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Soft Intersection-star Product of Groups

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Abstract

Soft set theory provides a mathematically rigorous and algebraically expressive framework for modeling systems characterized by epistemic uncertainty, vagueness, and parameter-dependent variability—phenomena central to decision theory, engineering, economics, and information science. Expanding on this foundation, the present study introduces and examines a novel binary operation, the soft intersection–star product, defined over soft sets with parameter domains possessing intrinsic group-theoretic structures. Developed within a formally consistent, axiomatic framework, this operation aligns with generalized concepts of soft subsethood and soft equality. A comprehensive algebraic analysis is conducted for the operation's core properties—closure, associativity, commutativity, and idempotency. The presence or absence of identity, inverse, and absorbing elements, and the soft product's behavior concerning the null and absolute soft sets, are precisely delineated. Two key contributions emerge: first, the operation substantially extends the algebraic toolkit of soft set theory within a rigorous operational framework; second, it lays the foundation for a generalized soft group theory, wherein soft sets indexed by group-structured parameters mimic classical group behavior through abstractly defined soft operations. Beyond its theoretical value, the proposed framework offers a principled basis for soft computational modeling grounded in abstract algebra. Such models are highly applicable to multi-criteria decision analysis, algebraic classification, and uncertainty-sensitive data analytics. Hence, this study not only strengthens the theoretical foundations of soft algebra but also reinforces its relevance to both mathematical research and practical computation.

Keywords: Soft sets; Soft subsets; Soft equalities; Soft intersection-star product.

1. INTRODUCTION

A wide array of mathematically sophisticated frameworks has been developed to model uncertainty, vagueness, and indeterminacy—features prevalent in engineering, economics, social sciences, and medical diagnostics. Yet, classical paradigms such as fuzzy set theory and probabilistic models face epistemological and algebraic limitations. Fuzzy set theory by Zadeh [1] relies on subjectively defined membership functions, while probabilistic models assume repeatable experiments and precise distributions—conditions often unmet in real-world settings.

To address these shortcomings, Molodtsov [2] introduced soft set theory as an axiomatically minimal yet structurally adaptable alternative, where uncertainty is captured through parameter dependence rather than probabilities or membership grades. Since then, its algebraic structure has evolved significantly. Foundational operations—including union, intersection, and AND/OR products—originally introduced by Maji et al. [3], were reformulated by Pei and Miao [4] through an information-theoretic perspective. Ali et al. [5] further enhanced this framework by defining restricted and extended variants, thereby increasing its algebraic granularity and expressive power. A substantial and evolving corpus of scholarship—including contributions from [6-19]—have addressed semantic ambiguities, introduced generalized notions of soft equality, and defined novel binary operations, thereby progressively enriching the algebraic landscape of the theory. More recent advances have

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extended this foundation through the systematic introduction and rigorous algebraic examination of new operations. Noteworthy among these are the contributions of [20-35] whose collective efforts have established a robust, extensible, and internally consistent algebraic framework underpinning ongoing developments in soft set theory.

A pivotal dimension of this progression concerns the formalization and generalization of soft subsethood and soft equality. The foundational concept of soft subsets introduced by Maji et al. [3] was generalized by Pei and Miao [4] and Feng et al. [7], while Qin and Hong [36] contributed soft congruences embedding equivalence relations within the soft universe. Jun and Yang [37] further enhanced the theoretical apparatus by proposing J-soft equalities alongside related distributive principles. Liu et al. [38] introduced L-soft subsets and L-equality, revealing violations of classical distributive laws within generalized soft contexts. Feng and Li [39] developed a comprehensive classification of soft subsets under L-equality and demonstrated that certain quotient structures satisfy associativity, commutativity, and distributivity, thereby exhibiting semigroup properties. Broader generalizations—including g-soft, gf-soft, and T-soft equalities—were subsequently advanced by Abbas et al. [40,41], Al-shami [42], and Al-shami and El-Shafei [43], who explored congruence-based and lattice-theoretic formulations of soft algebraic systems.

