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EXTENDED GRAM-SCHMIDT PROCESS ON SESQUILINEAR SPACES OVER FINITE FIELDS

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Abstract. Orthonormal bases play an important role in the geometric study of vector spaces. For inner product spaces over real or complex number fields, we can apply Gram-Schmidt algorithm to construct an orthonormal subset from a linearly independent subset. However, on sesquilinear spaces over finite fields, Gram-Schmidt algorithm fails to produce an orthonormal subset because of the presence of non-zero, self-orthogonal vectors. In fact, there is a subspace that does not contain an orthonormal basis. In this paper, we study sesquilinear spaces over finite fields and show that a non-zero subspace has an orthonormal basis if and only if it is non-degenerate. An Extended Gram-Schmidt Process (EG-SP) is then discussed to construct an orthogonal subset from a linearly independent subset having equal generated subspaces. An advantage of the proposed EG-SP is that the obtained orthogonal subset is orthonormal when the generated subspace is non-degenerate. In addition, we can also extend an orthonormal subset of a sesquilinear space to an orthonormal basis.

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1. Introduction

Inner product notion has a significant role in the geometric study of vector spaces, including the study concerning norm and orthogonality. In recent years, generalizations of inner product were developed and studied including semi-inner product, indefinite inner product and sesquilinear form [10]. Many of these notions are generalizations of inner product in vector spaces over real or complex fields. In the case of the underlying field being finite, inner product notion has been generalized to sesquilinear form [2] or Euclidean and Hermitian inner product [9]. Furthermore, geometric study including semi-norm and orthogonality can also be investigated in those spaces.

Meanwhile, vector spaces over finite fields play a fundamental role in some of the most fascinating applications of modern algebra to the real world. These applications occur in the general area of data communication, a vital concern in our information society [6]. Therefore, the study of vector spaces over finite fields is very important to be developed. In this article, we deal with finite dimensional sesquilinear spaces over the finite field $\mathbb{F}_{q^2}$, the finite field having order q^2 where $q = p^k$ for some odd prime p, and $k \in \mathbb{N}$. This field is unique up to isomorphism [5]. The field $\mathbb{F}_{q^2}$ can be considered as Galois extension of degree 2 over the field $\mathbb{F}_q$. This extension was introduced by Coons, *et al.* [4], where q prime and $q \equiv 3 \mod 4$ and generalized by Ballico in [1] for $\mathbb{F}_{q^2}$ where $q = p^k$, p is a prime number and $k \in \mathbb{N}$.

On the field $\mathbb{F}_{q^m}$ for some positive integer $m \geq 2$, if $\alpha \in \mathbb{F}_{q^m}$, the $m-1$ elements $\alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{m-1}}$ are called conjugates of α [6]. We define $\bar{\alpha} = \alpha^{q+q^2+\ldots+q^{m-1}}$ and norm of α denoted by $|\alpha|$ as $\alpha\bar{\alpha}$. With this definition we get $\overline{(\bar{\alpha})} = |\alpha|^{m-2}\alpha$. Thus, for the case $m = 2$, we obtain $\bar{\alpha} = \alpha^q$ and $\overline{(\bar{\alpha})} = \alpha$ and for any $\alpha \in \mathbb{F}_{q^2}$.

One of the applications of vector spaces over finite fields is in coding theory. Since linear codes can be viewed as vector spaces, their structures are easier to describe and handle. By knowing the basis of linear codes, we can express its codewords explicitly [7]. Having a basis for a given linear code or its dual can be useful in encoding and decoding algorithms. Wilson in [11] developed algorithm to find standard basis of finite-dimensional Hermitian form. A basis B of V is called standard if every $x \in B$ has at most one $y_x \in B$ such that their Hermitian product is non-zero. He also proved the existence of standard basis. Clear that orthonormal and orthogonal bases are a special case of standard basis. Moreover, if the basis of the code is orthonormal, then certain calculations become easier to handle. In this article, it is shown that any non-zero subspace of a sesquilinear space over finite fields has an orthogonal basis. In contrast, not every subspace of a sesquilinear space has an orthonormal basis. We will be more specific about orthonormal basis, investigate its existence, and develop an algorithm for obtaining it.

For the class of inner product spaces over real or complex fields, Gram-Schmidt process is a well known process to obtain an orthonormal basis of a given linear independent subset [8]. However, Gram-Schmidt method fails to transform a linearly independent subset of a sesquilinear space over a finite field to become an orthonormal subset because of the presence of self-orthogonal vectors. For sesquilinear spaces $\mathbb{F}_{q^2}^n$ where q is odd prime, equipped with dot product, Soules method was proposed to construct an orthonormal basis for a given non-self-orthogonal vector [3]. Surely, Soules method addressed the above issues. Nevertheless, it remains an open problem: can and how we construct an orthonormal basis for a given subspace? And also, can we extend an orthonormal subset to become an orthonormal basis for the whole vector space?[2]. Moreover, it also remains an open problem whether every non-zero subspace of a sesquilinear space over a finite field has an orthonormal basis. For example, one dimensional subspace generated by self-orthogonal vector does not have an orthonormal basis. In fact, we also find counterexample of [3, Lemma 6.2] for subspaces with dimension greater than one.

