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Some Estimates of the Growth of Polynomials in the Region with Piecewise Smooth Boundary

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ABSTRACT

In this paper, we investigate inequalities for higher order derivatives of algebraic polynomials in weighted Lebesgue space. In doing so, using the weighted $\mathcal{L}_p$ −(quasi)norm, we establish the growth of the modulus of the *m*-th derivatives of algebraic polynomials on the closure and outside of a given region of the complex plane bounded by a piecewise smooth curve with interior zero angles. As a result, we estimate the growth rate of the derivatives of algebraic polynomials on the whole complex plane depending on the parameters of the region under consideration.

Keywords: Bernstein inequality, Polynomial estimates, Quasiconformal curve, Piecewise smooth curve.

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Introduction and Preliminaries

Let $\mathbb{C}$ denote the complex plane and $\Gamma \subset \mathbb{C}$ be a closed, rectifiable Jordan curve on the plane, G be the region bounded by Γ such that $0 \in G$ without loss of generality and Ω be the exterior region of Γ with respect to extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Also, for $t \in \mathbb{C}$, let $D(t,\rho) := \{w: |w-t| > \rho,\ \rho > 0\}$. According to the Riemann mapping theorem, there exists a univalent conformal mapping $w = \Phi(z)$ that maps the region $D(0,1)$ onto the region Ω with the normalizations $\Phi(\infty) = \infty$ and $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$. For any $R \geq 1$, we denote the exterior level curve for the Γ by $\Gamma_R := \{z: |\Phi(z)| = R\}$. Let $G_R := int\Gamma_R$, $\Omega_R := ext\Gamma_R$.

Let $\mathcal{P}_n$ denote the collection of all algebraic polynomials with complex coefficients of degree not exceeding n.

For the curve $\Gamma = \partial G$ and a given weight function $h(z)$, the following can be written for $0 < p \leq \infty$:

$$\|P_n\|_p := \|P_n\|_{\mathcal{L}_p(h,\Gamma)} := \left(\int_C h(z)|P_n(z)|^p|dz|\right)^{1/p}, 0 < p < \infty, \quad (1)$$

$$\|P_n\|_\infty := \|P_n\|_{\mathcal{L}_\infty(1,\Gamma)} := \max_{z\in C}|P_n(z)|;\ \mathcal{L}_p(1,\Gamma) =: \mathcal{L}_p(\Gamma).$$

Here, it's clear that $\|\cdot\|_p$ is a norm for $1 \leq p \leq \infty$ and a quasi-norm for $0 < p < 1$.

Assume that $\{z_j\}_{j=1}^{l}$ denote a fixed collection of distinct points on Γ that are positively positioned in an ordered positive manner without loss of generality. The generalized Jacobian weight function $h(z)$ we use in this paper is as follows:

For $1 < R_0 < \infty$,

$$h(z) := \begin{cases} h_0(z)\prod_{j=1}^{l}|z-z_j|^{\gamma_j}, & z \in \overline{G}_{R_0}, \\ 0, & z \in \mathbb{C}\backslash\overline{G}_{R_0}, \end{cases} \quad (2)$$

where $\gamma_j > -1$, for all $j = 1,2,\dots,l$ and $h_0(z) \geq c_0(\Gamma) > 0$ with a constant $c_0(\Gamma) > 0$.

The Bernstein–Walsh inequality [1] given below gives an upper bound on the growth of any polynomial $P_n \in \mathcal{P}_n$ outside a certain region, and this is an important and fundamental result for comprehending the behavior of polynomials outside this specified region. The estimation given in this inequality shows that upon the extension of the region $\overline{G}$ to a given region $\overline{G}_R$, where $c = c(G) > 0$ is a constant and $R = 1 + \frac{c}{n}$, the norm $\|P_n\|_{C(\overline{G})}$ increases by no more than a constant:

$$\|P_n\|_{C(\overline{G}_R)} \leq R^n\|P_n\|_{C(\overline{G})}, \quad z \in \Omega. \quad (3)$$

The question of whether similar estimates to (3) hold when we replace the norm in (3) by a different norm was studied in [2, Lemma 2.2] for the space $\mathcal{L}_p$. Here we can see that the same effect occurs for a constant $c = c(G) > 0$ and $R = 1 + \frac{c}{n}$:

$$\|P_n\|_{\mathcal{L}_p(\Gamma_R)} \leq R^{n+\frac{1}{p}}\|P_n\|_{\mathcal{L}_p(\Gamma)}, \quad p > 0. \quad (4)$$

Moreover, for $h(z) \neq 1$, the estimate similar to that in (4) was obtained in generalized form in [3, Lemma 2.4] as follows:

$$\|P_n\|_{\mathcal{L}_p(h,\Gamma_R)} \leq R^{n+\frac{1+\gamma_{max}}{p}}\|P_n\|_{\mathcal{L}_p(h,\Gamma)},\ \gamma_{max} = \max\{0,\gamma_j: j = 1,2,\dots,l\}. \quad (5)$$

For the analogs of the estimates (3-5) in Bergman spaces, we first introduce the following:

Let σ be the area measure on the Jordan region G. For $0 < p \leq \infty$,

$$\|P_n\|_{A_p(h,G)} := \left(\iint_G h(z)|P_n(z)|^p d\sigma_z\right)^{1/p}, 0 < p < \infty, \quad (6)$$

$$\|P_n\|_{A_\infty(1,G)} := \max_{z\in\overline{G}} |P_n(z)|, p = \infty; \quad A_p(1,G) =: A_p(G).$$

Here too, as in (1) $\|\cdot\|_{A_p(h,G)}$ is a norm for $1 \le p \le \infty$ and a quasi-norm for $0 < p < 1$.

Subsequent to the below definition, we can give corresponding formulations of the estimations (3) and (5) pertaining to the A_p −(quasi-)norm for the regions with quasiconformal boundary.

Definition 1.1. *[4] Let $\Gamma \subset \overline{\mathbb{C}}$ be a Jordan arc or curve on the plane. We say that Γ is K −quasiconformal if there is a K −quasiconformal mapping f of a region $D \supset \Gamma$ such that $f(\Gamma)$ is a line segment or a circle, respectively.*

If Γ is K −quasiconformal for some K, it is called quasiconformal. We denote

$$K(\Gamma) := \inf\{K(f): f \in F(\Gamma)\},$$

where $K(f)$ is the maximal dilatation of f and, $F(\Gamma)$ is the set of all sense-preserving plane homeomorphisms f of regions $D \supset \Gamma$ such that $f(\Gamma)$ is a line segment or a circle, respectively. Then Γ is quasiconformal if $K(\Gamma) < \infty$, and Γ is K −quasiconformal if $K(\Gamma) \le K$.

In [5], it was showed that if Γ is rectifiable quasiconformal curve, then there exists a constant $c_1 = c_1(\Gamma) > 0$ such that for every $P_n \in \mathcal{P}_n$,

$$|P_n(z)| \le c_1 \frac{\sqrt{n}}{d(z,\Gamma)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \qquad z \in \Omega,$$

where $d(z,\Gamma) = dist(z,\Gamma)$.

Similar estimate to (5), provided $\Gamma = \partial G$ is quasiconformal and $h(z)$ is defined as in (2) for $\gamma_j > -2$, was obtained in [6] as follows:

$$\|P_n\|_{A_p(h,G_R)} \le c_2 R^{*\,n+\frac{1}{p}} \|P_n\|_{A_p(h,G)}, \qquad p > 0,$$

where $R^* := 1 + c_3(R-1)$ and $c_3 > 0$, $c_2 := c_2(G,p,c_3) > 0$ are constants independent of n and R. Furthermore, for arbitrary Jordan region G and for any $P_n \in \wp_n$ and $R_1 = 1 + \frac{1}{n}$, it was established in [7] that

$$\|P_n\|_{A_p(G_R)} \le cR^{\,n+\frac{2}{p}} \|P_n\|_{A_p(G_{R_1})}, p > 0,$$

holds for any $R > R_1 = 1 + \frac{1}{n}$, where $c = \left(\frac{2}{e^p-1}\right)^{\frac{1}{p}} \left[1 + O\left(\frac{1}{n}\right)\right]$ as $n \to \infty$ is an asymptotically exact constant.

