İstatistikçiler Dergisi: İstatistik & Aktüerya

Journal of Statisticians: Statistics and Actuarial Sciences

IDIA 11, 2018, 1, 13-22

Geliş/Received:20.12.2017, Kabul/Accepted: 20.04.2018

www.istatistikciler.org

Araştırma Makalesi / Research Article

Comparison of the $r-(k,d)$ class estimators to some estimators by the mean square error matrix criteria

Nilgün Yıldız
Marmara University, Department of
Mathematics, Faculty of Art-Sciences
34722, İstanbul, Turkey
ncelebiyil@gmail.com
0000-0002-4084-1969

Abstract

The ordinary least squares, the principal components regression and the Liu-type estimators are special cases of the *r-(k,d)* class estimators, for regression models with multicollinearity. In this article we derived conditions for the superiority of the *r-(k,d)* class estimator over other estimators such as ordinary least squares, principal component and Liu-type estimator based on the mean square error matrix *(MSEM)* criterion. Finally, a numerical example and a Monte Carlo simulation are also given to show the theoretical results.

Keywords Liu-type estimators; *r-(k,d)* class estimator, Principal Component Regression, Mean square error matrix, Multicollinearity

Öz

r-(k,d) sınıf tahmin edicisinin, ortalama karesel hata kriterine göre bazı yanlı tahmin ediciler ile karşılaştırılması

En küçük kareler, temel bileşenler ve Liu-tipi tahmin ediciler, çok değişkenli regresyon modelleri için r-(k,d) sınıf tahmin edicilerin özel durumlarıdır. Bu makalede r-(k,d) sınıf tahmin edicisini, en küçük kareler, temel bileşenler ve Liu-tipi tahmin ediciler ile Matris Hata kareler ortalaması kriterine göre karşılaştırılmıştır. Son olarak teorik sonuçları göstermek için sayısal bir örnek ve bir Monte Carlo simülasyonu verilmektedir.

Anahtar sözcükler *Liu-tipi tahmin edici; r-(k,d)Sınıf Tahmin Edici; Temel bileşenler regresyonu; Matris Hata Kareler Ortalaması; Çoklu İç İlişki*

1. Introduction

It is a common knowledge that variance of regression coefficients could be inflated, thus making the ordinary least squares (OLS) estimation unsuitable in the presence of multicollinearity. Through the years a lot of research has been dedicated to overcome this hurdle. While early studies have proposed new estimators to tackle the multicollinearity (e.g. the principal component regression (PCR) estimator and the ordinary ridge regression (ORR), in later years researchers combined various estimators to obtain better results (e.g. the $(r-k)$ class estimator, which combines the ORR and PCR; the Liu estimator (LE), which combines Stein [1] and ORR estimators; the $(r-d)$ class estimator, which combines the LE and PCR estimators, and the Liu-type estimator (LTE), which combines the ORR and LE estimators.

One of the latest additions to such efforts is the $r-(k,d)$ estimator, which is a combination of the Liu-type and PCR estimators. Inan [2], who proposed this approach, have also determined that this estimator was superior to OLS, ORR, Liu-type, PCR and $(r-k)$ class estimators based on the strong mean square error (SMSE) criterion. However, they did not compare the $r-(k,d)$ estimator with other estimators by using the MSEM criterion, which is often used to measure the performance of an estimator and is stronger than the SMSE criterion. We choose MSEM criterion instead of SMSE criteria. Since the diagonal elements of the MSEM are neglected when calculating the SMSE, we have set the MSEM as the criterion. But calculation of MSEM is more difficult than SMSE. In this study, we aimed to compare the $r-(k,d)$ estimator with OLS, PCR and Liu-type estimators based on the MSEM criteria. Furthermore, necessary and sufficient conditions for the $r-(k,d)$ class estimator to dominate the OLS, PCR and Liu-type estimators' sense are derived. A Monte Carlo simulation and and real data have been conducted to show the performance of estimators.