A significant reformulation of the definitional and operational calculus was realized through the axiomatic restructuring introduced by Çağman and Enginoğlu [44], which resolved structural inconsistencies inherent in the original theory and provided a logically coherent, algebraically tractable foundation for further inquiry. This enhanced formalism now supports a wide range of applications across algebra, decision theory, and soft computing. Parallel research extended binary soft products across algebraic domains, notably generalizing the soft intersection–union product to rings [45], semigroups [46], and groups [47], thereby establishing the notions of soft rings, soft semigroups, and soft groups. Its dual operation, the soft union–intersection product, has been similarly examined within group-theoretic [48], semigroup-theoretic [49], and ring-theoretic [50] frameworks, with the algebraic behavior depending critically on structural elements such as identities and inverses within the parameter domains.

Building upon this extensive foundation, the present study introduces a novel binary operation—termed the soft intersection-star product—defined on soft sets indexed by group-structured parameter domains. This operation is rigorously axiomatized and subjected to comprehensive algebraic scrutiny. We explore its core properties, including closure, associativity, commutativity, and idempotency. Furthermore, we analyze its interactions with identity and absorbing elements and verify its compatibility with generalized soft subsethood and soft equality, ensuring seamless integration into the existing algebraic architecture of soft set theory. A comparative evaluation against established soft binary operations highlights its representational expressiveness and algebraic coherence across stratified soft subset classifications. The operation's behavior vis-à-vis null and absolute soft sets is also formally characterized. By generalizing classical group-theoretic constructs within the soft set framework, this operation establishes a conceptual foundation for a generalized soft group theory—wherein soft sets emulate classical algebraic behavior under rigorously defined soft operations. The manuscript is organized as follows: Section 2 introduces fundamental definitions and preliminaries; Section 3 develops the algebraic theory of the soft intersection-star product in detail; and Section 4 synthesizes the principal findings and outlines prospective directions for advancing the algebraic foundations of soft sets and their applications in abstract algebra and uncertainty quantification.

2. PRELIMINARIES

This section presents a rigorous rearticulation of the foundational definitions and algebraic axioms underpinning this study. Originally introduced by Molodtsov [2] to model systems with epistemic uncertainty, soft set theory lacked the algebraic rigor needed for formal development. The axiomatic refinement by Çağman and Enginoğlu [44] addressed these limitations, resolving internal inconsistencies and establishing a coherent, algebraically sound framework. The present work adopts this refined structure as the basis for all subsequent developments, ensuring internal coherence, structural integrity, and alignment with established standards in soft algebra. Unless stated otherwise, all references to soft sets and operations are made within this axiomatic framework.

Definition 2.1. [44] Let E be a parameter set, U be a universal set, $P(U)$ be the power set of U, and $\mathcal{H} \subseteq E$. Then, the soft set $f_{\mathcal{H}}$ over U is a function such that $f_{\mathcal{H}}: E \rightarrow P(U)$, where for all $w \notin \mathcal{H}$, $f_{\mathcal{H}}(w) = \emptyset$. That is,

$$f_{\mathcal{H}} = \{(w, f_{\mathcal{H}}(w)): w \in E\}$$

From now on, the soft set over U is abbreviated by $\mathcal{SS}$.

Definition 2.2. [44] Let $f_{\mathcal{H}}$ be an $\mathcal{SS}$. If $f_{\mathcal{H}}(w) = \emptyset$ for all $w \in E$, then $f_{\mathcal{H}}$ is called a null $\mathcal{SS}$ and indicated by $\emptyset_E$, and if $f_{\mathcal{H}}(w) = U$, for all $w \in E$, then $f_{\mathcal{H}}$ is called an absolute $\mathcal{SS}$ and indicated by U_E.

Definition 2.3. [44] Let $f_{\mathcal{H}}$ and $g_{\aleph}$ be two $\mathcal{SS}$s. If $f_{\mathcal{H}}(w) \subseteq g_{\aleph}(w)$, for all $w \in E$, then $f_{\mathcal{H}}$ is a soft subset of $g_{\aleph}$ and indicated by $f_{\mathcal{H}} \tilde{\subseteq} g_{\aleph}$. If $f_{\mathcal{H}}(w) = g_{\aleph}(w)$, for all $w \in E$, then $f_{\mathcal{H}}$ is called soft equal to $g_{\aleph}$, and indicated by $f_{\mathcal{H}} = g_{\aleph}$.