The aim of this article is to derive an equivalent condition for the existence of an orthonormal basis for a subspace of a finite dimensional sesquilinear space over a finite field. Further, we upgrade the Gram-Schmidt Process to transform a linearly independent subset into an orthogonal subset having equal generated subspaces. As a result of the EG-SP, we obtain an orthonormal subset for a given linearly independent subset whose its generated subspace is non-degenerate. Another result is a positive answer to the problem that any orthonormal subset of a sesquilinear space can be extended to an orthonormal basis.

2. Sesquilinear spaces

Let p be an odd prime number and $q = p^k$ where $k \in \mathbb{N}$. Throughout this article, unless it is stated otherwise, F shall denote $\mathbb{F}_{q^2}$, the finite field having order q^2. The $\mathbb{F}_q$-conjugate of an element $\alpha \in F$ is defined to be the element $\bar{\alpha} = \alpha^q$ and $\mathbb{F}_q$-norm of α defined to be the product of α with its conjugate, $|\alpha| = \alpha\overline{\alpha}$.

Let $A \in F^{n\times m}$. Similar to the complex matrix case, the $m \times n$ matrix obtained by transpose-conjugate action on A is called the adjoint of A and denoted by A^*. Matrix $A \in F^{n\times n}$ is called self-adjoint if $A^* = A$. In this article we deal with finite dimensional sesquilinear spaces over finite fields defined as the following.

Definition 2.1. Let V be a finite dimensional vector space over the field F. A mapping $[\cdot,\cdot] : V \times V \to F$ is called a sesquilinear product on V if the following conditions hold.

(1) $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$ for all $x, y, z \in V$ and $\alpha, \beta \in F$.
(2) $[x, y] = \overline{[y, x]}$, for all $x, y \in V$.
(3) If $x \in V$ satisfies $[x, y] = 0$ for all $y \in V$, then $x = 0$.

A sesquilinear space is a vector space equipped with a sesquilinear product.

Remark 2.2. We say the mapping $[\cdot,\cdot] : V \times V \to F$ that satisfies condition 3 as non-degenerate product. A subspace of a sesquilinear space such that the condition 3 holds in this subspace is called a non-degenerate subspace. Hence, a subspace of a sesquilinear space is non-degenerate if and only if the restriction of the product on the subspace is sesquilinear product or equivalently, if it is a sesquilinear space with respect to the product on the space.

The n-dimensional vector space consists of all n-column vectors is denoted by F^n. The mapping $\langle\cdot,\cdot\rangle : F^n \times F^n \to F$ defined as $\langle x, y\rangle = y^*x,\ \forall x, y \in F^n$ is a sesquilinear product on F^n ([3], [9]) and it is called dot product or Hermitian inner product or the standard sesquilinear product on F^n. Given a self-adjoint matrix $A \in F^{n\times n}$ with $\det(A) \neq 0$, we can also define a sesquilinear product on F^n corresponding to A as follows.

Theorem 2.3. *Let $A \in F^{n\times n}$ be a non-singular self-adjoint matrix and define a mapping $\langle\cdot,\cdot\rangle_A : F^n \times F^n \to F$ by $\langle x, y\rangle_A = \langle Ax, y\rangle,\ \forall x, y \in F^n$. Then $\langle\cdot,\cdot\rangle_A$ is a sesquilinear product on F^n.*

Proof. Linearity on the first term is obvious. Further, $\forall x, y \in F^n$,

$$\overline{\langle y, x\rangle_A} = \overline{\langle Ay, x\rangle} = \overline{x^* Ay} = \bar{y}A^*\overline{x^*} = y^*Ax = \langle Ax, y\rangle = \langle x, y\rangle_A.$$

Note that if $x \in F^n$, satisfies $\langle x, y\rangle_A = 0$ for all $y \in F^n$, then $\langle Ax, y\rangle = 0$. According to the fact that $\langle\cdot,\cdot\rangle$ is a sesquilinear product, we get $Ax = 0$ and so $x = 0$ since A is non-singular. Thus, $\langle\cdot,\cdot\rangle_A$ is a sesquilinear product on F^n. □

Theorem 2.4. *Let $[\cdot,\cdot] : F^n \times F^n \to F$ be a sesquilinear product on F^n. Then there exists a unique non-singular self-adjoint matrix A such that $[\cdot,\cdot] = \langle\cdot,\cdot\rangle_A$.*

