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Some Novel Fractional Integral Inequalities for m- and (α,m)-Convex Functions

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Abstract

In this study, several novel integral inequalities are established for various classes of convex functions by employing the Caputo–Fabrizio fractional integral operator. Specifically, new inequalities are derived for m-convex and $(\alpha,m)-$convex functions. The presented results generalize and extend existing inequalities in the literature, reducing to known outcomes for certain specific parameter values. The derivations rely on the fundamental properties of the Caputo–Fabrizio fractional operator, formal definitions of different types of convexity, and standard analytical techniques.

Keywords: Caputo-Fabrizio fractional integral operator, $m-$convex functions, $(\alpha,m)-$convex functions.

1. *Introduction*

The theory of inequalities plays a fundamental role in mathematical analysis and has become a focal point of modern research due to its extensive applications in various fields such as statistics, approximation theory, optimization, and numerical analysis. Within this broad domain, convex functions occupy a central position, not only because of their intrinsic connection to inequalities through their definitions, but also due to their elegant mathematical structure and the wide range of generalizations they admit. Convexity, in its classical form, has inspired the development of numerous generalized convexity concepts, such as m-convexity and (α,m)-convexity, each tailored to capture specific functional behaviors and to extend the applicability of known inequalities. These generalized convex functions have proven useful in obtaining more refined and flexible forms of integral inequalities.

On the other hand, the theory of fractional calculus—particularly involving non-singular and non-local operators—has attracted increasing attention in recent years. Among these operators, the Caputo–Fabrizio fractional operator, characterized by its exponential kernel and absence of singularity, has emerged as a powerful tool in modeling memory-dependent processes and in deriving new analytical results.

Motivated by the growing interest in both generalized convex functions and fractional operators, this paper aims to establish several new integral inequalities for m-convex and $(\alpha,m)-$convex functions via the Caputo–Fabrizio fractional integral operator. The results presented herein not only extend known inequalities but also unify various existing findings as special cases under particular parameter choices. The derivations are based on the properties of the Caputo–Fabrizio operator, the defining characteristics of generalized convex functions, and classical tools of analysis.

We begin by recalling essential definitions and preliminary results that will be used throughout the paper.

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2. Materials and Methods

In this section, we present some essential definitions and foundational concepts that will be used in the subsequent analysis.

Definition 2.1. Let I be on interval in R. Then $f: I \to R$ is said to be convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$ (Pecaric et al. 1992).

Definition 2.2. (Toader 1984) The function $f: [0, b] \to R$, $b > 0$, is said to be $m-$convex, where $m \in [0,1]$, if we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0,1]$.

Definition 2.3. (Miheşan 1993) The function $f: [0, b] \to R$, $b > 0$ is said to be $(\alpha, m)-$ convex, where $(\alpha, m) \in [0,1]^2$, if we have

$$f(tx + m(1-t)y) \leq t^{\alpha}f(x) + m(1-t^{\alpha})f(y)$$

for all $x, y \in [0, b]$ and $t \in [0,1]$.

This class generalizes both convex and $m-$convex functions and has been widely studied due to its applications in integral inequalities (Mihesan 1993, Özdemir et all 2011).

These generalized convex functions provide greater flexibility than classical convex functions in the derivation of functional inequalities, especially when used in conjunction with fractional integral operators.

Definition 2.4. Let $f \in H^1(0, b), b > a, \alpha \in [0,1]$ then, the definition of the left and right Caputo-Fabrizio fractional integral is:

$$\left({}_{a}^{CF}I^{\alpha}\right)(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\int_{a}^{t} f(y)dy,$$

and

$$\left({}^{CF}I_{b}^{\alpha}\right)(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\int_{t}^{b} f(y)dy$$

where $B(\alpha) > 0$ is normalization function (Abdeljawad and Baleanu 2017).

Throughout the remainder of the paper, we shall refer to the normalization function by $B(\alpha)$, where it satisfies $B(0) = B(1) = 1$.

Tariq et al. (2022) established a Hermite-Hadamard type integral inequality for preinvex functions by employing the Caputo-Fabrizio fractional integral, as given below.