Definition 2.4. [44] Let $f_{\mathcal{H}}$ and $g_{\aleph}$ be two $\mathcal{SS}$s. Then, the union of $f_{\mathcal{H}}$ and $g_{\aleph}$ is the $\mathcal{SS}$ $f_{\mathcal{H}} \tilde{\cup} g_{\aleph}$, where $(f_{\mathcal{H}} \tilde{\cup} g_{\aleph})(w) = f_{\mathcal{H}}(w) \cup g_{\aleph}(w)$, for all $w \in E$.

Definition 2.5. [44] Let $f_{\mathcal{H}}$ be an $\mathcal{SS}$. Then, the complement of $f_{\mathcal{H}}$ denoted by ${f_{\mathcal{H}}}^{c}$, is defined by the soft set ${f_{\mathcal{H}}}^{c}: E \to P(U)$ such that ${f_{\mathcal{H}}}^{c}(e) = U \setminus f_{\mathcal{H}}(e) = \big(f_{\mathcal{H}}(e)\big)'$, for all $e \in E$.

Definition 2.6. [51] Let f_K and $g_{\aleph}$ be two $\mathcal{SS}$s. Then, f_K is called a soft S-subset of $g_{\aleph}$, denoted by $f_K \tilde{\subseteq}_S g_{\aleph}$ if for all $w \in E$, $f_K(w) = \mathcal{M}$ and $g_{\aleph}(w) = \mathcal{D}$, where $\mathcal{M}$ and $\mathcal{D}$ are two fixed sets and $\mathcal{M} \subseteq \mathcal{D}$. Moreover, two SSs f_K and $g_{\aleph}$ are said to be soft S-equal, denoted by $f_K =_S g_{\aleph}$, if $f_K \tilde{\subseteq}_S g_{\aleph}$ and $g_{\aleph} \tilde{\subseteq}_S f_K$.

It is obvious that if $f_K =_S g_{\aleph}$, then f_K and $g_{\aleph}$ are the same constant functions, that is, for all $w \in E$, $f_K(w) = g_{\aleph}(w) = \mathcal{M}$, where $\mathcal{M}$ is a fixed set .

Definition 2.7. [51] Let f_K and $g_{\aleph}$ be two $\mathcal{SS}$s. Then, f_K is called a soft A-subset of $g_{\aleph}$, denoted by $f_K \tilde{\subseteq}_A g_{\aleph}$, if, for each $a, b \in E$, $f_K(a) \subseteq g_{\aleph}(b)$.

Definition 2.8. [51] Let f_K and $g_{\aleph}$ be two $\mathcal{SS}$s. Then, f_K is called a soft S-complement of $g_{\aleph}$, denoted by $f_K =_S (g_{\aleph})^c$, if, for all $w \in E$, $f_K(w) = \mathcal{M}$ and $g_{\aleph}(w) = \mathcal{D}$, where $\mathcal{M}$ and $\mathcal{D}$ are two fixed sets and $\mathcal{M} = \mathcal{D}'$. Here, $\mathcal{D}' = U \setminus \mathcal{D}$.

From now on, let G be a group, and $S_G(U)$ denotes the collection of all $\mathcal{SS}$s over U, whose parameter sets are G; that is, each element of $S_G(U)$ is an $\mathcal{SS}$ parameterized by G.

Definition 2.9. [48] Let f_G and g_G be two $\mathcal{SS}$s. Then, the union-intersection product $f_G \otimes_{u/i} g_G$ is defined by

$$(f_G \otimes_{u/i} g_G)(x) = \bigcup_{x=yz} \big(f_G(y) \cap g_G(z)\big), \quad y, z \in G$$

for all $x \in G$.

For additional information on $\mathcal{SS}$s, we refer to [52-90].