Proof. Let $B = \{b_1, \ldots, b_n\}$ be a basis for F^n. Let $x = \sum_{j=1}^n \alpha_j b_j$ and $y = \sum_{i=1}^n \beta_i b_i \in F^n$. Note that

$$\begin{aligned}
[x, y] &= \left[\sum_{j=1}^n \alpha_j b_j, \sum_{i=1}^n \beta_i b_i\right] \\
&= \sum_i \sum_j \overline{\beta_i}\alpha_j [b_j, b_i] \\
&= \overline{\beta_1}\sum_j \alpha_j [b_j, b_1] + \overline{\beta_2}\sum_j \alpha_j [b_j, b_2] + \ldots + \overline{\beta_n}\sum_j \alpha_j [b_j, b_n] \\
&= (\bar{\beta}_1\bar{\beta}_2 \ldots \bar{\beta}_n)\begin{pmatrix} \sum_j \alpha_j [b_j, b_1] \\ \sum_j \alpha_j [b_j, b_2] \\ \vdots \\ \sum_j \alpha_j [b_j, b_n] \end{pmatrix} \\
&= (\bar{\beta}_1\bar{\beta}_2 \ldots \bar{\beta}_n)\begin{pmatrix} [b_1, b_1] & [b_2, b_1] & \ldots & [b_n, b_1] \\ [b_1, b_2] & [b_2, b_2] & \ldots & [b_n, b_2] \\ \vdots & \vdots & \ddots & \vdots \\ [b_1, b_n] & [b_2, b_n] & \ldots & [b_n, b_n] \end{pmatrix}\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.
\end{aligned}$$

Choose $A = (a_{ij})$ where $a_{ij} = [b_j, b_i],\ \forall i, j = 1, 2, \ldots, n$. We obtain

$$[x, y] = \langle Ax, y\rangle = \langle x, y\rangle_A.$$

Assume for the contrary, i.e., A is singular, then there exists $x \in F^n, x \neq 0$ such that $Ax = 0$. Therefore, for all $y \in F^n$, we get $[x, y] = \langle Ax, y\rangle = \langle 0, y\rangle = 0$, a contradiction to the non-degenerate condition of the product. We can see that

A is self-adjoint since for all $1 \le i, j \le n$, we get $[b_i, b_j] = \overline{[b_j, b_i]}$. Finally, the uniqueness of the matrix A is obtained from the fact that if A' is a matrix that satisfies the theorem, then the (i, j)-th component of A' is none other than $[b_j, b_i]$. Thus $A' = A$. □

Remark 2.5. The matrix A above depends on the choice of the basis. If the standard basis is used, then A is the identity matrix. For different sesquilinear product, A may take a different form.

Theorem 2.6. *For any n-dimensional sesquilinear space V over the field F, there exists an isomorphism $f : V \to (F^n, \langle \cdot, \cdot \rangle_A)$ that preserves product; that is,*

$$\langle f(x), f(y) \rangle_A = [x, y], \ \forall x, y \in V.$$

Proof. Let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis of V and the coordinate of any vector $x \in V$ with respect to the basis B be denoted by $[x]_B$. Certainly, the coordinate mapping $f(x) = [x]_B$, $\forall x \in V$ is an isomorphism from the space V to the space F^n. Let define $A = (a_{ij}) \in F^{n \times n}$, where $a_{ij} = [b_j, b_i]$, $\forall i, j = 1, 2, \ldots, n$. Similar to the proof of Theorem 2.4 above, it can be shown that A is a non-singular self-adjoint matrix such that for all $i, j = 1, 2, \ldots, n$ occur $[b_j, b_i] = \langle [b_j]_B, [b_i]_B \rangle_A = \langle f(b_j), f(b_i) \rangle_A$. As a consequence, we obtain $[x, y] = \langle f(x), f(y) \rangle_A$, $\forall x, y \in V$. □

As a result of the above explanation, one can represent an arbitrary sesquilinear product on F^n as a product corresponding to a non-singular self-adjoint matrix A. Any n-dimensional sesquilinear space V can be represented by a sesquilinear space F^n for a certain non-singular self-adjoint matrix. All these facts will be utilized to derive our results.

Let V be a sesquilinear space over the field F with sesquilinear product $[\cdot, \cdot]$. The norm of an element $x \in V$ is defined as $|x| = [x, x]$. A unit vector is a vector with norm 1. Elements $x, y \in V$ are called orthogonal if $[x, y] = 0$. Non-zero element $x \in V$ is called self-orthogonal if $[x, x] = 0$. By exploiting the field property F, every non-zero, non-self-orthogonal vector in V can be shown as a multiple of a unit vector.

A subset $S \subseteq V$ defined to be an orthogonal subset if for any pair $x, y \in S, x \neq y$, we have $[x, y] = 0$. An orthonormal subset is an orthogonal subset with all its elements are unit. Subsets S and T of V are orthogonal sets if for all $s \in S$, we have $[s, t] = 0$ for every $t \in T$.