To summarize, we can say that various analogues of inequalities of type (3) have been studied under properties of different regions by changing the maximal norm in the inequality.

Our aim in this paper is to establish the behavior of the modulus of the higher order derivative of $P_n(z)$ in weighted Lebesgue space by obtaining inequalities such as the following:

$$\left|P_n^{(m)}(z)\right| \le \mu_n \| P_n \|_{\mathcal{L}_p(h,\Gamma)}, \ z \in \Omega, \qquad (7)$$

where $\mu_n = \mu_n(|\Phi(z)|, d(z,\Gamma), h, m, p, \Gamma) \to \infty$ as $n \to \infty$ is a constant which depends on the characteristics of Γ as well as the weight function h. While doing this, we assume that Γ is piecewise smooth with cusps (see Definition 1.4) and derive the estimates using geometric properties of the curve Γ. The assumption here means that the curve Γ consists of a finite set of smooth arcs connecting at exterior non-zero angles $\lambda\pi$, $0 < \lambda < 2$ and at interior zero angles (i.e. $\lambda = 2$).

Estimates analogous to (7) have been studied, improved and generalized in various aspects in [5], [8]-[25], [26, p.418-428], [27-30], [31, Sect. 5.3], [32, p.122-133], [33]-[36] (also reference therein and others), where $m \ge 0$. In this paper, we develop the methods of previous works and continue to find results in the weighted Lebesgue space for $p \ge 1$, thus contributing to the growth behavior of polynomials.

We first introduce a number of definitions and notations that will be used in subsequent sections of the paper.

The constants symbolized by c–whether or not they include indices–represent positive values which may vary throughout a particular formula and in a general context depend on G and parameters that are not critical to the argument. Otherwise, any such dependencies will be articulated explicitly. For each $a, b > 0$, we use the notations $a \preceq b$ and $a \asymp b$ if the conditions $a \le cb$ and $c_1 a \le b \le c_2 x$ are satisfied, respectively. For any $k \ge 0$ and $m > k$, the notation $i = \overline{k,m}$ means $i = k, k+1, \dots, m$.

Assume that $z = z(s)$, $s \in [0, mes\Gamma]$ states the natural representation of Γ.

Definition 1.2. *Let Γ be a Jordan curve or arc. Γ is called smooth if it possesses a continuous tangent $\theta(z) := \theta(z(s))$ at each point $z(s)$. The collection of such smooth curves or arcs is represented by the notation C_θ. Consequently, we express $G \in C_\theta$ to indicate that $\partial G \in C_\theta$.*

According to [4], we have the following corollary:

Corollary 1.3. *If $\Gamma \in C_\theta$, then Γ is $(1+\varepsilon)$ − quasiconformal for arbitrary small $\varepsilon > 0$.*

We now recall the definition of regions bounded by a piecewise smooth curve with non-zero exterior angles at some connection points of the boundary arcs and zero interior angles at others.

Definition 1.4. *[8] Let $G \subset \overline{\mathbb{C}}$ be a Jordan region on the plane and the points $\{\xi_j\}_{j=1}^{l} \in \Gamma = \partial G$ have the properties given in the definition of $h(z)$ in (2). We say that $G \in PC_\theta(1; \lambda_1, \dots, \lambda_l)$, $0 < \lambda_j \le 2$, $j = \overline{1,l}$ if $\Gamma = \partial G$ is the union of $\{\Gamma_j\}_{j=1}^{l}$ smooth arcs such that they have exterior angles (with respect to $\overline{G}$) $\lambda_j\pi$, $0 < \lambda_j \le 2$, at the corner points $\{\xi_j\}_{j=1}^{l} \in \Gamma$, where two arcs Γ_{j-1} and Γ_j, $j = \overline{1,l}$, ($\Gamma_1 \equiv \Gamma_l$) are connected.*

For simplicity of calculations, we take the case $l_1 = 1$, $l = 2$. This means that, we have the region $G \in PC_\theta(1; \lambda_1, 2)$, $0 < \lambda_1 < 2$ such that G has an exterior non-zero angle $\lambda_1\pi$ at $\xi_1 \in \Gamma$ and an interior zero angle at $\xi_2 \in \Gamma$.

Let us give some notations we need in the proofs:

For $0 < \delta_j < \delta_0 := \frac{1}{4}\min\{|z_1 - z_2|: j = 1,2\}$,let $\delta := \min_{1\le j\le 2} \delta_j$, $\Omega(z_j, \delta_j) := \Omega \cap \{z: |z - z_j| \le \delta_j\}$ and $\Omega(\delta) := \bigcup_{j=1}^{2} \Omega(z_j, \delta)$. Let $w_j := \Phi(z_j), \varphi_j := \arg w_j$. We assume that $\varphi_2 < 2\pi$, without loss of generality. For $\eta_j = \min_{t\in\partial\Phi(\Omega(z_j,\delta_j))} |t - w_j| > 0$ and $\eta := \min\{\eta_j, j = 1,2\}$ let us set:

$$D(\eta_j):=\{t:|t-w_j|\le\eta_j\}\subset\Phi(\Omega(z_j,\quad\delta_j)),\qquad D(\eta):=\bigcup_{j=1}^{2}D(\eta_j),$$

$$\widehat{D}_j=D(0,1)\backslash D(\eta_j),\qquad \widehat{D}(\eta):=D(0,1)\backslash D(\eta),$$

$$D_1'(\rho):=\left\{t=R\cdot e^{i\theta}:R\ge\rho\ge1,\frac{\varphi_0+\varphi_1}{2}\le\theta<\frac{\varphi_1+\varphi_2}{2}\right\},$$

$$D_2'(\rho):=\left\{t=R\cdot e^{i\theta}:R\ge\rho\ge1,\frac{\varphi_1+\varphi_2}{2}\le\theta<\frac{\varphi_2+\varphi_0}{2}\right\}\text{ where }\ \varphi_0=2\pi-\varphi_2;\ \ \Omega_R^j:=\Psi(D_j'(R)),\ \Gamma_R^j:=\Gamma_R\cap\overline{\Omega}_R^j,\ R\ge1,\ \Omega_R=\cup_{j=1}^{2}\Omega_R^j.$$

Also, some notations that we use both in formulating the results and in the proof are below:

$$\tilde{\lambda}:=\begin{cases}\max\{1,\lambda\}+\varepsilon, & 0<\lambda<2,\\ 2, & \lambda=2;\end{cases}\tag{8}$$

$$\tilde{\lambda}_j:=\max\{1,\lambda_j\}+\varepsilon,\ \forall\varepsilon>0,0<\lambda_j<2,\ j=1,2;\ \ \lambda^*:=\max\{\tilde{\lambda}_1,\tilde{\lambda}_2\};$$

$$\tilde{\gamma}_j:=\max\{0;\gamma_j\},\ j=1,2;\ \ \tilde{\gamma}:=\begin{cases}\tilde{\gamma}_1, & 0<\lambda<2,\\ \tilde{\gamma}_2, & \lambda=2;\end{cases}$$

$$\gamma^*:=\max\{0,\gamma_1,\gamma_2\}.$$

Main Results

In the theorems and corollaries given throughout the this section, the weight function $h(z)$ is given by (2) for $l=2$, and, the constants $c_i>0,\ i=\overline{1,9}$ are independent of n and z but depend on G,p,γ,λ,m and ε, where $\gamma:=\begin{cases}\gamma_1, & 0<\lambda<2,\\ \gamma_2, & \lambda=2,\end{cases}$ and $0<\lambda\le2$.