3. SOFT INTERSECTION-STAR PRODUCT OF GROUPS

This section introduces and investigates a novel binary operation on soft sets, termed the soft intersection-star product, defined over group-structured parameter domains. A detailed algebraic analysis establishes key properties such as closure, associativity, commutativity, idempotency, and compatibility with generalized soft equality and subsethood. The operation's behavior is examined within established inclusion hierarchies and positioned within the broader algebraic framework of soft set theory. Comparative analysis with existing soft operations further highlights its expressive capacity, structural coherence, and integrability. To support the theoretical development, illustrative examples are provided, showcasing subtle operational dynamics. Together, these results affirm the soft intersection-star product as a robust and foundational construct for the ongoing algebraic expansion of soft set theory.

Definition 3.1. Let f_G and g_G be two $\mathcal{SS}$s. Then, the soft intersection-star product $f_G \otimes_{i/st} g_G$ is defined by

$$(f_G \otimes_{i/st} g_G)(x) = \bigcap_{x=yz} \big(f_G(y) * g_G(z)\big) = \bigcap_{x=yz} \big({f_G}^{c}(y) \cup {g_G}^{c}(z)\big), \qquad y, z \in G$$

for all $x \in G$.

Note here that since G is a group, there always exist $y, z \in G$ such that $x = yz$, for all $x \in G$. Let the order of the group G be n, that is, $|G| = n$. Then, it is obvious that there exist n distinct algebraic representations for expressing each $x \in G$ such that $x = yz$, where $y, z \in G$. Besides, for more on star $(*)$ operation of sets, we refer to [86].

Note 3.2. The soft intersection-star product is well-defined in $S_G(U)$. In fact, let f_G, g_G, o_G, $k_G \in S_G(U)$ such that $(f_G, g_G) = (o_G, k_G)$. Then, $f_G = o_G$ and $g_G = k_G$, implying that $f_G(x) = o_G(x)$ and $g_G(x) = k_G(x)$ for all $x \in G$. Thereby, for all $x \in G$,

$$\begin{aligned}(f_G \otimes_{i/st} g_G)(x) &= \bigcap_{x=yz} \left(f_G{}^c(y) \cup g_G{}^c(z)\right) \\ &= \bigcap_{x=yz} \left(o_G{}^c(y) \cup k_G{}^c(z)\right) \\ &= (o_G \otimes_{i/st} k_G)(x)\end{aligned}$$

Hence, $f_G \otimes_{i/st} g_G = o_G \otimes_{i/st} k_G$.

Example 3.3. Consider the group G = {Ձ, ɓ} with the following operation**:**

·	Ձ	ɓ
Ձ	Ձ	ɓ
ɓ	ɓ	Ձ

Let f_G and g_G be two $\mathcal{SS}$s over $U = D_2 = \{<x, y>: x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$ as follows:

$f_G = \{(Ձ, \{e, x, y\}), (ɓ, \{yx\})\}$ and $g_G = \{(Ձ, \{y\}), (ɓ, \{e, yx\})\}$

Since Ձ = ՁՁ = ɓɓ, $(f_G \otimes_{i/st} g_G)(Ձ) = \left(f_G{}^c(Ձ) \cup g_G{}^c(Ձ)\right) \cap \left(f_G{}^c(ɓ) \cup g_G{}^c(ɓ)\right) = \{e, x\}$ and since ɓ = Ձɓ = ɓՁ, $(f_G \otimes_{i/st} g_G)(ɓ) = \left(f_G{}^c(Ձ) \cup g_G{}^c(ɓ)\right) \cap \left(f_G{}^c(ɓ) \cup g_G{}^c(Ձ)\right) = \{x, y, yx\}$ is obtained. Hence,

$$f_G \otimes_{i/st} g_G = \{(Ձ, \{e, x\}), (ɓ, \{x, y, yx\})\}$$

Proposition 3.4. The set $S_G(U)$ is closed under the soft intersection-star product. That is, if f_G and g_G are two $\mathcal{SS}$s, then so is $f_G \otimes_{i/st} g_G$.

PROOF. It is obvious that the soft intersection-star product is a binary operation in $S_G(U)$. Thereby, $S_G(U)$ is closed under the soft intersection-star product.