The rest of the article is organized as follows. In the second section, we described the model and defined the $r-(k,d)$ estimator. In the third section, we compared the $r-(k,d)$ estimator with OLS, PCR and Liu-type estimators based on the MSEM criterion. Furthermore, a numerical example that illustrates some of the theoretical results is given in section 4 and the results of the Monte Carlo simulation are presented in section 5. The paper is wrapped up with some concluding remarks in section 6.

2. Model

We considered the linear regression model given as

$$y = X\beta + \varepsilon \tag{1}$$

Where y is $(n \times 1)$ observable random vector, X is a $(n \times p)$ matrix of non-stochastics variables of rank p; β is $(p \times 1)$ vector of unknown parameters associated with X, and ε is a $(n \times 1)$ vector of error terms. Let $T = [t_1, t_2, \dots, t_p]$ be an orthogonal matrix that consist of the eigenvalues of $X'X$.

$T'X'XT = \Lambda$, where $\Lambda = diag(\lambda_1, \lambda_2, \dots, \lambda_p)$ ranked according to the magnitude. Further let $T_r = [t_1, t_2, \dots, t_r]$ be remaining columns of T having deleted r columns where $r \le p$. Obviously, $T_r{}'X'XT_r = \Lambda_r = diag(\lambda_1, \lambda_2, \dots, \lambda_r)$ and $T_{p-r}{}'X'XT_{p-r} = \Lambda_{p-r} = diag(\lambda_1, \lambda_2, \dots, \lambda_{p-r})$ where $T_{p-r} = [t_{r+1}, t_{r+2}, \dots, t_p]$. The $r-(k,d)$ class estimator for β as proposed by Inan [2] is

$$\hat{\beta}_r(k,d) = T_r(T_r{}'X'XT_r + kI)^{-1}(T_r{}'X'XT_r - dI)\,T_r{}'\hat{\beta}_{PCR} \quad k \ge 0, -\infty < d < +\infty \tag{2}$$

where $\hat{\beta}_{PCR} = T_r(T_r{}'X'XT_r)^{-1}T_r{}'X'y$ is the *PCR* estimator. The $r-(k,d)$ class estimator is a general estimator, which includes the *OLS*, *PCR*, $r-k$, and Liu-type estimators as special cases:

$\hat{\beta}_p(0,0) = \hat{\beta} = (X'X)^{-1}Xy$ is the OLS estimator,

$\hat{\beta}_r(k,-k) = T_r(T_r'X'XT_r)^{-1}T_r'X'y$ is the PCR estimator,

$\hat{\beta}_r(k,0) = \hat{\beta}_r(k) = T_r(T_r'X'XT_r + kI)^{-1}T_r'X'y$ is the $r-k$ class estimator, and

$\hat{\beta}_p(k,d) = \hat{\beta}(k,d) = (X'X + kI)^{-1}(X'X - dI)\hat{\beta}$ is the Liu-type estimator *PCR*, $r-k$, and Liu-type estimators are all methods that address the multicollinearity. When proposing $r-(k,d)$ class estimator, Inan [2] showed that the MSE was increased by jointly utilizing Liu-type and PCR estimators. In other words, $r-(k,d)$ class estimator has smaller MSE than other estimators. Therefore, $r-(k,d)$ class estimator can be used as an alternative estimator for combating multicollinearity.

3. Superiority of the proposed estimators

In this section, we have discussed the superiority of the $r-(k,d)$ class estimator over some other estimators based on the MSEM criterion. We first listed some notations and definitions that will be needed in the following discussions. For a matrix $A, A', A^+, rank\ (A), R(A), and\ N(A)$ stand for the transpose, Moore-Penrose inverse, rank, column space and null space of M, respectively.