We can now start presenting the main results:

Theorem 2.1. *Assume that $p\ge1;G\in PC_\theta(1;\lambda_1,2)$ with $0<\lambda_1<2$. Then, for arbitrary $P_n\in\wp_n,\ n\in\mathbb{N},n\ge m$, any $m=1,2,\ldots$, and $z\in\Omega_{R_1}$, we have:*

$$\left|P_n^{(m)}(z)\right|\le c_1\frac{|\Phi^{n-m+1}(z)|}{d(z,\Gamma)}I_{n,p}^1(m)\|P_n\|_p\ ,$$

where

$$I_{n,p}^1(m):=\begin{cases}n^{\left(\frac{\gamma_1+1}{p}+m-1\right)\tilde{\lambda}_1}, & 1\le p<\frac{\tilde{\lambda}}{\tilde{\lambda}+1}\tilde{\gamma}+1,\ \gamma_1>\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right),\ \gamma_2>0,\\ n^{2\left(\frac{\gamma_2+1}{p}+m-1\right)}, & 1\le p<\frac{\tilde{\lambda}}{\tilde{\lambda}+1}\tilde{\gamma}+1,\ 0<\gamma_1<\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right),\ \gamma_2>0,\\ n^{m\tilde{\lambda}+1-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=\frac{\tilde{\lambda}}{\tilde{\lambda}+1}\tilde{\gamma}+1,\ \gamma_1=\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right),\ \gamma_2>0,\\ n^{m\tilde{\lambda}+1-\frac{1}{p}}, & p>\frac{2}{3}\gamma_2+1,\ 0<\gamma_1<\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right),\ \gamma_2>0,\\ n^{m\tilde{\lambda}+1-\frac{1}{p}}, & p>\frac{\tilde{\lambda}_1}{\tilde{\lambda}_1+1}\gamma_1+1,\ \gamma_1>\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right),\ \gamma_2>0,\\ n^{m\tilde{\lambda}+1-\frac{1}{p}}, & p\ge1,\ -1<\gamma_1,\gamma_2\le0.\end{cases}\tag{9}$$

In the situation of the region has zero or nonzero angles at both points simultaneously, we obtain the following results.

Corollary 2.2. *Under the conditions of Theorem 2.1, for $G\in PC_\theta(1;\lambda_1,\lambda_2),\ 0<\lambda_1,\lambda_2<2$, we have:*

$$\left|P_n^{(m)}(z)\right|\le c_2\frac{|\Phi^{n-m+1}(z)|}{d(z,\Gamma)}I_{n,p}^2(m)\|P_n\|_p,$$

where

$$I_{n,p}^2(m):=\begin{cases}n^{\left(\frac{\gamma^*+1}{p}+m-1\right)\lambda^*}, & 1\le p<\frac{\lambda^*}{\lambda^*+1}\gamma^*+1,\ \gamma^*>0,\\ n^{m\lambda^*+1-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=\frac{\lambda^*}{\lambda^*+1}\gamma^*+1,\ \gamma^*>0,\\ n^{m\lambda^*+1-\frac{1}{p}}, & p>\frac{\lambda^*}{\lambda^*+1}\gamma^*+1,\ \gamma^*>0,\ or\\ & p\ge1,\ -1<\gamma_1,\gamma_2\le0.\end{cases}\tag{10}$$

Corollary 2.3. *Under the conditions of Theorem 2.1, for $G\in PC_\theta(1;2,2)$, we have:*

$$\left|P_n^{(m)}(z)\right|\le c_3\frac{|\Phi^{n-m+1}(z)|}{d(z,\Gamma)}I_{n,p}^3(m)\|P_n\|_p,$$

where

$$I_{n,p}^{3}(m):=\begin{cases} n^{2\left(\frac{\gamma^{*}+1}{p}+m-1\right)}, & 1\le p<\frac{2\gamma^{*}}{3}+1,\ \gamma^{*}>0,\\ n^{2m+1-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=\frac{2\gamma^{*}}{3}+1,\ \gamma^{*}>0,\\ n^{2m+1-\frac{1}{p}}, & p>\frac{2\gamma^{*}}{3}+1,\ \gamma^{*}>0,\ or\\ & p\ge 1,\ -1<\gamma_1,\gamma_2\le 0. \end{cases} \tag{11}$$

After obtaining the estimates for the unbounded region above, we obtain the estimates of $\left|P_n^{(m)}(z)\right|$, $m\ge 0$, for the bounded region as follows:

Theorem 2.4. *Assume that* $p>0;\ \ G\in PC_{\theta}(1;\lambda_1,2)$ *with* $0<\lambda_1<2$. *Then, for arbitrary* $P_n\in\wp_n,\ n\in\mathbb{N}, n\ge m, m=0,1,2,\ldots$ *we have:*

$$\left|P_n^{(m)}(z)\right|\le c_4 n^{\tilde{\lambda}\left(m+\frac{\tilde{\gamma}+1}{p}\right)}\parallel P_n\parallel_p.$$

In the first result below, the case where the Γ has exterior nonzero angles at both points, and in the second result, the case where the Γ has interior zero angles at both points is examined.

Corollary 2.5. *Under the conditions of Theorem 2.4, for* $G\in PC_{\theta}(1;\lambda_1,\lambda_2)$, $0<\lambda_1,\lambda_2<2$, *we get:*

$$\left|P_n^{(m)}(z)\right|\le c_5 n^{\lambda^{*}\left(m+\frac{\gamma^{*}+1}{p}\right)}\parallel P_n\parallel_p.$$

Corollary 2.6. *Under the conditions of Theorem 2.4, for* $G\in PC_{\theta}(1;2,2)$ *we get:*

$$\left|P_n^{(m)}(z)\right|\le c_6 n^{2\left(m+\frac{\gamma^{*}+1}{p}\right)}\parallel P_n\parallel_p.$$

Combining Theorem 2.1 and Theorem 2.4, we obtain the following theorem in the whole complex plane:

Theorem 2.7. *Assume that* $p\ge 1;\ \ G\in PC_{\theta}(1;\lambda_1,2)$ *with* $0<\lambda_1<2$. *Then, for arbitrary* $P_n\in\wp_n,\ n\in\mathbb{N}, n\ge m, m=1,2,\ldots$ *we have:*

$$\left\|P_n^{(m)}\right\|_{\infty}\le c_7\|P_n\|_p\begin{cases}\frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,\Gamma)}I_{n,p}^{1}(m), & z\in\Omega_{1+\frac{1}{n}},\\ n^{\tilde{\lambda}\left(m+\frac{\tilde{\gamma}+1}{p}\right)}, & z\in\overline{G}_{1+\frac{1}{n}},\end{cases}$$

where $I_{n,p}^{1}(m)$ *is defined as in* (9).

Respectively, combining Corollaries 2.2 and 2.3 with Corollaries 2.5 and 2.6, we get:

Corollary 2.8. *Assume that* $p\ge 1; G\in PC_{\theta}(1;\lambda_1,\lambda_2)$ *with* $0<\lambda_1,\lambda_2<2$. *Then, for arbitrary* $P_n\in\wp_n,\ n\in\mathbb{N}, n\ge m$; *any* $m=1,2,\ldots$, *and* $z\in\Omega_R^{j}$, *we have:*

$$\left\|P_n^{(m)}\right\|_{\infty}\le c_8\|P_n\|_p\begin{cases}\frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,\Gamma)}I_{n,p}^{2}(m), & z\in\Omega_{1+\frac{1}{n}},\\ n^{\lambda^{*}\left(m+\frac{\gamma^{*}+1}{p}\right)}, & z\in\overline{G}_{1+\frac{1}{n}},\end{cases}$$

where $I_{n,p}^{2}(m)$ *is defined as in* (10).

Corollary 2.9. *Assume that* $p\ge 1; G\in PC_{\theta}(1;\lambda_1,\lambda_2)$ *with* $0<\lambda_1,\lambda_2<2$. *Then, for arbitrary* $P_n\in\wp_n,\ n\in\mathbb{N}, n\ge m$; *any* $m=1,2,\ldots$, *and* $z\in\Omega_R^{j}$, *we have:*

$$\left\|P_n^{(m)}\right\|_{\infty}\le c_9\|P_n\|_p\begin{cases}\frac{\left|\Phi^{n-m+1}(z)\right|}{d(z,\Gamma)}I_{n,p}^{3}(m), & z\in\Omega_{1+\frac{1}{n}},\\ n^{2\left(m+\frac{\gamma^{*}+1}{p}\right)}, & z\in\overline{G}_{1+\frac{1}{n}},\end{cases}$$

where $I_{n,p}^{3}(m)$ *is defined as in* (11).