Theorem 2.1. Let $f: I = [k_1, k_1 + \mu(k_2, k_1)] \to (0, \infty)$ be a preinvex function on I° and $f \in L[k_1, k_1 + \mu(k_2, k_1)]$. If $\alpha \in [0,1]$, then the following inequality holds

$$\begin{aligned} & f\left(\frac{2k_1 + \mu(k_2, k_1)}{2}\right) \\ & \leq \frac{B(\alpha)}{\alpha\mu(k_2, k_1)} \\ & \times \left[{}_{k_1}^{CF}I^{\alpha}\{f(k)\} + {}^{CF}I_{k_1+\mu(k_2,k_1)}^{\alpha}\{f(k)\} \right. \\ & \qquad \left. - \frac{2(1-\alpha)}{B(\alpha)} f(k) \right] \\ & \leq \frac{f(k_1) + f(k_2)}{2} \end{aligned}$$

where $k \in [k_1, k_1 + \mu(k_2, k_1)]$.

By utilizing the Mittag-Leffler function within the framework of the Caputo-Fabrizio derivative, Atangana and Baleanu formulated new fractional derivative operators, presented as follows

Definition 2.5. (Atangana and Baleanu 2017). Let $f \in H^1(0, b), b > a, \alpha \in [0,1]$ then, the definition of the new fractional derivative is given:

(1.1) $$\left({}_{a}^{ABC}D_{t}^{\alpha}\right)[f(t)] = \frac{B(\alpha)}{1-\alpha}\int_{a}^{t} f'(x)E_{\alpha}\left[-\alpha\frac{(t-x)^{\alpha}}{(1-\alpha)}\right]dx.$$

Definition 2.6. (Atangana and Baleanu 2017). Let $f \in H^1(0, b), b > a, \alpha \in [0,1]$ then, the definition of the new fractional derivative is given:

(1.2) $$\left({}_{a}^{ABR}D_{t}^{\alpha}\right)[f(t)] = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_{a}^{t} f(x)E_{\alpha}\left[-\alpha\frac{(t-x)^{\alpha}}{(1-\alpha)}\right]dx.$$

Both Equations (1.1) and (1.2) feature non-local kernels. Additionally, Equation (1.1) yields zero when the input function is constant.

The associated fractional integral operator was defined by Atangana and Baleanu as follows.

Definition 2.7. The fractional integral associate to the new fractional derivative with non-local kernel of a function $f \in H^1(a, b)$ as defined:

$$
{}^{AB}_{a}I^{\alpha}\{f(t)\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t f(y)(t-y)^{\alpha-1}dy
$$

where, $b > a,\ \alpha \in [0,1]$ (Atangana and Baleanu 2017).

Abdeljawad and Baleanu introduced right hand side of integral operator as follows; The right fractional new integral with ML kernel of order $\alpha \in [0,1]$ is defined by

$$
{}^{AB}I^{\alpha}_{b}\{f(t)\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_t^b f(y)(y-t)^{\alpha-1}dy.
$$

where, $b > a,\ \alpha \in [0,1]$ (Abdeljawad and Baleanu 2017).

To explore the diverse formulations and applications of fractional operators, we direct interested readers to the following references, which offer detailed discussions and significant contributions to the field (Abdeljawad 2015, Abdeljawad and Baleanu 2017, Akdemir et al. 2021, Akdemir et al. 2022-Akdemir et al. 2017, Butt et al. 2020, Caputo and Fabrizio 2015, Gürbüz et al. 2020, Rashid et al 2020, Samko et al. 1993, Set 2012, Set et al. 2017).

3. *Results*

Theorem 3.1. Let $I \subseteq \mathbb{R}$. Suppose that $f: [a,b] \subseteq I \longrightarrow \mathbb{R}$ is a m-convex function on $[a,b]$ such that $f \in L_1[a,b]$. Then, we have the following inequality for Caputo-Fabrizio fractional integrals:

$$
\left({}^{CF}_{a}I^{\alpha}f\right)(k) + \left({}^{CF}I^{\alpha}_{b}f\right)(k) \leq \frac{4(1-\alpha)f(k) + \alpha(b-a)(f(a)+mf(b))}{2B(\alpha)}
$$

where $B(\alpha) > 0$ is normalization function $m \in (0,1]$ and $\alpha \in [0,1]$.

Proof. By using the definition of convex function, we can write

$$
f(ta + m(1-t)b) \leq tf(a) + m(1-t)f(b).
$$

By integrating both sides of the inequality over [0,1] with respect to t, we get

$$
\int_0^1 f(ta + m(1-t)b)\,dt \leq \int_0^1 tf(a)\,dt + \int_0^1 m(1-t)f(b)\,dt.
$$

By changing the variable as $x = ta + (1-t)b$ and by calculating the right hand side, we obtain

$$
\frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+mf(b)}{2}.
$$

By multiplying both sides of the above inequality with $\frac{\alpha(b-a)}{B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}f(k)$, we have

$$
\frac{2(1-\alpha)}{B(\alpha)}f(k) + \frac{\alpha}{B(\alpha)}\int_a^b f(x)\,dx \leq \frac{2(1-\alpha)}{B(\alpha)}f(k) + \frac{\alpha(b-a)}{B(\alpha)}\frac{|f(a)| + m|f(b)|}{2}.
$$

