $\alpha=15$, $\eta=0,2=\frac{1}{5}$, $T=\frac{3}{4}$, $0<\alpha=15<37,5=\frac{2T}{\eta^2}$, $f(t,u)=u-\frac{u}{\sqrt{1+u}}\in C([0,T]\times[0,\infty),[0,\infty))$ *and* $a(t)=e^t>0$ *for* $t\in\left[\frac{1}{5},\frac{3}{4}\right]$. *Obviously* $f_0=0$. *From Theorem 3.1, the (4.3)-(4.4) has at least one positive solution. On the other hand, we have* $f_0=1$ *, then the function f is not superlinear. Consequently, we cannot apply the Krasnoselskii's fixed point theorem like in [10, 17, 19]*

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