Remark 2.7. By the above definition, 1-dimensional sesquilinear product space must be generated by a non-zero, non-self-orthogonal vector.

[3, Lemma 6.2] says that any subspace of F^n with dimension at least 2 contains a vector with non-zero norm. Here is a counterexample that this lemma is not true.

Example 2.8. Let $F = \mathbb{Z}_7[i]$, the extension field of $\mathbb{Z}_7$ with irreducible polynomial x^2+1. Let $V = F^4$ be the dot product space over the field F. Let $S = \text{span}(b_1, b_2)$ be the 2 dimensional subspace of V where

$$b_1 = \begin{pmatrix} 2+i & 1+i & 0 & 0 \end{pmatrix}^T, b_2 = \begin{pmatrix} 0 & 0 & 2+i & 1+i \end{pmatrix}^T.$$

Clearly $\langle b_i, b_j \rangle = 0$ for all $i, j = 1, 2$ and hence all non-zero elements in S are self-orthogonal.

In the following, we provide a modified result concerning that lemma, that is, by restricting the result to non-degenerate subspaces.

Lemma 2.9. *Let V be a non-zero sesquilinear space over the field F. Then V contains a non-zero vector that is not self-orthogonal.*

Proof. If V is a 1-dimensional sesquilinear space, then obviously, V is generated by a vector with non-zero norm. If V is an n-dimensional sesquilinear space where $n \geq 2$, then it has a basis with n-vectors, say $B = \{b_1, \dots, b_n\}$. If there is $b_i \in B$ with non-zero norm for some $i \in \{1, 2, \dots, n\}$, then we are done. Now suppose that $[b_i, b_i] = 0$ for all $i \in \{1, 2, \dots, n\}$. Since $b_1 \neq 0$, there exists $x \in V, x \neq 0$ such that $[b_1, x] \neq 0$. As a result of x being a linear combination of B and b_1 being self-orthogonal, there exists $j \in \{2, \dots, n\}$ such that $[b_1, b_j] \neq 0$. Without loss of generality, we can assume $[b_1, b_j] = 1$. Hence, for any $\mu \in F$, we obtain the norm of $b_1 + \mu b_j$ is

$$\begin{aligned} [b_1 + \mu b_j, b_1 + \mu b_j] &= [b_1, b_1] + \bar{\mu}[b_1, b_j] + \mu[b_j, b_1] + \mu\bar{\mu}[b_j, b_j] \\ &= \bar{\mu} + \mu. \end{aligned}$$

Consequently, we have $[b_1 + \mu b_j, b_1 + \mu b_j] = \bar{\mu} + \mu$ is non-zero by the choice of μ. Of course $b_1 + \mu b_j \neq 0$. Therefore, V contains a non-zero vector that is not self-orthogonal. □

Definition 2.10. Let V be a sesquilinear space and $S \subseteq V, S \neq \emptyset$. The orthogonal complement of S is the set defined as follows

$$S^\perp = \{x \in V \mid [x, s] = 0, \forall s \in S\}.$$

Remark 2.11. It is a routine to show that $S^\perp$ is a subspace of V.

In the following theorem, we provide a characterization of non-degenerate subspace through its orthogonal complement.

Theorem 2.12. *Let V be a sesquilinear space over the field F and S be a subspace of V. Then S is non-degenerate if and only if $S \cap S^\perp = \{0\}$.*

Proof. If $S \cap S^{\perp} = \{0\}$, then any vector in S which is orthogonal to all vectors $x \in S$ is only 0. Thus, S is non-degenerate. Conversely, if S is a non-degenerate subspace, then an element $x \in S$ that satisfies $[s, x] = 0$ for all $s \in S$ is only $x = 0$. This fact means that 0 is the only element in $S \cap S^{\perp}$. □

Remark 2.13. In the area of Coding Theory, a k-dimensional subspace of the space F^n is called a linear $[n, k]$ code over the field F. A linear $[n, k]$ code S on the Hermitian inner product F^n is called Hermitian Linear Complementary Dual (LCD) if $S \cap S^{\perp} = \{0\}$ [9]. Hence, we can conclude that a Hermitian LCD code is a linear code that is non-degenerate.

According to Theorem 2.12, the subspace S is non-degenerate if $S \cap S^{\perp} = \{0\}$. An alternative means to evaluate the non-degeneracy property of a subspace of V is by utilizing its basis, particularly examining a matrix containing products of the basis elements called as Gram matrix [11].