Some Auxiliary Results

Lemma 3.1. *[37] Let* $\Gamma=\partial G$ *be a* K *−quasiconformal curve,* $z_1\in\Gamma,\ z_2,z_3\in\Omega\cap\{z:|z-z_1|\preceq d(z_1,\Gamma_{R_0})\};\ w_j=\Phi(z_j), j=1,2,3$. *Then*

a) *The statements* $|z_1-z_2|\preceq|z_1-z_3|$ *and* $|w_1-w_2|\preceq|w_1-w_3|$ *are equivalent. So* $|z_1-z_2|\asymp|z_1-z_3|$ *and* $|w_1-w_2|\asymp|w_1-w_3|$ *are also equivalent.*

b) *If* $|z_1 - z_2| \preceq |z_1 - z_3|$, *then*
$\left|\frac{w_1-w_3}{w_1-w_2}\right|^{K^{-2}} \preceq \left|\frac{z_1-z_3}{z_1-z_2}\right| \preceq \left|\frac{w_1-w_3}{w_1-w_2}\right|^{K^2}$,
where $R_0 > 1$ *is a constant, depending on* G.

Corollary 3.2. *Under the assumptions of Lemma 3.1, if* $z_3 \in \Gamma_{R_0}$, *then*
$|w_1 - w_2|^{K^2} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{K^{-2}}$.

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Corollary 3.3. *If* $\Gamma \in C_\theta$, *then*
$|w_1 - w_2|^{1+\varepsilon} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{1-\varepsilon}$,

for all $\varepsilon > 0$.

From the results in [38] and [39], we have the following lemma as a consequence:

Lemma 3.4. *Let* $G \in PC_\theta(1; \lambda_1, \ldots, \lambda_l)$, $0 < \lambda_j < 2$, $j = 1,2,\ldots, l$,. *Then*

a) *for any* $w \in D_j(\eta)$, $|w - w_j|^{\lambda_j+\varepsilon} \preceq |\Psi(w) - \Psi(w_j)| \preceq |w - w_j|^{\lambda_j-\varepsilon}$, $|w - w_j|^{\lambda_j-1+\varepsilon} \preceq |\Psi'(w)| \preceq |w - w_j|^{\lambda_j-1-\varepsilon}$;

b) *for any* $w \in \widehat{D}(\eta)$, $(|w| - 1)^{1+\varepsilon} \preceq d(\Psi(w), \Gamma)| \preceq (|w| - 1)^{1-\varepsilon}$, $(|w| - 1)^{\varepsilon} \preceq |\Psi'(w)| \preceq (|w| - 1)^{-\varepsilon}$.

Let $\{z_j\}_{j=1}^{l}$ be a fixed system of distinct points on curve Γ ordered in the positive direction and the weight function $h(z)$ be defined as in (2).

Lemma 3.5. *[3] Let* Γ *be a rectifiable Jordan curve*; $h(z)$ *as defined in (2). Then, for arbitrary* $P_n(z) \in \wp_n$, *any* $R > 1$ *and* $n \in \mathbb{N}$, *we have:*

$$\|P_n\|_{\mathcal{L}_p(h,\Gamma_R)} \preceq R^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h,\Gamma)}, \ p > 0, \tag{12}$$

where $\gamma^* = \max\{\gamma_k : k \leq m\}$.

Proofs

Proof of Theorem 2.1. Let $G \in C_\theta(1; \lambda_1, 2)$ with $0 < \lambda_1 < 2$, $R = 1 + \frac{\varepsilon_0}{n}$, $R_1 := 1 + \frac{R-1}{3}$ and $R_2 := 1 + \frac{2(R-1)}{3}$. For $z \in \Omega$ and $0 \leq m < n$, let us define

$$S_{n,m}(z) := \frac{P_n^{(m)}(z)}{\Phi^{n-m+1}(z)}.$$

Since the function $S_{n,m}(z)$ is analytic in Ω and continuous on $\overline{\Omega}$ and moreover $S_{n,m}(\infty) = 0$, then by the Cauchy integral formula for unbounded region, we have:

$$S_{n,m}(z) = -\frac{1}{2\pi i}\int_{\Gamma_{R_1}} S_{n,m}(\xi)\frac{d\xi}{\xi - z}, \ z \in \Omega_{R_1}.$$

If we consider the fact that $|\Phi^{n-m+1}(\xi)| = R_1^{n-m+1} > 1$ for $\xi \in \Gamma_{R_1}$, we obtain the following:

$$\left|\frac{P_n^{(m)}(z)}{\Phi^{n-m+1}(z)}\right| \leq \frac{1}{2\pi}\int_{\Gamma_{R_1}} \left|\frac{P_n^{(m)}(\xi)}{\Phi^{n-m+1}(\xi)}\right| \frac{|d\xi|}{|\xi - z|} \leq \frac{1}{2\pi d(z,\Gamma_{R_1})}\int_{\Gamma_{R_1}} \left|P_n^{(m)}(\xi)\right||d\xi|,$$

and so,

$$\left|P_n^{(m)}(z)\right| \leq \frac{|\Phi^{n-m+1}(z)|}{2\pi d(z,\Gamma_{R_1})}\int_{\Gamma_{R_1}} \left|P_n^{(m)}(\xi)\right||d\xi|. \tag{13}$$

Let us express the Cauchy $m - th$ derivative formula for $P_n(\xi)$:

$$P_n^{(m)}(\xi) = \frac{m!}{2\pi i}\int_{\Gamma_{R_2}} P_n(t)\frac{dt}{(t-\xi)^{m+1}}, \ \xi \in G_{R_2}, \tag{14}$$

and take $\xi \in \Gamma_{R_1}$, so that by substituting (14) into (13), we obtain the following:

$$\left|P_n^{(m)}(z)\right| \leq \frac{|\Phi^{n-m+1}(z)|}{d(z,\Gamma_{R_1})}\int_{\Gamma_{R_1}} \left(\int_{\Gamma_{R_2}} |P_n(t)|\frac{|dt|}{|t-\xi|^{m+1}}\right)|d\xi| \tag{15}$$

$$\leq \frac{|\Phi^{n-m+1}(z)|}{d(z,\Gamma_{R_1})} \sup_{t\in\Gamma_{R_2}} \left(\int_{\Gamma_{R_1}} \frac{|d\xi|}{|t-\xi|^{m+1}}\right)\int_{\Gamma_{R_2}} |P_n(t)||dt|. \tag{16}$$

Denote by

$$\int_{\Gamma_{R_1}} \frac{|d\xi|}{|t-\xi|^{m+1}} =: I_1(\Gamma_{R_1}); \int_{\Gamma_{R_2}} |P_n(t)||dt| =: I_2(\Gamma_{R_2}), \tag{17}$$

and estimate these integrals separately.

Before we move on to the integral estimations, let us mention that we have the following estimation for $|\Psi'(\tau)|$ ([40, Th.2.8]):

$$|\Psi'(\tau)| \asymp \frac{d(\Psi(\tau),\Gamma)}{|\tau|-1}. \tag{18}$$

1) Estimation of $I_1(\Gamma_{R_1}) = \int_{\Gamma_{R_1}} \frac{|d\xi|}{|t-\xi|^{m+1}}$.

Replacing the variable $\tau = \Phi(\xi), \xi \in \Gamma_{R_1}, w = \Phi(t), t \in \Gamma_{R_2}$, we have:

$$I_1(\Gamma_{R_1}) = \int_{|\tau|=R_1} \frac{|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w)|^{m+1}}. \tag{19}$$

Considering the corner points $z_j = \Psi(w_j), j = 1,2$ to estimate the $I_1(\Gamma_{R_1})$, we define the following sets for which we can use the geometric properties of the curve Γ:

$$H_{\rho,j}^1 := \{\tau: \tau \in \Phi(\Gamma_\rho^j), |\tau - w_j| < c_j(\rho-1)\}, H_{\rho,j}^{11} := \{\tau \in H_{\rho,j}^1: |\tau - w_j| \geq |\tau - w|\},$$
$$H_{\rho,j}^{12} := H_{\rho,j}^1 \backslash H_{\rho,j}^{11};$$
$$H_{\rho,j}^2 := \{\tau: \tau \in \Phi(\Gamma_\rho^j), c_j(\rho-1) \leq |\tau - w_j| < \eta\}, H_{\rho,j}^{21} := \{\tau \in H_{\rho,j}^2: |\tau - w_j| \geq |\tau - w|\},$$
$$H_{\rho,j}^{22} := H_{\rho,j}^2 \backslash H_{\rho,j}^{21};$$
$$H_{\rho,j}^3 := \{\tau: \tau \in \Phi(\Gamma_\rho)\} \backslash \left(H_{\rho,j}^1 \cup H_{\rho,j}^2\right), j = 1,2$$

where $\rho \geq 1$. For $\rho = R_1$, we perform the estimation of $I_1(\Gamma_{R_1})$ separately for the sets specified above:

$$I_1(\Gamma_{R_1}) = \sum_{i,j=1}^2 I_1(H_{R_1,j}^i) + I_1(H_{R_1,j}^3). \tag{20}$$