Proposition 3.5. The soft intersection-star product is not associative in $S_G(U)$.
PROOF. Consider the $\mathcal{SS}$s f_G and g_G over $U = \{e, x, y, yx\}$ in Example 3.3. Let $\hbar_G = \{(Ձ, \{x\}), (ɓ, \{y, yx\})\}$ be a $\mathcal{SS}$. Since $f_G \otimes_{i/st} g_G = \{(Ձ, \{e, x\}), (ɓ, \{x, y, yx\})\}$, then

$$(f_G \otimes_{i/st} g_G) \otimes_{i/st} \hbar_G = \{(Ձ, \{e\}), (ɓ, \{e, y, yx\})\}$$

Moreover, since $g_G \otimes_{i/st} \hbar_G = \{(Ձ, \{e, x, y\}), (ɓ, \{e, x, yx\})\}$, then

$$f_G \otimes_{i/st} (g_G \otimes_{i/st} \hbar_G) = \{(Ձ, \emptyset), (ɓ, \{y, yx\})\}$$

Thereby, $(f_G \otimes_{i/st} g_G) \otimes_{i/st} \hbar_G \neq f_G \otimes_{i/st} (g_G \otimes_{i/st} \hbar_G)$. □

Proposition 3.6. The soft intersection-star product is not commutative in $S_G(U)$. However, if G is an abelian group, then the intersection-star product is commutative in $S_G(U)$.
PROOF. Let f_G and g_G be two $\mathcal{SS}$s and G be an abelian group. Then, for all $x \in G$,

$$(f_G \otimes_{i/st} g_G)(x) = \bigcap_{x=yz} \left(f_G{}^c(y) \cup g_G{}^c(z)\right)$$

$$= \bigcap_{x=zy} \left(g_G{}^c(z) \cup f_G{}^c(y) \right)$$
$$= \left(g_G \otimes_{i/st} f_G \right)(x)$$

implying that $f_G \otimes_{i/st} g_G = g_G \otimes_{i/st} f_G$. □

Example 3.7. Consider the $\mathcal{SS}$s f_G and g_G over $U = \{e, x, y, yx\}$ in Example 3.3. Then,

$f_G \otimes_{i/st} g_G = \{(\mathfrak{a}, \{e, x\}), (\mathfrak{b}, \{x, y, yx\})\}$, and $g_G \otimes_{i/st} f_G = \{(\mathfrak{a}, \{e, x\}), (\mathfrak{b}, \{x, y, yx\})\}$

implying that $f_G \otimes_{i/st} g_G = g_G \otimes_{i/st} f_G$.

Proposition 3.8. The soft intersection-star product is not idempotent in $S_G(U)$.

PROOF. Consider the $\mathcal{SS}$ f_G in Example 3.3. Then, for all $x \in G$,

$$f_G \otimes_{i/st} f_G = \{(\mathfrak{a}, U), (\mathfrak{b}, \emptyset)\}$$

implying that $f_G \otimes_{i/st} f_G \neq f_G$. □

Proposition 3.9. Let f_G be a constant $\mathcal{SS}$. Then, $f_G \otimes_{i/st} f_G = f_G{}^c$.

PROOF. Let f_G be a constant $\mathcal{SS}$ such that, for all $x \in G$, $f_G(x) = A$, where A is a fixed set. Hence, for all $x \in G$,

$$\left(f_G \otimes_{i/st} f_G \right)(x) = \bigcap_{x=yz} \left(f_G{}^c(y) \cup f_G{}^c(z) \right) = f_G{}^c(x)$$

Thereby, $f_G \otimes_{i/st} f_G = f_G{}^c$. □

Remark 3.10. Let $S_G{}^*(U)$ be the collection of all constant $\mathcal{SS}$s. Then, the soft intersection-star product is not idempotent in $S_G{}^*(U)$ either.

Proposition 3.11. Let f_G be a constant $\mathcal{SS}$. Then, $U_G \otimes_{i/st} f_G = f_G \otimes_{i/st} U_G = f_G{}^c$.