By simplifying the inequality, we get the result as:

$$
\left(\frac{1-\alpha}{B(\alpha)}f(k) + \frac{\alpha}{B(\alpha)}\int_a^k f(x)\,dx\right) + \left(\frac{1-\alpha}{B(\alpha)}f(k) + \frac{\alpha}{B(\alpha)}\int_k^b f(x)\,dx\right) \leq \frac{2(1-\alpha)}{B(\alpha)}f(k) + \frac{\alpha(b-a)}{B(\alpha)}\frac{f(a)+mf(b)}{2}.
$$

Therefore, we can write

$$
\left({}^{CF}_{a}I^{\alpha}f\right)(k) + \left({}^{CF}I^{\alpha}_{b}f\right)(k) \leq \frac{4(1-\alpha)f(k) + \alpha(b-a)(f(a)+mf(b))}{2B(\alpha)}.
$$

This completes the proof.

Theorem 3.2 Let $I \subseteq \mathbb{R}$. Suppose that $f: [a,b] \subseteq I \longrightarrow \mathbb{R}$ is a m-convex function on $[a,b]$ such that $f \in L_1[a,b]$. Then, we have the following inequality for Caputo-Fabrizio fractional integrals:

$$
\left({}^{CF}_{a}I^{\alpha}|f|\right)(k) + \left({}^{CF}I^{\alpha}_{b}|f|\right)(k) \leq \frac{2(1-\alpha)|f(k)|(p+1)^{\frac{1}{p}} + \alpha(b-a)(|f(a)| + m|f(b)|)}{B(\alpha)(p+1)^{\frac{1}{p}}}
$$

where $B(\alpha) > 0$ is normalization function $q > 1,\ \frac{1}{p} + \frac{1}{q} = 1, m \in (0,1]$ and $\alpha \in [0,1]$.

Proof. By using the definition of convex function, we can write

$$
f(ta + (1-t)b) \leq tf(a) + m(1-t)f(b).
$$

By taking the absolute values of both sides of the inequality, the integration is performed with respect to t in the interval [0,1].

$$\int_0^1 |f(ta+(1-t)b)|\,dt$$
$$\leq |f(a)|\int_0^1 t\,dt + m|f(b)|\int_0^1 1-t\,dt.$$

If we apply the Hölder's inequality to the right-hand side of the inequality, we get

$$\int_0^1 |f(ta+(1-t)b)|\,dt$$
$$\leq |f(a)|\left(\int_0^1 t^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 1dt\right)^{\frac{1}{q}}$$
$$+m|f(b)|\left(\int_0^1 (1-t)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 1dt\right)^{\frac{1}{q}}.$$

By changing the variable as $x = ta + (1-t)b$ and by calculating the right hand side, we obtain

$$\frac{1}{b-a}\int_a^b |f(x)|\,dx$$
$$\leq |f(a)|\left(\frac{1}{p+1}\right)^{\frac{1}{p}} + m|f(b)|\left(\frac{1}{p+1}\right)^{\frac{1}{p}}.$$

By multiplying both sides of the above inequality with $\frac{\alpha(b-a)}{B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}|f(k)|$, we have

$$\frac{2(1-\alpha)}{B(\alpha)}|f(k)| + \frac{\alpha}{B(\alpha)}\int_a^b |f(x)|\,dx$$
$$\leq \frac{2(1-\alpha)}{B(\alpha)}|f(k)|$$
$$+\frac{\alpha(b-a)(|f(a)|+m|f(b)|)}{B(\alpha)}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}.$$

By simplifying the inequality, we get the result.

$$\left(\frac{1-\alpha}{B(\alpha)}|f(k)| + \frac{\alpha}{B(\alpha)}\int_a^k |f(x)|\,dx\right)$$
$$+\left(\frac{1-\alpha}{B(\alpha)}|f(k)| + \frac{\alpha}{B(\alpha)}\int_k^b |f(x)|\,dx\right)$$
$$\leq \frac{2(1-\alpha)}{B(\alpha)}|f(k)|$$
$$+\frac{\alpha(b-a)(|f(a)|+m|f(b)|)}{B(\alpha)}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}.$$

The desired result is obtained as:

$$\left({}^{CF}_{a}I^{\alpha}|f|\right)(k) + \left({}^{CF}I^{\alpha}_{b}|f|\right)(k)$$
$$\leq \frac{2(1-\alpha)|f(k)|(p+1)^{\frac{1}{p}} +}{B(\alpha)(p+1)^{\frac{1}{p}}}$$
$$+\frac{\alpha(b-a)(|f(a)|+m|f(b)|)}{B(\alpha)(p+1)^{\frac{1}{p}}}.$$

This completes the proof.