By using Lemma 3.1 and Lemma 3.4 we get the following estimates:

$$I_1(H_{R_1,1}^1) = \int\limits_{H_{R_1,1}^1} \frac{|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w)|^{m+1}} \asymp \int\limits_{H_{R_1,1}^1} \frac{|\tau - w_1|^{\lambda_1-1-\varepsilon}}{|\tau - w|^{(m+1)(\lambda_1+\varepsilon)}}|d\tau|;$$

$$I_1(H_{R_1,1}^2) = \int_{H_{R_1,1}^2} \frac{|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w)|^{m+1}} \asymp \int_{H_{R_1,1}^2} \frac{|\tau-w_1|^{\lambda_1-1-\varepsilon}}{|\tau-w|^{(m+1)(\lambda_1+\varepsilon)}}|d\tau|$$
$$= \int_{H_{R_1,1}^{21}} \frac{|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w)|^{m+1}}|d\tau| + \int_{H_{R_1,1}^{22}} \frac{|\tau-w_1|^{\lambda_1-1-\varepsilon}}{|\tau-w|^{(m+1)(\lambda_1+\varepsilon)}}|d\tau|.$$

Now, we can estimate these integrals for $1 \leq \lambda_1 < 2$ and $0 < \lambda_1 < 1$ separately.

If $1 \leq \lambda_1 < 2$, then we have:

$$I_1(H_{R_1,1}^1) \asymp \int\limits_{H_{R_1,1}^1} \frac{|\tau - w_1|^{\lambda_1-1-\varepsilon}}{|\tau - w|^{(m+1)(\lambda_1+\varepsilon)}}|d\tau| \preceq \left(\frac{1}{n}\right)^{\lambda_1-1-\varepsilon} \int\limits_{H_{R_1,1}^1} \frac{|d\tau|}{|\tau - w|^{(m+1)(\lambda_1+\varepsilon)}} \preceq \left(\frac{1}{n}\right)^{\lambda_1-1-\varepsilon} n^{(m+1)(\lambda_1+\varepsilon)-1} = n^{m\lambda_1+\varepsilon}.$$

$$I_1(H_{R_1,1}^{21}) = \int\limits_{H_{R_1,1}^{21}} \frac{|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w)|^{m+1}}|d\tau| = \int\limits_{\Psi\left(H_{R_1,1}^{21}\right)} \frac{|d\xi|}{|\xi - t|^{m+1}} \preceq \int\limits_{d(\xi,\Gamma_{R_2})}^{c} \frac{|d\xi|}{|\xi - t|^{m+1}} \preceq \frac{1}{d^m(\xi,\Gamma_{R_2})} \preceq n^{m\lambda_1+\varepsilon};$$

$$I_1(H_{R_1,1}^{22}) = \int\limits_{H_{R_1,1}^{22}} \frac{|\tau - w_1|^{\lambda_1-1-\varepsilon}}{|\tau - w|^{(m+1)(\lambda_1+\varepsilon)}}|d\tau| \preceq \int\limits_{H_{R_1,1}^{22}} \frac{|d\tau|}{|\tau - w|^{(m+1)\lambda_1-\lambda_1+1+\varepsilon}} \preceq n^{m\lambda_1+\varepsilon}.$$

Let $0 < \lambda_1 < 1$. In this case we have:

$$I_1(H_{R_1,1}^1) \asymp \int\limits_{H_{R_1,1}^1} \frac{|\tau - w_1|^{\lambda_1-1-\varepsilon}}{|\tau - w|^{(m+1)(\lambda_1+\varepsilon)}}|d\tau| = \int\limits_{H_{R_1,1}^1} \frac{|d\tau|}{|\tau - w|^{1-\lambda_1+(m+1)(\lambda_1+\varepsilon)}}$$
$$\preceq n^{1-\lambda_1+(m+1)(\lambda_1+\varepsilon)} mes(H_{R_1,1}^1) \preceq n^{1-\lambda_1+(m+1)(\lambda_1+\varepsilon)-1} = n^{m\lambda_1+\varepsilon}).$$

$$I_1(H_{R_1,1}^2) = \int\limits_{H_{R_1,1}^2} \frac{|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w)|^{m+1}} \asymp \int\limits_{H_{R_1,1}^2} \frac{|\tau - w_1|^{\lambda_1-1-\varepsilon}}{|\tau - w|^{(m+1)(\lambda_1+\varepsilon)}}|d\tau|$$
$$= \int_{H_{R_1,1}^{21}} \frac{|d\tau|}{|\tau-w|^{1-\lambda_1+(m+1)(\lambda_1+\varepsilon)}} + \int_{H_{R_1,1}^{22}} \frac{|d\tau|}{|\tau-w_1|^{1-\lambda_1+(m+1)(\lambda_1+\varepsilon)}} \preceq n^{m\lambda_1+\varepsilon} + n^{m\lambda_1+\varepsilon} \preceq n^{m\lambda_1+\varepsilon}.$$

Finally, applying (18) and Corollary 3.3, for $k = 1,2$, we have:

$$I_1\left(H_{R_1,2}^k\right) \asymp \int_{H_{R_1,2}^k} \frac{d(\Psi(\tau),\Gamma)}{|\Psi(\tau)-\Psi(w)|^{m+1}(|\tau|-1)}|d\tau| \preceq n\int_{H_{R_1,2}^k} \frac{|d\tau|}{|\tau-w|^{2m}} \preceq n^{2m},$$

$$I_1(H_{R_1,k}^3) \asymp \int_{H_{R_1,k}^3} \frac{d(\Psi(\tau),\Gamma)}{|\Psi(\tau)-\Psi(w)|^{m+1}(|\tau|-1)}|d\tau| \preceq n\int_{H_{R_1,2}^1} \frac{|d\tau|}{|\tau-w|^{m(1+\varepsilon)}} \preceq n^{2(1+\varepsilon)}.$$

Combining all this estimates, we obtain for all $0 < \lambda_1 \leq 2$:

$$I_1(\Gamma_{R_1}) \preceq n^{m\tilde{\lambda}}. \tag{21}$$

2) Estimation of $I_2(\Gamma_{R_2}) = \int_{\Gamma_{R_2}} |P_n(t)||dt|$.

We consider the cases $p > 1$ and $p = 1$ separately. Firstly, we assume that $p > 1$ for the weighted $\mathcal{L}_p$ −norm . Multiplying and dividing the integrand in $I_2(\Gamma_{R_2})$ by $h^{\frac{1}{p}}(t)$, using Hölder inequality and (16), we get:

$$I_2(\Gamma_{R_2}) = \int_{\Gamma_{R_2}} |P_n(t)||dt| \leq \left(\int_{\Gamma_{R_2}} h(t)|P_n(t)|^p|dt|\right)^{\frac{1}{p}} \times \left(\int_{\Gamma_{R_2}} \frac{|dt|}{\frac{q}{h^p(t)}}\right)^{\frac{1}{q}}$$

$$\leq \|P_n\|_p \cdot \left(\int_{\Gamma_{R_2}} \frac{|dt|}{h^{q-1}(t)}\right)^{\frac{1}{q}} =: \|P_n\|_p \times \tilde{I}_2(\Gamma_{R_2}) \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{22}$$

To estimate the last integral, first we replace the variable $w = \Phi(t)$ and after we set:

$H^1_{R_2,j} := \{w: w \in \Phi(\Gamma^j_{R_2}), |w - w_j| < c_j(R_2 - 1)\}$,

$H^2_{R_2,j} := \{w: w \in \Phi(\Gamma^j_{R_2}), c_j(R_2 - 1) \leq |w - w_j| < \eta\}$,