Theorem 2.14. *Let V be an n-dimensional sesquilinear space over the field F and S be a subspace of V. Let $B = \{b_1, \ldots, b_k\}$ be an order basis of S and Gram matrix $A = (a_{ij})$ where $a_{ij} = [b_j, b_i],\ i, j \in \{1, 2, \ldots, n\}$. Then $\det(A) \neq 0$ if and only if S is non-degenerate.*

Proof. Similar to the proof of Theorem 2.6, we consider the coordinate mapping with respect to the basis B, $[x]_B$ for any $x \in S$, is an isomorphism from S to the space F^k. If S is non-degenerate, then S is a sesquilinear space with respect to the sesquilinear product of V. Hence, following the proof of Theorem 2.6, we obtain $\det(A) \neq 0$.

Conversely, let $det(A) \neq 0$ which implies that the product $\langle \cdot, \cdot \rangle_A$ is a sesquilinear on F^k. Let x, y be arbitrary elements in S. Then, $x = \sum_{j=1}^{k} \alpha_j b_j$ and $y = \sum_{i=1}^{k} \beta_i b_i$ for some $\alpha_j, \beta_i \in F$. Thus, we obtain

$$[x, y] = \left[\sum_{j=1}^{k} \alpha_j b_j, \sum_{i=1}^{k} \beta_i b_i \right] = [y]_B^* A [x]_B = \langle [x]_B, [y]_B \rangle_A.$$

Therefore, if $\langle \cdot, \cdot \rangle_A$ is non-degenerate on F^k, then $[\cdot, \cdot]$ is also non-degenerate on S. □

3. Orthonormal bases

This section explains necessary and sufficient conditions for a subspace of a sesquilinear space to have an orthonormal basis. Suppose S is a non-degenerate subspace of F^n. The only vector in S that is orthogonal to all vectors in S is the zero vector. Consequently, if $B = \{b_1, \ldots, b_k\}$ is a basis of S and $x \in S$ such that

$[x, b_i] = 0$ for all $b_i \in B$, then we must have $x = 0$. This observation gives us the following lemma.

Lemma 3.1. *Let V be a non-zero sesquilinear space over the field F and $S \subseteq V$ be a non-degenerate subspace of V. Then S does not have a basis which is simultaneously orthogonal and self-orthogonal.*

Proof. Suppose on the contrary, there exists $B = \{b_1, \ldots, b_k\}$ a basis of S which is simultaneously orthogonal and self-orthogonal. Particularly, we have $[b_1, b_i] = 0$ for all $b_i \in B$. By the above discussion, this implies $b_1 = 0$, a contradictory fact to the assumption that b_1 is an element of the basis B. □

The following theorem will be useful for our purpose to get an orthonormal subset of vectors that spans a subspace of F^n. Moreover, every vector in that subset is not self-orthogonal.

Theorem 3.2. *Let V be an n-dimensional sesquilinear space and $S = span\{u\}$ for some $u \in V$ with $[u, u] \neq 0$. Then $V = S \oplus S^\perp$, $\dim(S^\perp) = n - 1$ and $S^\perp$ is non-degenerate.*

Proof. Clearly, S is non-degenerate, hence $S \cap S^\perp = \{0\}$. Define a linear functional $\lambda(v) = [v, u]$, $\forall v \in V$. Since $[u, u] \neq 0$, λ is a non-zero mapping. According to Rank-Nullity Theorem, $null(\lambda) = n - 1$. Since $ker(\lambda) = S^\perp$, we get $dim(S^\perp) = n - 1$. Hence $V = S \oplus S^\perp$. Further, we will show that $S^\perp$ is non-degenerate. Suppose on the contrary, $S^\perp$ being degenerate. Then, there exists $x \in S^\perp, x \neq 0$ which is orthogonal to all elements in $S^\perp$. Since $V = S \oplus S^\perp$, we obtain x being orthogonal to all elements in V, a contradictory fact to the assumption that V is a sesquilinear space. Thus $S^\perp$ must be non-degenerate. □

There is a necessary and sufficient condition for the existence of an orthonormal basis in sesquilinear subspace. The details are given by two theorems below.

Theorem 3.3. *Any n-dimensional sesquilinear space with $n \geq 1$ has an orthonormal basis.*

Proof. The theorem will be proved by induction on the dimension of the sesquilinear space V. If $\dim(V) = 1$, then the non-degeneracy of V implies $V = \text{span}\{v\}$ for some non self-orthogonal vector v. Thus V has an orthonormal basis. Assume that the theorem is true for every space V with $1 \leq \dim(V) \leq k - 1$. Let $\dim(V) = k$. According to Lemma 2.9, there exists $x_1 \in V$, $x_1 \neq 0$ and $[x_1, x_1] \neq 0$. Define $S_1 = \text{span}\{x_1\}$. According to Theorem 2.12, $V = S_1 \oplus S_1^\perp, \dim(S_1^\perp) = k - 1$ and $S_1^\perp$ is non-degenerate. Hence, $S_1^\perp$ is a $k-1$-dimensional sesquilinear space and so, according to the induction hypothesis, we know that $S_1^\perp$ has an orthonormal basis.