Theorem 3.3. Let $I \subseteq \mathbb{R}$. Suppose that $f: [a,b] \subseteq I \longrightarrow \mathbb{R}$ is a m-convex function on $[a,b]$ such that $f \in L_1[a,b]$. Then, we have the following inequality for Caputo-Fabrizio fractional integrals:

$$\left({}^{CF}_{a}I^{\alpha}|f|\right)(k) + \left({}^{CF}I^{\alpha}_{b}|f|\right)(k)$$
$$\leq \frac{2(1-\alpha)}{B(\alpha)}|f(k)|$$
$$+\frac{\alpha(b-a)(|f(a)|+m|f(b)|)}{B(\alpha)}\left(\frac{q+p(p+1)}{qp(p+1)}\right)$$

where $B(\alpha) > 0$ is normalization function $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1, m \in (0,1]$ and $\alpha \in [0,1]$.

Proof : By using the definition of convex function, we can write

$$f(ta+(1-t)b) \leq tf(a) + m(1-t)f(b).$$

By taking the absolute values of both sides of the inequality, the integration is performed with respect to t in the interval [0,1], then we get

$$\int_0^1 |f(ta+(1-t)b)|\,dt$$
$$\leq |f(a)|\int_0^1 t\,dt + m|f(b)|\int_0^1 (1-t)\,dt.$$

If we apply the Young's inequality to the right-hand side of the inequality, we get

$$\int_0^1 |f(ta+(1-t)b)|\,dt$$
$$\leq |f(a)|\left(\frac{1}{p}\left(\int_0^1 t^p\,dt\right) + \frac{1}{q}\left(\int_0^1 1^q dt\right)\right)$$
$$+m|f(b)|\left(\frac{1}{p}\left(\int_0^1 (1-t)^p\,dt\right) + \frac{1}{q}\left(\int_0^1 1^q dt\right)\right).$$

By changing the variable as $x = ta + (1-t)b$ and by calculating the right hand side, we obtain

$$\frac{1}{b-a}\int_a^b |f(x)|\,dx$$
$$\leq |f(a)|\left(\frac{1}{p(p+1)}+\frac{1}{q}\right)+m|f(b)|\left(\frac{1}{p(p+1)}+\frac{1}{q}\right).$$

By multiplying both sides of the above inequality with $\frac{\alpha(b-a)}{B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}|f(k)|$, we have

$$\frac{2(1-\alpha)}{B(\alpha)}|f(k)|+\frac{\alpha}{B(\alpha)}\int_a^b |f(x)|\,dx$$
$$\leq \frac{2(1-\alpha)}{B(\alpha)}|f(k)|$$
$$+\frac{\alpha(b-a)(|f(a)|+m|f(b)|)}{B(\alpha)}\left(\frac{1}{p(p+1)}+\frac{1}{q}\right).$$

By simplifying the inequality, we get the result as:

$$\left(\frac{1-\alpha}{B(\alpha)}|f(k)|+\frac{\alpha}{B(\alpha)}\int_a^k |f(x)|\,dx\right)$$
$$+\left(\frac{1-\alpha}{B(\alpha)}|f(k)|+\frac{\alpha}{B(\alpha)}\int_k^b |f(x)|\,dx\right)$$
$$\leq \frac{2(1-\alpha)}{B(\alpha)}|f(k)|$$
$$+\frac{\alpha(b-a)(|f(a)|+m|f(b)|)}{B(\alpha)}\left(\frac{q+p(p+1)}{qp(p+1)}\right).$$

After necessary operations, the result will be written as follows:

$$\left({}^{CF}_{a}I^{\alpha}|f|\right)(k)+\left({}^{CF}I^{\alpha}_{b}|f|\right)(k)$$
$$\leq \frac{2(1-\alpha)}{B(\alpha)}|f(k)|$$
$$+\frac{\alpha(b-a)(|f(a)|+m|f(b)|)}{B(\alpha)}\left(\frac{q+p(p+1)}{qp(p+1)}\right).$$