PROOF. Let f_G be a constant $\mathcal{SS}$ such that, for all $x \in G$, $f_G(x) = A$, where A is a fixed set. Hence, for all $x \in G$,

$$\left(U_G \otimes_{i/st} f_G \right)(x) = \bigcap_{x=yz} \left(U_G{}^c(y) \cup f_G{}^c(z) \right)$$
$$= \bigcap_{x=yz} \left(\emptyset \cup f_G{}^c(z) \right)$$
$$= f_G{}^c(x)$$

Similarly, for all $x \in G$,

$$\left(f_G \otimes_{i/st} U_G \right)(x) = \bigcap_{x=yz} \left(f_G{}^c(y) \cup U_G{}^c(z) \right)$$
$$= \bigcap_{x=yz} \left(f_G{}^c(y) \cup \emptyset \right)$$
$$= f_G{}^c(x)$$

Thereby, $U_G \otimes_{i/st} f_G = f_G \otimes_{i/st} U_G = f_G{}^c$. □

Proposition 3.12. Let f_G be a constant $\mathcal{SS}$. Then, $\emptyset_G \otimes_{i/st} f_G = f_G \otimes_{i/st} \emptyset_G = U_G$.

PROOF**.** Let f_G be a constant $\mathcal{SS}$ such that, for all $x \in G$, $f_G(x) = A$, where A is a fixed set. Hence, for all $x \in G$,

$$\left(\emptyset_G \otimes_{i/st} f_G \right)(x) = \bigcap_{x=yz} \left(\emptyset_G{}^c(y) \cup f_G{}^c(z) \right)$$

$$= \bigcap_{x=yz} \left(U \cup f_G{}^c(z)\right)$$
$$= U_G(x)$$

Similarly, for all $x \in G$,

$$\left(f_G \otimes_{i/st} \emptyset_G\right)(x) = \bigcap_{x=yz} \left(f_G{}^c(y) \cup \emptyset_G{}^c(z)\right)$$
$$= \bigcap_{x=yz} (f_G{}^c(y) \cup U)$$
$$= U_G(x)$$

Thereby, $\emptyset_G \otimes_{i/st} f_G = f_G \otimes_{i/st} \emptyset_G = U_G$. □

Proposition 3.13. Let f_G be a constant SS. Then, $f_G \otimes_{i/st} f_G{}^c = f_G{}^c \otimes_{i/st} f_G = U_G$.

PROOF**.** Let f_G be a constant SS such that, for all $x \in G$, $f_G(x) = A$, where A is a fixed set. Hence, for all $x \in G$,

$$\left(f_G \otimes_{i/st} f_G{}^c\right)(x) = \bigcap_{x=yz} \left(f_G{}^c(y) \cup (f_G{}^c)^c(z)\right)$$
$$= \bigcap_{x=yz} \left(f_G{}^c(y) \cup f_G(z)\right)$$
$$= U_G(x)$$

Similarly,

$$\left(f_G{}^c \otimes_{i/st} f_G\right)(x) = \bigcap_{x=yz} \left((f_G{}^c)^c(y) \cup f_G{}^c(z)\right)$$
$$= \bigcap_{x=yz} \left(f_G(y) \cup f_G{}^c(z)\right)$$
$$= U_G(x)$$

Thereby, $f_G \otimes_{i/st} f_G{}^c = f_G{}^c \otimes_{i/st} f_G = U_G$. □

Proposition 3.14. Let f_G and g_G be two SSs. If $f_G \tilde{\subseteq}_S g_G$, then $f_G \otimes_{i/st} g_G = f_G{}^c$.

PROOF. Let f_G and g_G be two SSs and $f_G \tilde{\subseteq}_S g_G$. Hence, for all $x \in G$, $f_G(x) = A$ and $g_G(x) = B$, where A and B are two fixed sets and $A \subseteq B$. Thus, for all $x \in G$, $g_G{}^c(x) \subseteq f_G{}^c(x)$. Then,

$$\left(f_G \otimes_{i/st} g_G\right)(x) = \bigcap_{x=yz} \left(f_G{}^c(y) \cup g_G{}^c(z)\right) = f_G{}^c(x)$$

for all $x \in G$. Thereby, $f_G \otimes_{i/st} g_G = f_G{}^c$. □

Proposition 3.15. Let f_G and g_G be two SSs. If $g_G \tilde{\subseteq}_S f_G$, then $f_G \otimes_{i/st} g_G = g_G{}^c$.