If we denote the covariance matrix of an estimator of $\hat{\beta}$ by $Cov(\hat{\beta})$

Let $Cov(\hat{\beta}) = E\left[(\hat{\beta} - E(\hat{\beta}))(\hat{\beta} - E(\hat{\beta}))'\right]$

and the bias by $Bias(\hat{\beta})$

$Bias(\hat{\beta}) = E(\hat{\beta}) - \beta$

then

$MSEM(\hat{\beta}) = Cov(\hat{\beta}) + Bias(\hat{\beta})Bias(\hat{\beta})'$

Lemma 1. Let $\hat{\beta}_1$ and $\hat{\beta}_2$ be two competing estimators. The estimator $\hat{\beta}_2$ is said to be better than $\hat{\beta}_1$ by the $MSEM$ criterion if and if only $MSEM(\hat{\beta}_1) - MSEM(\hat{\beta}_2) \geq 0$

From (1) and (2), we find that,

$\hat{\beta}_r(k,d) = T_rS_r(k)^{-1}S_r(-d)\,T_r'\hat{\beta}_{PCR}$

$$Bias\left(\hat{\beta}_r(k,d)\right) = E\left(\hat{\beta}_r(k,d)\right) - \beta$$
$$= T_rS_r(k)^{-1}S_r(-d)\,T_r'\beta - \beta$$
$$= [(-k-d)T_rS_r(k)^{-1}\,T_r' - T_{p-r}\,T_{p-r}']\beta$$

$$Cov\left(\hat{\beta}_r(k,d)\right) = T_rS_r(k)^{-1}S_r(-d)\,T_r'Cov(\hat{\beta}_{PCR})T_r\,T_r'S_r(-d)S_r(k)^{-1}$$
$$= \sigma^2T_rS_r(k)^{-1}S_r(-d)\Lambda_r^{-1}S_r(-d)S_r(k)^{-1}\,T_r'$$

Where $Cov(\hat{\beta}_{PCR}) = \sigma^2T_r\Lambda_r^{-1}T_r'$, $S_r(k) = (\Lambda_r + kI_r)$. Thus $MSEM(\hat{\beta}_r(k,d))$ is

$$MSEM(\hat{\beta}_r(k,d)) = \sigma^2T_rS_r(k)^{-1}S_r(-d)\Lambda_r^{-1}S_r(-d)S_r(k)^{-1}\,T_r' + [(-k-d)T_rS_r(k)^{-1}\,T_r' - T_{p-r}\,T_{p-r}']\beta \times \beta'[(-k-d)T_rS_r(k)^{-1}\,T_r' - T_{p-r}\,T_{p-r}'] \quad (3)$$

3.1. Comparison of the $\mathbf{r-(k,d)}$ *class estimator to the OLS estimator*

In this section, we have compared the $r-(k,d)$ class estimator with the OLS estimator. In the following theorem, we have obtained a necessary and sufficient condition for the $r-(k,d)$ class estimator to be superior to the OLS estimator in the MSEM sense.

Theorem 1. *The $r-(k,d)$ class estimator $\hat{\beta}_r(k,d)$ is superior to the OLS estimator $\hat{\beta}$ if and only if*

$$\beta' T_r (k+d)^2 \left[2(k+d)I_r + (k^2-d^2)\Lambda_r^{-1}\right]^{-1} T_r'\beta + \beta' T_{p-r}\Lambda_{p-r}^{-1}T_{p-r}'\beta \le \sigma^2$$

Proof. We first compared the $r-(k,d)$ class estimator with the OLS estimator of β by the matrix mean squares criterion. We obtain the *MSEM* matrix of the OLS estimator $(\hat{\beta})$, by substituting $k=0$ and $r=p$ in (3). We thus have

$MSEM(\hat{\beta}) = \sigma^2 S^{-1}$ where $S = X'X$. If we write $\Lambda = \begin{pmatrix} \Lambda_r & 0 \\ 0' & \Lambda_{p-r} \end{pmatrix}$ and

$T = (T_r \quad T_{p-r})$ then

$$MSEM(\hat{\beta}) = \sigma^2\left(T_r\Lambda_r^{-1}T_r' + T_{p-r}\Lambda_{p-r}^{-1}T_{p-r}'\right) \tag{4}$$