Theorem 3.4. Let I ⊆ ℝ. Suppose that $f: [a,b] \subseteq \mathrm{I} \longrightarrow \mathbb{R}$ is a (α_1,m)-convex function on $[a,b]$ such that $f \in L_1[a,b]$. Then, we have the following inequality for Caputo-Fabrizio fractional integrals:

$$\left({}^{CF}_{a}I^{\alpha}|f|\right)(k)+\left({}^{CF}I^{\alpha}_{b}|f|\right)(k)$$
$$\leq \frac{2(1-\alpha)|f(k)|(\alpha_1+1)}{B(\alpha)(\alpha_1+1)}$$
$$+\frac{\alpha(b-a)(|f(a)|+m|f(b)|)}{B(\alpha)(\alpha_1+1)}$$

where $B(\alpha) > 0$ is normalization function $m \in (0,1]$ and $\alpha, \alpha_1 \in [0,1]$.

Proof : By using the definition of (α_1, m)- convex function, we can write

$$f(ta+(1-t)b) \leq t^{\alpha_1}f(a)+m(1-t^{\alpha_1})f(b).$$

By taking the absolute values of both sides of the inequality, the integration is performed with respect to t in the interval [0,1], then one has

$$\int_0^1 |f(ta+(1-t)b)|\,dt$$
$$\leq \int_0^1 t^{\alpha_1}|f(a)|\,dt+\int_0^1 (1-t^{\alpha_1})m|f(b)|\,dt.$$

By changing the variable as $x = ta + (1-t)b$ and by calculating the right hand side, we obtain

$$\frac{1}{b-a}\int_a^b |f(x)|\,dx = \frac{|f(a)|+\alpha_1 m|f(b)|}{\alpha_1+1}.$$

By multiplying both sides of the above inequality with $\frac{\alpha(b-a)}{B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}|f(k)|$, we have

$$\frac{2(1-\alpha)}{B(\alpha)}|f(k)|+\frac{\alpha}{B(\alpha)}\int_a^b |f(x)|\,dx$$
$$\leq \frac{2(1-\alpha)}{B(\alpha)}|f(k)|+\frac{\alpha(b-a)}{B(\alpha)}\frac{|f(a)|+\alpha_1 m|f(b)|}{\alpha_1+1}.$$

By simplifying the inequality, we get the following result

$$\left(\frac{1-\alpha}{B(\alpha)}|f(k)|+\frac{\alpha}{B(\alpha)}\int_a^k |f(x)|\,dx\right)$$
$$+\left(\frac{1-\alpha}{B(\alpha)}|f(k)|+\frac{\alpha}{B(\alpha)}\int_k^b |f(x)|\,dx\right)$$
$$\leq \frac{2(1-\alpha)}{B(\alpha)}|f(k)|+\frac{\alpha(b-a)}{B(\alpha)}\frac{|f(a)|+\alpha_1 m|f(b)|}{\alpha_1+1}.$$

After necessary operations, the result will be given as follows:

$$\left({}^{CF}_{a}I^{\alpha}|f|\right)(k)+\left({}^{CF}I^{\alpha}_{b}|f|\right)(k)$$
$$\leq \frac{2(1-\alpha)|f(k)|(\alpha_1+1)}{B(\alpha)(\alpha_1+1)}$$
$$+\frac{\alpha(b-a)(|f(a)|+\alpha_1 m|f(b)|)}{B(\alpha)(\alpha_1+1)}$$

Theorem 3.5. Let $I \subseteq \mathbb{R}$. Suppose that $f: [a,b] \subseteq I \longrightarrow \mathbb{R}$ is a (α_1,m)-convex function on $[a,b]$ such that $f \in L_1[a,b]$. Then, we have the following inequality for Caputo-Fabrizio fractional integrals:

$$\begin{aligned}&\left({}_{a}^{CF}I^{\alpha}|f|\right)(k) + \left({}^{CF}I_b^{\alpha}|f|\right)(k) \\ &\le \frac{2(1-\alpha)}{B(\alpha)}|f(k)| \\ &+ \frac{\alpha(b-a)}{B(\alpha)}\left(\frac{|f(a)| + \alpha_1 m|f(b)|(p\alpha_1)^{\frac{1}{p}}}{(p\alpha_1+1)^{\frac{1}{p}}}\right).\end{aligned}$$

where $B(\alpha) > 0$ is normalization function $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1, m \in (0,1]$ and $\alpha, \alpha_1 \in [0,1]$.