By combining the orthonormal bases of S_1 and $S_1^{\perp}$, we obtain an orthonormal basis of V. □

Theorem 3.4. *Let V be an n-dimensional sesquilinear space and $S \subset V$ be a non-zero subspace of V. Then S has an orthonormal basis if and only if S is non-degenerate.*

Proof. If S is non-degenerate, then S equipped with sesquilinear product restricted on S is a sesquilinear space. Hence according to Theorem 3.3, S has an orthonormal basis. Conversely, let S has an orthonormal basis $B = \{b_1, b_2, \ldots, b_k\}$. For any vector $x \in S, x \neq 0$, then $x = \sum_{i=1}^{k} \alpha_i b_i$ for some $\alpha_i \in F, i \in \{1, 2, \ldots, k\}$ with $\alpha_j \neq 0$ for some j. In this case, we obtain $[x, b_j] = \alpha_j \neq 0$. Thus S is non-degenerate. □

Remark 3.5. From Example 2.8, we can conclude that S is not only two-dimensional space with self-orthogonal vectors but also does not have orthonormal bases. Based on Theorem 2.14, we can also said that a non-zero space with non-singular Gram matrix has orthonormal bases.

Let $S \subset V$ be a non-zero non-degenerate subspace of a finite dimensional sesquilinear space V. The existence of an orthonormal basis of S is enable us to define an orthogonal projection on S. Let $B = \{b_1, b_2, \ldots, b_k\}$ be an orthonormal basis of S. The mapping defined on V, $p(v) = \sum_{i=1}^{k} [v, b_i] b_i$, $\forall v \in V$ is an orthogonal projection on S. Particularly, we have $V = S \oplus S^{\perp}$ where $S = im(p)$ and $S^{\perp} = ker(p)$. Hence, we obtain the following corollary which extends Theorem 2.12.

Corollary 3.6. *Let V be an n-dimensional sesquilinear space and $S \subset V$ be a non-zero subspace of V. Then $V = S \oplus S^{\perp}$ if and only if S is non-degenerate.*

Further consequences is shown in the following corollary which answers one of the open problems mentioned in the introduction section.

Corollary 3.7. *Let V be an n-dimensional sesquilinear space and $B = \{b_1, \ldots, b_k\}$ be an orthonormal subset, i.e., $[b_i, b_i] = 1$ and $[b_i, b_j] = 0, \forall i \neq j$. Then B can be extended into an orthonormal basis of V.*

Proof. Define $S = \mathrm{Span}(B)$, then S is a non-degenerate subspace. Hence $V = S \oplus S^{\perp}$. Since any non-degenerate subspace has an orthonormal basis, the proof is complete if we can show that $S^{\perp}$ is also non-degenerate. Suppose on the contrary, that is, $S^{\perp}$ being degenerate, then there exists $x \in S^{\perp}$ non-zero element that is orthogonal to all elements in $S^{\perp}$. Since x is also orthogonal to all elements in S, we obtain x is orthogonal to all elements in V, a contradictory statement to the fact that V is non-degenerate. Thus $S^{\perp}$ is a non-degenerate subspace. □

Remark 3.8. We can generalized Corollary 3.7 as follows: Suppose S, T be non-degenerate subspaces of a sesquilinear space V with $S \subseteq T$. If B is an orthonormal basis of S, then B can be extended to an orthonormal basis of T.

4. Extended Gram-Schmidt process

In this section we upgrade the Gram-Schmidt Process to transform a linearly independent subset of a finite dimensional sesquilinear space to an orthogonal subset with equal generated subspaces. Further, if the subspace generated by the given linearly independent subset is non-degenerate, the obtained orthogonal subset is orthonormal.

On inner product spaces over real or complex number fields, Gram-Schmidt Process is an algorithm to construct an orthonormal subset from a given linearly independent subset. The basic principle of Gram-Schmidt Process is the construction of a chain of orthonormal subsets recursively by utilizing orthogonal projections on subspaces generated by the obtained orthonormal subsets. Our proposed Extended Gram-Schmidt Process (EG-SP) maintains this principle by modifying the utilized orthogonal projections and paying attention to the obtained vectors when they are used to construct the orthogonal projections. EG-SP will transform a linearly independent subset of a sesquilinear space to an orthogonal subset whose generated subspace is equal to the subspace generated by the given independent subset.

Extended Gram-Schmidt Process
Let V be a sesquilinear space over the field F with sesquilinear product $[\cdot, \cdot]$ and $B = \{b_1, b_2, \ldots, b_k\}$ be a linearly independent subset of V.
Let $B_1 = B$.