$H^3_{R_2,j} := \{w: w \in \Phi(\Gamma^j_{R_2}), \eta \leq |w - w_j| < diam\overline{G}\}$,

where $0 < c_j < \eta$ is chosen so that $\{w: |w - w_j| < c_j(R_2 - 1)\} \cap D(0,1) \neq \emptyset$ and $\Phi(\Gamma_{R_2}) = \Phi(\cup^2_{j=1} \Gamma^j_2) = \cup^2_{j=1} \Phi(\Gamma^j_{R_2}) = \cup^2_{j=1} \cup^3_{i=1} H^j_{R_2,i}$. Considering all these and also taking into account the points z_1 and z_2 are isolated, we get:

$$\left[\tilde{I}_2(\Gamma_{R_2})\right]^q = \int_{|w|=R_2} \frac{|\Psi'(w)|}{|\Psi(w)-\Psi(w_1)|^{\gamma_1(q-1)}|\Psi(w)-\Psi(w_2)|^{\gamma_2(q-1)}}|dw|$$

$$\leq \sum^2_{j=1} \int_{\Phi(\Gamma^j_{R_2})} \frac{|\Psi'(w)|}{|\Psi(w)-\Psi(w_1)|^{\gamma_1(q-1)}|\Psi(w)-\Psi(w_2)|^{\gamma_2(q-1)}}|dw|$$

$$\asymp \sum^2_{j=1} \int_{\Phi(\Gamma^j_{R_2})} \frac{|\Psi'(w)|}{|\Psi(w)-\Psi(w_j)|^{\gamma_j(q-1)}}|dw| =: \sum^2_{j=1} L_{n,2}(w_j).$$

Therefore, we need to estimate $L_{n,2}(w_j)$ for $j = 1,2$:

$$L_{n,2}(w_j) = \sum^3_{i=1} \int_{H^i_{R_2,j}} \frac{|\Psi'(w)|}{|\Psi(w)-\Psi(w_j)|^{\gamma_j(q-1)}}|dw| = \sum^3_{i=1} L^i_{n,2}(w_j). \tag{23}$$

For $j = 1,2$, we evaluate integrals in (23) separately for $\gamma_j > 0$ and $\gamma_j \leq 0$.

$$L^1_{n,2}(w_1) = \int_{H^1_{R_2,1}} \frac{|\Psi'(w)|}{|\Psi(w)-\Psi(w_1)|^{\gamma_1(q-1)}}|dw| \leq \int_{H^1_{R_2,1}} \frac{|w-w_1|^{\lambda_1-1-\varepsilon}}{|w-w_1|^{\gamma_1(q-1)(\lambda_1+\varepsilon)}}|dw|$$

$$\leq n^{\gamma_1(q-1)(\lambda_1+\varepsilon)-(\lambda_1-1-\varepsilon)} \int_{H^1_{R_2,1}} |dw| \leq n^{[\gamma_1(q-1)-1]\lambda_1+1+\varepsilon} mes(H^1_{R_2,1})$$

$$\leq n^{[\gamma_1(q-1)-1]\lambda_1+\varepsilon},$$

$$\left(L^1_{n,2}(w_1)\right)^{\frac{1}{q}} \leq n^{\left(\frac{\gamma_1+1}{p}-1\right)\lambda_1+\varepsilon} \tag{24}$$

for $\gamma_1 > 0$ and

$$L^1_{n,2}(w_1) \leq \int_{H^1_{R_2,1}} \frac{|w-w_1|^{\lambda_1-1-\varepsilon}}{|w-w_1|^{\gamma_1(q-1)(\lambda_1+\varepsilon)}}|dw| \leq \int_{H^1_{R_2,1}} |w - w_1|^{(\lambda_1-1)+(-\gamma_1)(q-1)\lambda_1+\varepsilon}|dw|$$

$$\leq \left(\frac{1}{n}\right)^{\lambda_1+(-\gamma_1)(q-1)\lambda_1+\varepsilon} \leq n,$$

$$\left(L^1_{n,2}(w_1)\right)^{\frac{1}{q}} \leq n^{1-\frac{1}{p}} \tag{25}$$

for $\gamma_1 \leq 0$.

$$L^2_{n,2}(w_1) = \int_{H^2_{R_2,1}} \frac{|\Psi'(w)|}{|\Psi(w)-\Psi(w_1)|^{\gamma_1(q-1)}}|dw| \asymp \int_{H^2_{R_2,1}} \frac{d(\Psi(w),\Gamma)}{|w|-1} \frac{|dw|}{|\Psi(w)-\Psi(w_1)|^{\gamma_1(q-1)}}$$

$$\leq n \int_{H^2_{R_2,1}} \frac{|dw|}{|w-w_1|^{[\gamma_1(q-1)-1](\lambda_1+\varepsilon)}} \leq n \int_{H^2_{R_2,1}} \frac{|dw|}{|w-w_1|^{[\gamma_1(q-1)-1]\lambda_1+\varepsilon}}$$

$$\leq \begin{cases} n^{[\gamma_1(q-1)-1]\lambda_1+\varepsilon}, & [\gamma_1(q-1)-1]\lambda_1+\varepsilon > 1, \\ n\ln n, & [\gamma_1(q-1)-1]\lambda_1+\varepsilon = 1, \\ n, & [\gamma_1(q-1)-1]\lambda_1+\varepsilon < 1, \end{cases}$$

$$\left(L^2_{n,2}(w_1)\right)^{\frac{1}{q}} \leq \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)\lambda_1+\varepsilon}, & p < \frac{\lambda_1(\gamma_1+1)+1}{\lambda_1+1} + \varepsilon, \\ (n\ln n)^{1-\frac{1}{p}}, & p = \frac{\lambda_1(\gamma_1+1)+1}{\lambda_1+1} + \varepsilon, \\ n^{1-\frac{1}{p}}, & p > \frac{\lambda_1(\gamma_1+1)+1}{\lambda_1+1} + \varepsilon, \end{cases} \tag{26}$$

for $\gamma_1 > 0$;

$$L_{n,2}^{2}(w_1) \preceq \int_{H_{R_2,1}^{2}} \frac{|w-w_1|^{\lambda_1-1-\varepsilon}}{|w-w_1|^{\gamma_1(q-1)(\lambda_1+\varepsilon)}}|dw| = \int_{H_{R_2,1}^{2}} |w-w_1|^{(\lambda_1+(-\gamma_1)(q-1)\lambda_1-1+\varepsilon}|dw|$$
$$\preceq \int_{H_{R_2,1}^{2}} |w-w_1|^{-1}|dw| \preceq n\int_{H_{R_2,1}^{2}} |dw| \preceq n,$$

$$\left(L_{n,2}^{2}(w_1)\right)^{\frac{1}{q}} \preceq n^{1-\frac{1}{p}} \tag{27}$$

for $\gamma_1 \leq 0$.

$$L_{n,2}^{1}(w_2) = \int_{H_{R_2,2}^{1}} \frac{|\Psi'(w)|}{|\Psi(w)-\Psi(w_2)|^{\gamma_2(q-1)}}|dw| \asymp \int_{H_{R_2,2}^{1}} \frac{d(\Psi(w),\Gamma)}{|w|-1}\frac{|dw|}{|\Psi(w)-\Psi(w_2)|^{\gamma_2(q-1)}}$$
$$\preceq n\int_{H_{R_2,2}^{1}} \frac{|dw|}{|\Psi(w)-\Psi(w_2)|^{\gamma_2(q-1)-1}} \preceq n\int_{H_{R_2,2}^{1}} \frac{|dw|}{|w-w_2|^{2[\gamma_2(q-1)-1]}}$$
$$\preceq n^{1+2[\gamma_2(q-1)-1]} mes\left(H_{R_2,2}^{1}\right) \preceq n^{2[\gamma_2(q-1)-1]},$$

$$\left(L_{n,2}^{1}(w_2)\right)^{\frac{1}{q}} \preceq n^{2\left(\frac{\gamma_2+1}{p}-1\right)} \tag{28}$$

for $\gamma_2 > 0$;

$$L_{n,2}^{1}(w_2) = n\int_{H_{R_2,2}^{1}} |\Psi(w)-\Psi(w_2)|^{-(\gamma_2)(q-1)+1}|dw| \preceq n\ mes(H_{R_2,2}^{1}) \preceq 1;$$

$$\left(L_{n,2}^{1}(w_2)\right)^{\frac{1}{q}} \preceq 1 \tag{29}$$

for $\gamma_2 \leq 0$.