Proof : By using the definition of α_1,m)- convex function, we can write

$$f(ta + (1-t)b) \le t^{\alpha_1} f(a) + m(1 - t^{\alpha_1}) f(b).$$

By taking the absolute values of both sides of the inequality, the integration is performed with respect to t in the interval [0,1], then we have

$$\begin{aligned}&\int_0^1 |f(ta + (1-t)b)|\, dt \\ &\le \int_0^1 t^{\alpha_1}|f(a)|\, dt + \int_0^1 (1 - t^{\alpha_1}) m|f(b)|\, dt.\end{aligned}$$

If we apply the Hölder's inequality to the right-hand side of the inequality, we get

$$\begin{aligned}&\int_0^1 |f(ta + (1-t)b)|\, dt \\ &\le |f(a)|\left(\int_0^1 |t^{\alpha_1}|^p\, dt\right)^{\frac{1}{p}}\left(\int_0^1 1^q\, dt\right)^{\frac{1}{q}} \\ &+ m|f(b)|\left(\int_0^1 |1 - t^{\alpha_1}|^p\, dt\right)^{\frac{1}{p}}\left(\int_0^1 1^q\, dt\right)^{\frac{1}{q}}.\end{aligned}$$

By using the fact that $|1 - t^{\alpha_1}|^p \le 1 - t^{p\alpha_1}$ for $\alpha_1 > 0, p > 1$, we can write

$$\begin{aligned}&\int_0^1 |f(ta + (1-t)b)|\, dt \\ &\le f(a)\left(\int_0^1 |t^{\alpha_1}|^p\, dt\right)^{\frac{1}{p}}\left(\int_0^1 1^q\, dt\right)^{\frac{1}{q}} \\ &+ mf(b)\left(\int_0^1 1 - t^{p\alpha_1}\, dt\right)^{\frac{1}{p}}\left(\int_0^1 1^q\, dt\right)^{\frac{1}{q}}.\end{aligned}$$

By changing the variable as $x = ta + (1-t)b$ and by calculating the right hand side, we obtain

$$\begin{aligned}&\frac{1}{b-a}\int_a^b |f(x)|\, dx \\ &\le |f(a)|\left(\frac{1}{p\alpha_1 + 1}\right)^{\frac{1}{p}} + mf(b)\left(\frac{p\alpha_1}{p\alpha_1+1}\right)^{\frac{1}{p}}.\end{aligned}$$

By multiplying both sides of the above inequality with $\frac{\alpha(b-a)}{B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}|f(k)|$, we have

$$\begin{aligned}&\frac{2(1-\alpha)}{B(\alpha)}|f(k)| + \frac{\alpha}{B(\alpha)}\int_a^b |f(x)|\, dx \\ &\le \frac{2(1-\alpha)}{B(\alpha)}|f(k)| + \frac{\alpha(b-a)}{B(\alpha)} \\ &\times\left(|f(a)|\left(\frac{1}{p\alpha_1+1}\right)^{\frac{1}{p}} + m|f(b)|\left(\frac{p\alpha_1}{p\alpha_1+1}\right)^{\frac{1}{p}}\right).\end{aligned}$$

By simplifying the inequality, we get

$$\begin{aligned}&\left(\frac{1-\alpha}{B(\alpha)}|f(k)| + \frac{\alpha}{B(\alpha)}\int_a^k |f(x)|\, dx\right) \\ &+ \left(\frac{1-\alpha}{B(\alpha)}|f(k)| + \frac{\alpha}{B(\alpha)}\int_k^b |f(x)|\, dx\right) \\ &\le \frac{2(1-\alpha)}{B(\alpha)}|f(k)| \\ &+ \frac{\alpha(b-a)}{B(\alpha)}\left(\frac{|f(a)| + m|f(b)|(p\alpha_1)^{\frac{1}{p}}}{(p\alpha_1+1)^{\frac{1}{p}}}\right).\end{aligned}$$

After necessary operations, we provide

$$\begin{aligned}&\left({}_{a}^{CF}I^{\alpha}|f|\right)(k) + \left({}^{CF}I_b^{\alpha}|f|\right)(k) \\ &\le \frac{2(1-\alpha)}{B(\alpha)}|f(k)| \\ &+ \frac{\alpha(b-a)}{B(\alpha)}\left(\frac{|f(a)| + m|f(b)|(p\alpha_1)^{\frac{1}{p}}}{(p\alpha_1+1)^{\frac{1}{p}}}\right).\end{aligned}$$