PROOF. Let f_G and g_G be two SSs and $g_G \tilde{\subseteq}_S f_G$. Hence, for all $x \in G$, $f_G(x) = A$ and $g_G(x) = B$, where A and B are two fixed sets and $B \subseteq A$. Thus, for all $x \in G$, $f_G{}^c(x) \subseteq g_G{}^c(x)$. Then,

$$\left(f_G \otimes_{i/st} g_G\right)(x) = \bigcap_{x=yz} \left(f_G{}^c(y) \cup g_G{}^c(z)\right) = g_G{}^c(x)$$

Thereby, $f_G \otimes_{i/st} g_G = g_G{}^c$. □

Proposition 3.16. Let f_G and g_G be two SSs. If $f_G \tilde{\subseteq}_S g_G{}^c$, then $f_G \otimes_{i/st} g_G = U_G$.

PROOF. Let f_G and g_G be two SSs and $f_G \tilde{\subseteq}_S g_G{}^c$. Hence, for all $x \in G$, $f_G(x) = A$ and $g_G(x) = B$, where A and B are two fixed sets and $A \subseteq B'$. Thus, for all $x \in G$,

$$\left(f_G \otimes_{i/st} g_G\right)(x) = \bigcap_{x=yz} \left(f_G{}^c(y) \cup g_G{}^c(z)\right) = U_G(x)$$

Thereby, $f_G \otimes_{i/st} g_G = U_G$. Here, note that, in classical set theory, if $A \subseteq B'$, then $A \cap B = \emptyset$, thus, $(A \cap B)' = A' \cup B' = U$. □

Note 3.17. Proposition 3.16 is also satisfied for the soft A-subset condition.

Proposition 3.18. Let f_G and g_G be two SSs. If one of the following assertions is satisfied, then $f_G \otimes_{i/st} g_G = U_G$.

i. $f_G = \emptyset_G$ or $g_G = \emptyset_G$

ii. $f_G \tilde{\subseteq}_A {g_G}^c$

iii. $f_G \tilde{\subseteq}_S {g_G}^c$

PROOF. (i) follows by Proposition 3.12, (ii) follows by Note 3.17, and (iii) follows by Proposition 3.16.

Proposition 3.19. Let f_G and g_G be two SSs. Then, $\left(f_G \otimes_{i/st} g_G\right)^c = f_G \otimes_{u/i} g_G$.

PROOF. Let f_G and g_G be two SSs. Then, for all $x \in G$,

$$\begin{aligned}\left(f_G \otimes_{i/st} g_G\right)^c(x) &= \left(\bigcap_{x=yz}\left({f_G}^c(y) \cup {g_G}^c(z)\right)\right)' \\ &= \bigcup_{x=yz}\left({f_G}^c(y) \cup {g_G}^c(z)\right)' \\ &= \bigcup_{x=yz}\left(f_G(y) \cap g_G(z)\right) \\ &= \left(f_G \otimes_{u/i} g_G\right)(x)\end{aligned}$$

Thereby. $\left(f_G \otimes_{i/st} g_G\right)^c = f_G \otimes_{u/i} g_G$. □

Proposition 3.20. Let f_G , g_G, and $\hbar_G$ be three SSs. If $f_G \tilde{\subseteq} g_G$, then $g_G \otimes_{i/st} \hbar_G \tilde{\subseteq} f_G \otimes_{i/st} \hbar_G$ and $\hbar_G \otimes_{i/st} g_G \tilde{\subseteq} \hbar_G \otimes_{i/st} f_G$.

PROOF. Let f_G , g_G, and $\hbar_G$ be three SSs such that $f_G \tilde{\subseteq} g_G$. Then, for all $x \in G$, $f_G(x) \subseteq g_G(x)$, and so ${g_G}^c(x) \subseteq {f_G}^c(x)$. Thus, for all $x \in G$,

$$\begin{aligned}\left(g_G \otimes_{i/st} \hbar_G\right)(x) &= \bigcap_{x=yz}\left({g_G}^c(y) \cup {\hbar_G}^c(z)\right) \\ &\subseteq \bigcap_{x=yz}\left({f_G}^c(y) \cup {\hbar_G}^c(z)\right) \\ &= \left(f_G \otimes_{i/st} \hbar_G\right)(x)\end{aligned}$$