$$L_{n,2}^{2}(w_2) \preceq n\int_{H_{R_2,2}^{2}} \frac{|dw|}{|\Psi(w)-\Psi(w_2)|^{\gamma_2(q-1)-1}} \preceq n\int_{H_{R_2,2}^{2}} \frac{|dw|}{|w-w_2|^{2[\gamma_2(q-1)-1]}}$$
$$\preceq \begin{cases} n^{2[\gamma_2(q-1)-1]}, & 2[\gamma_2(q-1)-1] > 1, \\ n\ln n, & 2[\gamma_2(q-1)-1] = 1, \\ n, & 2[\gamma_2(q-1)-1] < 1, \end{cases}$$

$$\left(L_{n,2}^{2}(w_2)\right)^{\frac{1}{q}} \preceq \begin{cases} n^{2\left(\frac{\gamma_2+1}{p}-1\right)}, & p < \frac{2(\gamma_2+1)+1}{3}, \\ (n\ln n)^{1-\frac{1}{p}}, & p = \frac{2(\gamma_2+1)+1}{3}, \\ n^{1-\frac{1}{p}}, & p > \frac{2(\gamma_2+1)+1}{3}, \end{cases} \tag{30}$$

for $\gamma_2 > 0$ and

$$L_{n,2}^{2}(w_2) = n\int_{H_{R_2,2}^{2}} |\Psi(w)-\Psi(w_2)|^{-(\gamma_2)(q-1)+1}|dw| \preceq n\ mes\left(H_{R_2,2}^{2}\right) \preceq n,$$

$$\left(L_{n,2}^{2}(w_2)\right)^{\frac{1}{q}} \preceq n^{1-\frac{1}{p}} \tag{31}$$

for $\gamma_2 \leq 0$.

For $w \in H_{R_2,j}^{3}$, we have $\left|\Psi(w)-\Psi(w_j)\right| \geq 1$, then for $j = 1,2$, we get:

$$L_{n,2}^{3}(w_j) \preceq \int_{H_{R_2,j}^{3}} \frac{d(\Psi(w),\Gamma)}{|w|-1}|dw| \preceq \int_{H_{R_2,j}^{3}} (|w|-1)^{-\varepsilon}|dw| \preceq n^{\varepsilon},$$

$$\left(L_{n,2}^{3}(w_j)\right)^{\frac{1}{q}} \preceq n^{\varepsilon\left(1-\frac{1}{p}\right)}. \tag{32}$$

Combining estimates (23)-(32), we get:

$$I_2(\Gamma_{R_2}) \preceq \|P_n\|_p \left[\begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)\tilde{\lambda}_1}, & p < \frac{\tilde{\lambda}_1(\gamma_1+1)+1}{\tilde{\lambda}_1+1}, \\ (n\ln n)^{1-\frac{1}{p}}, & p = \frac{\tilde{\lambda}_1(\gamma_1+1)+1}{\tilde{\lambda}_1+1}, \\ n^{1-\frac{1}{p}}, & p > \frac{\tilde{\lambda}_1(\gamma_1+1)+1}{\tilde{\lambda}_1+1}, \end{cases} + \begin{cases} n^{2\left(\frac{\gamma_2+1}{p}-1\right)}, & p < \frac{2(\gamma_2+1)+1}{3}, \\ (n\ln n)^{1-\frac{1}{p}}, & p = \frac{2(\gamma_2+1)+1}{3}, \\ n^{1-\frac{1}{p}}, & p > \frac{2(\gamma_2+1)+1}{3}, \end{cases} \right]$$

$$\preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)\tilde{\lambda}_1}, & 1<p<\frac{2}{3}\gamma_2+1,\ \gamma_1>\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right), \\ n^{\left(\frac{\gamma_1+1}{p}-1\right)\tilde{\lambda}_1}, & 1<p<\frac{\tilde{\lambda}_1}{\tilde{\lambda}_1+1}\gamma_1+1,\ \gamma_1>\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right), \\ n^{2\left(\frac{\gamma_2+1}{p}-1\right)}, & 1<p<\frac{2}{3}\gamma_2+1,\ 0<\gamma_1<\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right), \\ n^{2\left(\frac{\gamma_2+1}{p}-1\right)}, & 1<p<\frac{\tilde{\lambda}_1}{\tilde{\lambda}_1+1}\gamma_1+1,\ 0<\gamma_1<\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right), \\ (n\ln n)^{1-\frac{1}{p}}, & p=\frac{\tilde{\lambda}}{\tilde{\lambda}+1}\tilde{\gamma}+1,\ \gamma_1=\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right), \\ n^{1-\frac{1}{p}}, & p>\frac{2}{3}\gamma_2+1,\ 0<\gamma_1<\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right), \\ n^{1-\frac{1}{p}}, & p>\frac{\tilde{\lambda}_1}{\tilde{\lambda}_1+1}\gamma_1+1, \gamma_1>\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right), \end{cases}$$

for $\gamma_1,\gamma_2>0$ and

$$I_2(\Gamma_{R_2}) \preceq \|P_n\|_p n^{1-\frac{1}{p}}$$

for $\gamma_1,\gamma_2\leq 0$. Then, we obtain:

$$I_2(\Gamma_{R_2}) \preceq \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)\tilde{\lambda}_1}, & 1<p<\frac{\tilde{\lambda}}{\tilde{\lambda}+1}\tilde{\gamma}+1,\ \gamma_1>\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right),\ \gamma_2>0, \\ n^{2\left(\frac{\gamma_2+1}{p}-1\right)}, & 1<p<\frac{\tilde{\lambda}}{\tilde{\lambda}+1}\tilde{\gamma}+1,\ 0<\gamma_1<\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right),\ \gamma_2>0, \\ (n\ln n)^{1-\frac{1}{p}}, & p=\frac{\tilde{\lambda}}{\tilde{\lambda}+1}\tilde{\gamma}+1,\ \gamma_1=\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right),\ \gamma_2>0, \\ n^{1-\frac{1}{p}}, & p>\frac{2}{3}\gamma_2+1,\ 0<\gamma_1<\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right),\ \gamma_2>0, \\ n^{1-\frac{1}{p}}, & p>\frac{\tilde{\lambda}_1}{\tilde{\lambda}_1+1}\gamma_1+1,\ \gamma_1>\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right),\ \gamma_2>0, \\ n^{1-\frac{1}{p}}, & p>1,\ -1<\gamma_1,\gamma_2\leq 0. \end{cases} \tag{33}$$

Therefore, combining (16), (17), (21) and (33), for the case $p>1,\gamma_1>-1,\gamma_2>-1$, and all $z\in\Omega_{R_1}$, we obtain:

$$\left|P_n^{(m)}(z)\right| \preceq \frac{|\Phi^{n-m+1}(z)|}{d(z,\Gamma_{R_1})}\|P_n\|_p \times \begin{cases} n^{\left(\frac{\gamma_1+1}{p}+m-1\right)\tilde{\lambda}_1}, & 1<p<\frac{\tilde{\lambda}}{\tilde{\lambda}+1}\tilde{\gamma}+1,\ \gamma_1>\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right),\ \gamma_2>0, \\ n^{2\left(\frac{\gamma_2+1}{p}+m-1\right)}, & 1<p<\frac{\tilde{\lambda}}{\tilde{\lambda}+1}\tilde{\gamma}+1,\ 0<\gamma_1<\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right),\ \gamma_2>0, \\ n^{m\tilde{\lambda}+1-\frac{1}{p}}(\ln n)^{1-\frac{1}{p}}, & p=\frac{\tilde{\lambda}}{\tilde{\lambda}+1}\tilde{\gamma}+1,\ \gamma_1=\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right),\ \gamma_2>0, \\ n^{m\tilde{\lambda}+1-\frac{1}{p}}, & p>\frac{2}{3}\gamma_2+1,\ 0<\gamma_1<\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right),\ \gamma_2>0, \\ n^{m\tilde{\lambda}+1-\frac{1}{p}}, & p>\frac{\tilde{\lambda}_1}{\tilde{\lambda}_1+1}\gamma_1+1,\ \gamma_1>\frac{2}{3}\gamma_2\left(1+\frac{1}{\tilde{\lambda}_1}\right),\ \gamma_2>0, \\ n^{m\tilde{\lambda}+1-\frac{1}{p}}, & p>1,\ -1<\gamma_1,\gamma_2\leq 0. \end{cases}$$