By using (4), $MSEM(\hat{\beta}) - MSEM(\hat{\beta}_r(k,d))$ can be expressed as

$$\begin{aligned} MSEM(\hat{\beta}) - MSEM(\hat{\beta}_r(k,d)) = {} & \sigma^2 T_r S_r(k)^{-1}\left[2(k+d)I_r + (k^2-d^2)\Lambda_r^{-1}\right]S_r(k)^{-1}T_r' \\ & + T_{p-r}\left[\sigma^2\Lambda_{p-r}^{-1} - T_{p-r}'\beta\beta' T_{p-r}\right]T_{p-r}' \\ & - (k+d)^2 T_r S_r(k)^{-1}T_r'\,\beta\beta' T_r S_r(k)^{-1}T_r' \\ & - (k+d)T_r S_r(k)^{-1}T_r'\,\beta\beta' T_{p-r}\,T_{p-r}' \\ & - (k+d)T_{p-r}\,T_{p-r}'\,\beta\beta' T_r S_r(k)^{-1}T_r' \end{aligned} \tag{5}$$

Let us define $S^*(k,d)$ and $\Lambda^*(k,d)^{-1}$ as follows

$$S^*(k,d) = \begin{pmatrix} \frac{1}{k+d}S_r(k) & 0 \\ 0' & I_{p-r} \end{pmatrix}$$

and

$$\Lambda^*(k,d)^{-1} = \begin{pmatrix} \frac{1}{(k+d)^2}\left[2(k+d)I_r + (k^2-d^2)\Lambda_r^{-1}\right] & 0 \\ 0' & \Lambda_{p-r}^{-1} \end{pmatrix}$$

Then the expression in (5) equals

$$MSEM(\hat{\beta}) - MSEM(\hat{\beta}_r(k,d)) = TS^*(k,d)^{-1}[\sigma^2\Lambda^*(k,d)^{-1} - T'\beta\beta' T]S^*(k,d)^{-1}T' \tag{6}$$

So that $MSEM(\hat{\beta}) - MSEM(\hat{\beta}_r(k,d))$ is a nnd matrix if and only if $\sigma^2\Lambda^*(k,d)^{-1} - T'\beta\beta' T$ is nnd matrix. $\sigma^2\Lambda^*(k,d)^{-1} - T'\beta\beta' T$ is nnd if and only if $\beta' T\Lambda^*(k,d)T'\beta \le \sigma^2$ (Rao and Toutenburg, [3]). By substituting the expression for $\Lambda^*(k+d)$ and simplifying this condition, we conclude the proof.

3.2. Comparison of the $r-(k,d)$ class estimator to the PCR estimator

In this section we have compared the $r-(k,d)$ class estimator with the PCR estimator.

There are many different theorems that compare two biased estimators in terms of MSEM such as Trenkler [4], Trenkler and Toutenburg [5] and etc. We used the theorem by Baksalary and Trenkler [6].

Theorem 2. *Let $\varphi_{n\times p}$ be a set of $n\times p$ complex matrices and let H_n be subset of $\varphi_{n\times n}$ consisting of Hermitian matrices. Further, given $\mathcal{L}^* \in \varphi_{n\times n}$, the symbols $\mathcal{L}^*, \mathfrak{R}(\mathcal{L})$ and $\varphi(\mathcal{L})$ stand for conjugate transpose, the range, and the set of all generalized inverses with respect to $\mathcal{L}$. Now, let $A \in H_n$, a_1 and $a_2 \in \varphi_{n\times 1}$ be linearly independent,*

$f_{ij} = a_i^* A^- a_j$, $i, j = 1,2$ for $A^- \in \varphi(A)$ and $s = [a_1^*(I_n - AA^-)^*(I_n - AA^-)a_2]/[a_1^*(I_n - AA^-)^*(I_n - AA^-)a_1]$ provided that $a_1 \notin \Re(A)$ then $A + a_1 a_1^* - a_2 a_2^*$ is non-negative definite if and only if any one of the following sets of conditions hold.