is obtained, implying that $g_G \otimes_{i/st} \hbar_G \tilde{\subseteq} f_G \otimes_{i/st} \hbar_G$. Similarly, for all $x \in G$,

$$\begin{aligned}\left(\hbar_G \otimes_{i/st} g_G\right)(x) &= \bigcap_{x=yz}\left({\hbar_G}^c(y) \cup {g_G}^c(z)\right) \\ &\subseteq \bigcap_{x=yz}\left({\hbar_G}^c(y) \cup {f_G}^c(z)\right) \\ &= \left(\hbar_G \otimes_{i/st} f_G\right)(x)\end{aligned}$$

implying that $\hbar_G \otimes_{i/st} g_G \tilde{\subseteq} \hbar_G \otimes_{i/st} f_G$. □

Proposition 3.21. Let f_G, g_G, σ_G, and k_G be four SSs. If $k_G \tilde{\subseteq} \sigma_G$, and $f_G \tilde{\subseteq} g_G$, then $\sigma_G \otimes_{i/st} g_G \tilde{\subseteq} k_G \otimes_{i/st} f_G$ and $g_G \otimes_{i/st} \sigma_G \tilde{\subseteq} f_G \otimes_{i/st} k_G$.

PROOF. Let f_G, g_G, σ_G, and k_G be four SSs such that $k_G \tilde{\subseteq} \sigma_G$, and $f_G \tilde{\subseteq} g_G$. Then, for all $x \in G$, $k_G(x) \subseteq \sigma_G(x)$ and $f_G(x) \subseteq g_G(x)$. Thus, ${\sigma_G}^c(x) \subseteq {k_G}^c(x)$ and ${g_G}^c(x) \subseteq {f_G}^c(x)$, for all $x \in G$. Then,

$$\begin{aligned}\left(\sigma_G \otimes_{i/st} g_G\right)(x) &= \bigcap_{x=yz}\left({\sigma_G}^c(y) \cup {g_G}^c(z)\right) \\ &\subseteq \bigcap_{x=yz}\left({k_G}^c(y) \cup {f_G}^c(z)\right) \\ &= \left(k_G \otimes_{i/st} f_G\right)(x)\end{aligned}$$

for all $x \in G$, implying that $\sigma_G \otimes_{i/st} g_G \widetilde{\subseteq} k_G \otimes_{i/st} f_G$. Similarly, for all $x \in G$,

$$\begin{aligned}(g_G \otimes_{i/st} \sigma_G)(x) &= \bigcap_{x=yz} \left(g_G{}^c(y) \cup \sigma_G{}^c(z) \right) \\ &\subseteq \bigcap_{x=yz} \left(f_G{}^c(y) \cup k_G{}^c(z) \right) \\ &= (f_G \otimes_{i/st} k_G)(x)\end{aligned}$$

is obtained, implying that $g_G \otimes_{i/st} \sigma_G \widetilde{\subseteq} f_G \otimes_{i/st} k_G$.

4. CONCLUSION

This study introduces a novel binary operation on soft sets—the soft intersection-star product—defined over parameter domains with an intrinsic group-theoretic structure. A thorough algebraic analysis explores its structural behavior within layered hierarchies of soft subsethood and its compatibility with generalized soft equalities. The operation is positioned within the lattice of soft subset classifications through a rigorous comparative framework, offering deeper insights into its expressive power and algebraic coherence relative to existing soft products. Further investigation covers its interaction with null and absolute soft sets and other group-based binary operations, clarifying its integrative role within the broader algebraic topology of soft systems. Developed within a strict axiomatic framework grounded in abstract algebra, the study investigates the core properties—closure, associativity, commutativity, idempotency, and the presence or absence of identity, inverse, and absorbing elements. The results confirm the operation's structural consistency and theoretical robustness, establishing it as a foundational tool for advancing generalized soft group theory. In doing so, the framework opens avenues for further research into soft algebraic structures, generalized equalities, and applications in logic, abstract modeling, and decision-making under uncertainty.

Conflict of Interest

The authors declare no conflicts of interest.

Authors' Contributions

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1- Study design 2- Data collection 3- Data analysis and interpretation 4- Manuscript writing 5- Critical revision			

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