Theorem 3.6. Let $I \subseteq \mathbb{R}$. Suppose that $f: [a,b] \subseteq I \longrightarrow \mathbb{R}$ is a (α_1,m)-convex function on $[a,b]$ such that $f \in L_1[a,b]$. Then, we have the following inequality for Caputo-Fabrizio fractional integrals:

$$\begin{aligned}&\left({}_{a}^{CF}I^{\alpha}|f|\right)(k) + \left({}^{CF}I_b^{\alpha}|f|\right)(k) \\ &\le \frac{2(1-\alpha)}{B(\alpha)}|f(k)| + \frac{\alpha(b-a)}{B(\alpha)} \times \\ &\left(\frac{|f(a)|\big(q + p(p\alpha_1+1)\big) + m|f(b)|\big(pq\alpha_1 + p(p\alpha_1+1)\big)}{pq(p\alpha_1+1)}\right).\end{aligned}$$

where $B(\alpha) > 0$ is normalization function $q > 1$, $\frac{1}{p}+\frac{1}{q} = 1, m \in (0,1]$ and $\alpha, \alpha_1 \in [0,1]$.

Proof : By using the definition of convex function, we can write

$$f(ta+(1-t)b) \le t^{\alpha_1} f(a) + m(1-t^{\alpha_1}) f(b).$$

By taking the absolute values of both sides of the inequality, the integration is performed with respect to t in the interval [0,1], then we have

$$\int_0^1 |f(ta+(1-t)b)|\,dt \le \int_0^1 t^{\alpha_1}|f(a)|\,dt + \int_0^1 (1-t^{\alpha_1}) m |f(b)|\,dt.$$

If we apply the Young's inequality to the right-hand side of the inequality, we get

$$\int_0^1 |f(ta+(1-t)b)|\,dt \le |f(a)|\left(\frac{1}{p}\left(\int_0^1 |t^{\alpha_1}|^p\,dt\right) + \frac{1}{q}\left(\int_0^1 1^q\,dt\right)\right) + m|f(b)|\left(\frac{1}{p}\left(\int_0^1 |1-t^{\alpha_1}|^p\,dt\right) + \frac{1}{q}\left(\int_0^1 1^q\,dt\right)\right).$$

By using the fact that $|1-t^{\alpha}|^p \le 1-t^{p\alpha}$ for $\alpha > 0, p > 1$, we can write

$$\int_0^1 |f(ta+(1-t)b)|\,dt \le |f(a)|\left(\frac{1}{p}\left(\int_0^1 |t^{\alpha_1}|^p\,dt\right) + \frac{1}{q}\left(\int_0^1 1^q\,dt\right)\right) + m|f(b)|\left(\frac{1}{p}\left(\int_0^1 1-t^{p\alpha_1}\,dt\right) + \frac{1}{q}\left(\int_0^1 1^q\,dt\right)\right).$$

By changing the variable as $x = ta+(1-t)b$ and by calculating the right hand side, we obtain

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le |f(a)|\left(\frac{1}{p}\left(\frac{1}{p\alpha_1+1}\right)+\frac{1}{q}\right) + m|f(b)|\left(\frac{1}{p}\left(\frac{p\alpha_1}{p\alpha_1+1}\right)+\frac{1}{q}\right).$$

By multiplying both sides of the above inequality with $\frac{\alpha(b-a)}{B(\alpha)}$ and adding $\frac{2(1-\alpha)}{B(\alpha)}|f(k)|$, we have

$$\frac{2(1-\alpha)}{B(\alpha)}|f(k)| + \frac{\alpha}{B(\alpha)}\int_a^b |f(x)|\,dx \le \frac{2(1-\alpha)}{B(\alpha)}|f(k)| + \frac{\alpha(b-a)}{B(\alpha)} \times \left(|f(a)|\left(\frac{1}{p(p\alpha_1+1)}+\frac{1}{q}\right) + m|f(b)|\left(\frac{p\alpha_1}{p(p\alpha_1+1)}+\frac{1}{q}\right)\right).$$

By simplifying the inequality, we get

$$\left(\frac{1-\alpha}{B(\alpha)}|f(k)| + \frac{\alpha}{B(\alpha)}\int_a^k |f(x)|\,dx\right) + \left(\frac{1-\alpha}{B(\alpha)}|f(k)| + \frac{\alpha}{B(\alpha)}\int_k^b |f(x)|\,dx\right) \le \frac{2(1-\alpha)}{B(\alpha)}|f(k)| + + \frac{\alpha(b-a)}{B(\alpha)} \times \left(|f(a)|\left(\frac{1}{p(p\alpha_1+1)}+\frac{1}{q}\right) + m|f(b)|\left(\frac{p\alpha_1}{p(p\alpha_1+1)}+\frac{1}{q}\right)\right).$$

After necessary operations, the proof is completed.

References

ABDELJAWAD, THABET. On conformable fractional calculus. *Journal of computational and Applied Mathematics*, 2015, 279: 57-66.