a) *A is non-negative definite, $a_1 \in \Re(A)$, $a_2 \in \Re(A)$ and $(f_{11} + 1)(f_{22} - 1) \leq |f_{12}|^2$*

b) *A is non-negative definite, $a_1 \notin \Re(A)$, $a_2 \in \Re(A\colon a_1)$ and $(a_2 - sa_1)^* A^- (a_2 - sa_1) \leq 1 - |s|^2$*

c) *$A = U\Delta U^* - \delta v v^*$, $a_1 \in \Re(A)$, $a_2 \in \Re(A)$, $v^* a_1 \neq 0$ and $(f_{11} + 1) \leq 0$*

$(f_{22} - 1) \leq 0 (f_{11} + 1)(f_{22} - 1) \geq |f_{12}|^2$

where $(U\colon v)$ (where U is possibly absent) is a subunitary matrix, Δ is a positive definite diagonal matrix (occuring when U is present) and δ is a positive scalar. Furthermore, the conditions (a)-(c) are all independent of the choice of $A^- \in \varphi(A)$. In the following theorem, we have obtained a necessary and sufficient condition for the $r - (k, d)$ class estimator to be superior to the PCR estimator in the MSEM sense.

Theorem 3. *$MSEM(\hat{\beta}_r) - MSEM(\hat{\beta}_r(k, d))$ is nnd if and only if $\beta \in N(W_1)$, where $N(W_1)$ is the null space of $W_1 = \left(\frac{k+d}{\sigma}\right)\left[2(k+d)I_r + (k^2 - d^2)\Lambda_r^{-1}\right]^{-\frac{1}{2}} T_r'$*

Proof. The expression for $MSEM(\hat{\beta}_r)$, the *MSEM* matrix of $\hat{\beta}_r$, the *PCR* estimator of β, can easily be obtained from (5) by putting $d = -k$. Hence, we have

$$MSEM(\hat{\beta}_r) = \sigma^2 T_r \Lambda_r^{-1} T_r' + (T_r T_r' - I_p)\beta\beta'(T_r T_r' - I_p) \tag{7}$$

$MSEM(\hat{\beta}_r) - MSEM(\hat{\beta}_r(k, d))$ can be expressed as

$$\begin{aligned} MSEM(\hat{\beta}_r) - MSEM(\hat{\beta}_r(k, d)) = {} & \sigma^2 T_r S_r(k)^{-1}\left[2(k+d)I_r + (k^2 - d^2)\Lambda_r^{-1}\right] S_r(k)^{-1} T_r' \\ & + (T_r T_r' - I_p)\beta\beta'(T_r T_r' - I_p) + \\ & \left[-(k+d)T_r S_r(k)^{-1} T_r' - T_{p-r} T_{p-r}'\right]\beta \\ & \times \beta'\left[-(k+d)T_r S_r(k)^{-1} T_r' - T_{p-r} T_{p-r}'\right] \end{aligned} \tag{8}$$

Using matrix to compare the $r - (k, d)$ class estimator with PCR estimator based on the MSEM criterion would have been very difficult, so we used a theorem proposed by Baksalary and Trenkler [6].

We noted from (8) that in our case $A = \sigma^2 T_r B T_r'$, where

$B = S_r(k)^{-1}\left[2(k+d)I_r + (k^2 - d^2)\Lambda_r^{-1}\right] S_r(k)^{-1}$

Let us now consider the Moore-Penrose inverse A^+ of A, which is given by,

$$A^+ = \frac{1}{\sigma^2} T_r B^{-1} T_r'$$

And also $A A^+ = T_r T_r'$. Hence $a_1 = (T_r T_r' - I_p)\beta$ and $a_2 = \left[-(k+d)T_r S_r(k)^{-1} T_r' - T_{p-r} T_{p-r}'\right]\beta$. $a_1 \in \Re(A)$ if and only if $a_1 = 0$.