(1) Construction vector o_1
If $[b_1, b_1] \neq 0$, define o_1 to be the unit vector correspondence to b_1.
Otherwise, we divide into two cases.
If $[b_1, b_i] = 0$ for all $i \in \{2, 3, \ldots, k\}$, define $o_1 = b_1$.
Otherwise, there is $i \in \{2, 3, \ldots, k\}$ such that $[b_1, b_i] \neq 0$. In this case, there exists $\mu \in F$ such that $[b_1 + \mu b_i, b_1 + \mu b_i] \neq 0$. Let o_1 be the unit vector correspondence to $b_1 + \mu b_i$.

(2) Construction p_1, a linear operator on $S_1 = \text{span}(B_1)$
Let $O_1 = \{o_1\}$, $U_1 = \text{span}(O_1)$. Let p_1 be the linear operator on $S_1 = \text{span}(B_1)$ defined as

$$p_1(s) = [s, o_1]o_1, \ \forall s \in S_1.$$

Note that, if U_1 is non-degenerate, p_1 is the orthogonal projection operator on the space U_1, otherwise p_1 is the zero mapping on S_1.

(3) Construction subset B_2
For any $i \in \{2, \dots, k\}$, replace b_i with $v_i = b_i - p_1(b_i)$.
Clear that the subset $B_2 = \{b_2, \dots, b_k\}$ is linearly independent such that

$$S_1 = U_1 \oplus S_2$$

where $S_2 = \text{span}(B_2)$. Moreover, U_1 and S_2 are orthogonal.

(4) Repeat the above process on the subset B_i to get vector o_i and subset B_{i+1} for $i \geq 2$ until we get $O = \{o_1, o_2, \dots, o_k\}$ an orthogonal linearly independent subset such that $\text{span}(O) = \text{span}(B)$.

Remark 4.1. Below are some important facts about EG-SP.

(1) The above EG-SP produces an orthogonal and linearly independent subset O such that $\text{span}(O) = \text{span}(B)$. As a consequence, every nonzero subspace of a sesquilinear space has an orthogonal basis.
(2) If the subspace $\text{span}(B)$ is non-degenerate, then point 1. of EG-SP will produce a unit vector o_i, for $i = 1, 2, \dots, k$. Hence, the obtained subset O will be orthonormal. Conversely, if the obtained subset O is orthonormal, then the subspace $\text{span}(B)$ is non-degenerate. Hence, EG-SP can be used to evaluate the non-degeneracy property of a subspace.
(3) EG-SP is nothing but Gram-Schmidt Process when we apply it to a linearly independent subset of an inner product space over real or complex fields.

Consider the following table to see the difference between EG-SP and Gram-Schmidt.

Properties	EG-SP	Gram-Schmidt
Space	Sesquilinear Space	Inner Product Space
Output	Transform linearly independent set to orthogonal set and it becomes orthonormal if and only if the spanning set is non-degenerate	Always transform independent set into orthonormal set
Workflow	Considering non-zero norm vector in the algorithm	No need to considering non-zero norm vector in the algorithm

TABLE 1. Comparison table between EG-SP and Gram-Schmidt

The following two examples illustrate the application of EG-SP to obtain an orthonormal basis for a non-degenerate subspace.

Example 4.2. Consider $F = \mathbb{Z}_7[i]$, the extension field of the field $\mathbb{Z}_7$ with irreducible polynomial x^2+1 and $V = F^5$ is the dot product space over F. Let S be the subspace of V generated by linearly independent subset $B = \{b_1, b_2\}$ where $b_1 = \begin{pmatrix} 0 & 1-i & 2+i & 0 & 0 \end{pmatrix}^T$ and $b_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \end{pmatrix}^T$. Clear that b_1 and b_2 are self-orthogonal vectors. However, $\langle b_1, b_2 \rangle = 3$. Since $b_1 - b_2 = \begin{pmatrix} 0 & -i & 1+i & -1 & -2 \end{pmatrix}^T$ is a unit vector, let $o_1 = b_1 - b_2$ and $v_2 = b_2 - \langle b_2, o_1 \rangle o_1 = \begin{pmatrix} 0 & 1-4i & 5+4i & -3 & -6 \end{pmatrix}^T$. Clear that $\langle v_2, v_2 \rangle = 5$. Now it is left to normalize the vector v_2 by multiplying v_2 with $6-3i$, we get $o_2 = \begin{pmatrix} 0 & -6+i & 2i & 3+2i & 6+4i \end{pmatrix}^T$ and $\langle o_2, o_2 \rangle = 1$. Thus we get $O = \{o_1, o_2\}$ is an orthonormal basis of S and we conclude that S in non-degenerate.