For the case $p>1$, we have completed the estimations and can move on to the case $p=1$. For the estimation, we first multiply and divide the integrand in (15) by h, then by some operations we obtain the following:

$$\begin{aligned} \left|P_n^{(m)}(z)\right| &\preceq \frac{|\Phi^{n-m+1}(z)|}{d(z,\Gamma_{R_1})}\cdot\sup_{t\in\Gamma_{R_2}}\left(\int_{\Gamma_{R_1}}\frac{|d\xi|}{h(t)|t-\xi|^{m+1}}\right)\left(\int_{\Gamma_{R_2}}h(t)|P_n(t)||dt|\right) \\ &\preceq \frac{|\Phi^{n-m+1}(z)|}{d(z,\Gamma_{R_1})}\|P_n\|_{\mathcal{L}_1(h,\Gamma_{R_2})}\cdot\sup_{t\in\Gamma_{R_2}}\left(\int_{\Gamma_{R_1}}\frac{|d\xi|}{\prod_{j=1}^{2}|t-z_j|^{\gamma_j}|t-\xi|^{m+1}}\right). \end{aligned} \tag{34}$$

By changing the variable $t=\Psi(w)$, we obtain the following according to Lemma 3.1 and Lemma 3.4:

$$\begin{aligned} \frac{1}{|t-z_1|^{\gamma_1}}\cdot\frac{1}{|t-z_2|^{\gamma_2}} &= \frac{1}{|\Psi(w)-\Psi(w_1)|^{\gamma_1}}\cdot\frac{1}{|\Psi(w)-\Psi(w_2)|^{\gamma_2}} \\ &\preceq \frac{1}{|w-w_1|^{\gamma_1(\lambda_1+\varepsilon)}}\cdot\frac{1}{|w-w_2|^{2\gamma_2}} \\ &\preceq \begin{cases} n^{\tilde{\gamma}\tilde{\lambda}}, & \text{if } \gamma>0, \\ 1, & \text{if } \gamma\leq 0; \end{cases} \end{aligned}$$

so, we get:

$$\frac{1}{\prod_{j=1}^{2}|t-z_j|^{\gamma_j}} \preceq n^{\tilde{\gamma}^*\tilde{\lambda}}, \tag{35}$$

where $\tilde{\gamma}^* := \max\{0;\tilde{\gamma}\}$.

Considering the methods for estimating the integral (19) for $p=1$, we have:

$$\int_{\Gamma_{R_1}} \frac{|d\xi|}{|t-\xi|^{m+1}} \preceq n^{m\tilde{\lambda}}. \tag{36}$$

Also, from Lemma 3.5,

$$\|P_n\|_{\mathcal{L}_1(h,\Gamma_{R_2})} \preceq \|P_n\|_{\mathcal{L}_1(h,\Gamma_{R_1})}. \tag{37}$$

Finally, combining (34)-(37), we take the estimation for $p=1$ as follows:

$$\left|P_n^{(m)}(z)\right| \preceq \frac{|\Phi^{n-m+1}(z)|}{d(z,\Gamma_{R_1})} n^{(\tilde{\gamma}^*+m)\tilde{\lambda}} \|P_n\|_1.$$

Now, it remains to establish that $d(z,\Gamma_{R_1}) \succeq d(z,\Gamma)$ is satisfied for any $z \in \Omega_R$. Let $U_\infty(\Gamma,\delta) := \bigcup_{\xi\in\Gamma} U(\xi,\ \delta)$ and $U(\Gamma,\delta) := \bigcup_{j=1}^{N} U_j(\Gamma,\delta) \subset U_\infty(\Gamma,\delta)$ be defined as infinite open cover and finite open cover of Γ, respectively; $\Omega(\delta) := \Omega(\Gamma,\delta) := \Omega \cap U_N(\Gamma,\delta)$. For the points $z \notin \Omega(\Gamma_{R_1}, d(\Gamma_{R_1},\Gamma_R))$, it is obvious that $d(z,\Gamma_{R_1}) \succeq \delta \succeq d(z,\Gamma)$. Now consider the arbitrary point $z \in \Omega(\Gamma_{R_1}, d(\Gamma_{R_1},\Gamma_R))$. Let $\xi_1 \in \Gamma_{R_1}$ and $\xi_2 \in \Gamma$ be such points that $d(z,\Gamma_{R_1}) = |z-\xi_1|$ and $d(z,\Gamma) = |z-\xi_2|$. For $t=\Phi(z), t_1=\Phi(\xi_1), t_2=\Phi(\xi_2)$, we have: $|t-t_1| \geq \big||t-t_2|-|t_2-t_1|\big| \geq \left||t-t_2|-\frac{1}{2}|t-t_2|\right| \geq \frac{1}{2}|t-t_2|$. Then, according to Lemma 3.1, $d(z,\Gamma_{R_1}) \succeq d(z,\Gamma)$.

Proof of Theorem 2.4. The estimation of $\left|P_n^{(m)}(z)\right|$ is proved for $z \in \overline{G}$, for each $m \geq 0$ using the following theorem:

Theorem A [33] *Let $G \in PC_\theta(1,\lambda,2)$ for some $0<\lambda<2$, $h(z)$ be defined as in (2). Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, $\gamma_i > 1$, $i=1,2$ and arbitrary small $\varepsilon > 0$, we have:*

$$\|P_n\|_{C(\overline{G})} \leq c_{10} n^{\frac{\tilde{\gamma}+1}{p}\tilde{\lambda}} \|P_n\|_p$$

where $c_{10} = c_{10}(G,\gamma_i,\lambda_i,p,\varepsilon) > 0$ is the constant, independent of n and z, $\tilde{\gamma}$ and $\tilde{\lambda}$ are defined as in (8).

Let $z \in \Gamma$ is an arbitrary fixed point and, $B(z,d(z,\Gamma_{R_1}))$ denote the $d(z,\Gamma_{R_1})-$ neighborhood of z. According to the Cauchy integral formula for derivatives, we have:

$$P_n^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial B(z,d(z,\Gamma_{R_1}))} \frac{P_n(t)}{(t-z)^{m+1}}dt, m = 0,1,2,\ldots$$

Applying the Bernstein-Walsh inequality given in (3) and Lemmas 3.1 and 3.4, we get

$$\left|P_n^{(m)}(z)\right| \leq \frac{m!}{2\pi}\max_{z\in\partial B(z,d(z,\Gamma_{R_1}))}|P_n(t)| \cdot \int_{\partial B(z,d(z,\Gamma_{R_1}))}\frac{|dt|}{|t-z|^{m+1}}$$

$$\leq \frac{m!}{2\pi}\max_{t\in\overline{G}_{R_1}}|P_n(t)| \cdot \frac{2\pi d(z,\Gamma_R)}{d^{m+1}(z,\Gamma_R)} \preceq \frac{\|P_n\|_{C(\overline{G})}}{d^m(z,\Gamma_R)} \preceq n^{m\tilde{\lambda}}\|P_n\|_{C(\overline{G})}$$

Then, according to Theorem A, for $p>0$ and $\forall z \in \Gamma$, we obtain:

$$\left|P_n^{(m)}(z)\right| \preceq n^{m\tilde{\lambda}} \cdot n^{\frac{\tilde{\gamma}+1}{p}\tilde{\lambda}}\|P_n\|_p \preceq n^{m\tilde{\lambda}+\frac{\tilde{\gamma}+1}{p}\tilde{\lambda}} \parallel P_n \parallel_p.$$

Since $z \in \Gamma$ is arbitrary, we have:

$$\|P_n\|_\infty \preceq n^{\tilde{\lambda}\left(m+\frac{\tilde{\gamma}+1}{p}\right)} \parallel P_n \parallel_p.$$

Thus, we complete the proof of Theorem 2.4.

Conflict of interest

There are no conflicts of interest in this work.

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