ABDELJAWAD, THABET; BALEANU, DUMITRU. Integration by parts and its applications of a new nonlocal fractional derivative with Mittag-Leffler nonsingular kernel. *J. Nonlinear Sci. Appl., 10 (2017), 1098–1107.*

ABDELJAWAD, THABET; BALEANU, DUMITRU. On fractional derivatives with exponential kernel and their discrete versions. *Reports on Mathematical Physics*, 2017, 80.1: 11-27.

AKDEMİR, A. O.; ASLAN, S.; ÇAKALOĞLU, M.N. EKİNCİ, A. Some New Results for Different Kinds of Convex Functions CaputoFabrizio Fractional Operators. 4th International Conference on Mathematical and Related Sciences. Page (92) ICMRS 2021.

AKDEMİR, A. O.; ASLAN, S.; EKİNCİ, A. Novel Approaches for s-Convex Functions via Caputo-Fabrizio Fractional Integrals. *Proceedings of IAM,* 2022, 11.1: 3-16.

AKDEMIR, AHMET OCAK; BUTT, S.I; NADEEM, M; RAGUSA, M.A. New general variants of Chebyshev type inequalities via generalized fractional integral operators. *Mathematics*, 2021, 9.2: 122.

AKDEMIR, AHMET OCAK; EKINCI, ALPER; SET, ERHAN. Conformable fractional integrals and related new integral inequalities. 2017.

ATANGANA, ABDON; BALEANU, DUMITRU. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. *THERMAL SCIENCE, Year 2016, Vol. 20, No. 2, pp. 763-769.*

BUTT, SAAD IHSAN, UMAR, MUHAMMAD, RASHID, SAIMA, AKDEMIR, AHMET OCAK, CHU, YU-MING. New Hermite–Jensen–Mercer-type inequalities via k-fractional integrals. *Advances in Difference Equations*, 2020, 2020: 1-24.

CAPUTO, MICHELE; FABRIZIO, MAURO. A new definition of fractional derivative without singular kernel. *Progress in Fractional Differentiation & Applications*, 2015, 1.2: 73-85.

EKINCI, ALPER; OZDEMIR, MUHAMET. Some new integral inequalities via Riemann-Liouville integral operators. *Applied and Computational Mathematics*, 2019, 18.3.

GÜRBÜZ, MUSTAFA, AKDEMİR, AHMET OCAK, RASHID, SAIMA, SET, ERHAN. Hermite–Hadamard inequality for fractional integrals of Caputo–Fabrizio type and related inequalities. *Journal of Inequalities and Applications*, 2020, 2020: 1-10.

MIHESAN, V. G. (1993, May). A generalization of the convexity. In *Seminar on Functional Equations, Approx. and Convex., Cluj-Napoca, Romania*.

ÖZDEMIR, M. E.; AVCI, M.; KAVURMACI, H. (2011). Hermite–Hadamard-type inequalities via $(\alpha, m)-$convexity. *Computers and Mathematics with Applications*, *61*(9), 2614-2620.

PECARIC, J.E.; TONG, Y.L. *Convex functions, partial orderings, and statistical applications*. Academic Press, 1992.

RASHID, SAIMA, HAMMOUCH, ZAKIA, KALSOOM, HUMAIRA, ASHRAF, REHANA, CHU, YU MING. New investigation on the generalized K-fractional integral operators. *Frontiers in Physics*, 2020, 8: 25.

SAMKO, STEFAN G., et al. *Fractional integrals and derivatives*. Yverdon-les-Bains, Switzerland: Gordon and breach science publishers, Yverdon, 1993.

SET, ERHAN. New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. *Computers & Mathematics with Applications*, 2012, 63.7: 1147-1154.

SET, ERHAN; AKDEMIR, AHMET OCAK; ÖZDEMIR, EMİN. M. Simpson type integral inequalities for convex functions via Riemann-Liouville integrals. *Filomat*, 2017, 31.14: 4415-4420.

TARIQ, MUHAMMAD, AHMAD, HIJAZ, SHAIKH, ABDUL GHAFOOR, SAHOO, SOUBHAGYA KUMAR, KHEDHER, KHALED MOHAMED, NGUYEN, TUAN GIA. New fractional integral inequalities for preinvex functions involving Caputo-Fabrizio operator. AIMS Mathematics, 2022, 7(3): 3440-3455. doi: 10.3934/math.2022191

TOADER, G. (1984, October). Some generalizations of the convexity. In *Proceedings of the Colloquium on Approximation and Optimization* (Vol. 329, p. 338). Cluj-Napoca, Romania: University of Cluj-Napoca.