Example 4.3. Consider $V = F^5$ the dot product space over the field $F = \mathbb{Z}_7[i]$, the extension field of $\mathbb{Z}_7$ with irreducible polynomial x^2+1. Let S be a subspace of V which has a basis $B = \{b_1, b_2, b_3\}$ where $b_1 = \begin{pmatrix} 1+2i & 1-i & 0 & 0 & 0 \end{pmatrix}^T$, $b_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \end{pmatrix}^T$, and $b_3 = \begin{pmatrix} 1 & 2i & -1 & i & 0 \end{pmatrix}^T$. We obtain all vectors b_1, b_2 and b_3 are self-orthogonal vectors. First, we have to find o_1. Since $\langle b_1, b_2 \rangle \neq 0$, consider $b_1 - b_2 = \begin{pmatrix} 1+2i & -i & -1 & -1 & -2 \end{pmatrix}^T$ with norm 5. Normalize it, we get $o_1 = \begin{pmatrix} 5+2i & -3-6i & -6+3i & -6+3i & 2+6i \end{pmatrix}^T$. Define

$$v_2 = b_2 - \langle b_2, o_1 \rangle o_1 = \begin{pmatrix} 3+5i & -2+3i & 4+3i & 4+3i & 1+6i \end{pmatrix}^T,$$

we obtain $|v_2| = 1$. Define

$$v_3 = b_3 - \langle b_3, o_1 \rangle o_1 = \begin{pmatrix} -3+2i & 2+2i & -1+5i & -i & 3i \end{pmatrix}^T,$$

we also obtain $|v_3| = 1$. Now, we are working on the subset $B_2 = \{v_2, v_3\}$. Since v_2 is a unit vector, define $o_2 = v_2$. Further, define

$$w_3 = v_3 - \langle v_3, o_2 \rangle o_2 = \begin{pmatrix} 5+3i & -2+2i & 3+3i & 4+4i & 1-i \end{pmatrix}^T,$$

we obtain the norm of w_3 is 3. Now it is left to normalize the vector w_3 and we get the corresponding unit vector of w_3 is

$$o_3 = \begin{pmatrix} 6+6i & 1+5i & 4+2i & 3+5i & 3+i \end{pmatrix}^T.$$

Finally, we get that $O = \{o_1, o_2, o_3\}$ is an orthonormal basis of S.

As we have mentioned before, the presence of an orthonormal basis of a subspace can be used to evaluate non-degeneracy property of the subspace. We can also examine the non-degeneracy property of a subspace through the Gram matrix. The above EG-SP can also be used to examine the degeneracy property of a subspace as shown in the following example.

Example 4.4. Consider $V = F^5$ the dot product space over $F = \mathbb{Z}_7[i]$, the extension field of $\mathbb{Z}_7$ with irreducible polynomial $x^2 + 1$, and let S be the subspace of V generated by a linear independent subset $B = \{b_1, b_2, b_3\}$ where $b_1 = \begin{pmatrix} 1 & 2i & -1 & i & 1 \end{pmatrix}^T$, $b_2 = \begin{pmatrix} 0 & 0 & 0 & 1+2i & 1-i \end{pmatrix}^T$, and $b_3 = \begin{pmatrix} 1+2i & 1-i & 0 & 0 & 0 \end{pmatrix}^T$. We have b_1 is a unit vector and b_2, b_3 are self-orthogonal vectors. So, let $o_1 = b_1$. Define

$$v_2 = b_2 - \langle b_2, o_1 \rangle o_1 = \begin{pmatrix} -3+2i & -4-6i & 3-2i & -1-i & -2+i \end{pmatrix}^T$$

and

$$v_3 = b_3 - \langle b_3, o_1 \rangle o_1 = \begin{pmatrix} 2+2i & 1+i & -1 & i & 1 \end{pmatrix}^T.$$

Now we are working on the subset $B_2 = \{v_2, v_3\}$. Since v_2 is a unit vector, let $o_2 = v_2$ and construct

$$w_3 = v_3 - \langle v_3, o_2 \rangle o_2 = \begin{pmatrix} 1+2i & 1-i & 0 & 1-i & 2+i \end{pmatrix}^T.$$

Now we get $B_3 = \{w_3\}$, but w_3 is a self-orthogonal vector. Hence we define $o_3 = w_3$ and we obtain that $O = \{o_1, o_2, o_3\}$ is an orthogonal basis of S.

Note that o_3 is self-orthogonal in O, so it is orthogonal to all elements in O. As a consequence, o_3 is a non-zero vector in S that orthogonal to all elements in S. Thus, we conclude that S is a degenerate subspace.

5. Conclusion

In this paper we showed that Gram-Schmidt Process can be generalized to the class of finite dimensional sesquilinear spaces over finite fields to transform any linearly independent subset to an orthogonal subset with the generated subspaces are equal. Further, the obtained orthogonal subset is orthonormal when the generated subspace is non-degenerate. Those results were derived in the restriction that the characteristic of the underlying fields are odd prime order. Hence, it is of interest for further studies, what and how are the generalization of the obtained results in the context of spaces over finite fields having characteristic 2.

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