We cannot apply part (a) and (c) of the theorem, but part (b) is applicable. From the part (b) of theorem we can obtain the definition of s:

$$s = \frac{[a_1^*(I_n - AA^-)^*(I_n - AA^-)a_2]}{[a_1^*(I_n - AA^-)^*(I_n - AA^-)a_1]} = 1$$

In addition, $a_2 - a_1 = A\eta_1$, where

$$\eta_1 = -\frac{(k+d)}{\sigma^2} T_r S_r(k)\left[2(k+d)I_r + (k^2 - d^2)\Lambda_r^{-1}\right]^{-1} T_r' \beta$$

which indicates $a_2 \in \Re(A\colon a_1)$. Thus, using this theorem a necessary and sufficient condition for superiority of the $r - (k, d)$ class estimator over the PCR estimator is obtained as

$(a_2-a_1)'A^-(a_2-a_1)=\eta_1'A\eta_1$

Then, $r-(k,d)$ class estimator dominates the *PCR* estimator if and only if

$\eta_1'A\eta_1=\left(\frac{k+d}{\sigma}\right)^2\beta'T_r\left[2(k+d)I_r+(k^2-d^2){\Lambda_r}^{-1}\right]^{-1}{T_r}'\beta\le 0.$

Under the assumption $(k-d+2)(k+d)\ge 0$, we have

$(a_2-a_1)'A^-(a_2-a_1)\ge 0.$

Accordingly, since $-\infty<d<\infty$

if we let $Q_1=W_1\beta$ where

$W_1=\left(\frac{k+d}{\sigma}\right)\left[2(k+d)I_r+(k^2-d^2){\Lambda_r}^{-1}\right]^{-\frac{1}{2}}{T_r}'$

$MSEM(\hat{\beta}_r)-MSEM(\hat{\beta}_r(k,d))$ is nnd if and only if $Q_1=0$. $Q_1=W_1\beta=0$ means that $\beta\in N(W_1)$, where $N(W_1)$ is the null space of W_1.

3.3. Comparison of the $\boldsymbol{r-(k,d)}$ class estimator to the Liu-type estimator

In this section we have compared the $r-(k,d)$ class estimator with the Liu-type estimator. In the following theorem, we have obtained a necessary and sufficient condition for the $r-(k,d)$ class estimator to be superior to the Liu-type estimator in the MSEM sense.

Theorem 4. *The $r-(k,d)$ class estimator dominates the Liu-type estimator if and only if $\beta\in N(W_2)$, where $N(W_2)$ is the null space of* $W_2=\left(\frac{1}{\sigma}\right)(\Lambda_{p-r})^{\frac{1}{2}}{T_{p-r}}'$

Proof. We obtain $MSEM(\hat{\beta}(k,d))$ by substituting $r=p$ in (3). Hence, we have

$$MSEM(\hat{\beta}(k,d))=\sigma^2TS(k)^{-1}S(-d)\Lambda^{-1}S(-d)S(k)^{-1}T'+(k+d)^2TS(k)^{-1}T'\beta\beta'TS(k)^{-1}T' \quad (9)$$

where $S(k)=(\Lambda+kI_p)$. In consequence of (3) and (9), it can be seen that

$$\begin{aligned}MSEM\left(\hat{\beta}_r(k)\right)-MSEM((\hat{\beta}_r(k,d))=\;&\sigma^2T_{p-r}S_{p-r}(k)^{-1}S_{p-r}(-d){\Lambda_{p-r}}^{-1}S_{p-r}(-d)S_{p-r}(k)^{-1}{T_{p-r}}'\\&+(k+d)^2TS(k)^{-1}T'\beta\beta'TS(k)^{-1}T'\\&-[(-k-d)T_rS_r(k)^{-1}{T_r}'-T_{p-r}{T_{p-r}}']\beta\\&\times\beta'[(-k-d)T_rS_r(k)^{-1}{T_r}'-T_{p-r}{T_{p-r}}']\end{aligned}$$

As in the previous case, we use the theorem by Baksalarly and Trenkler (6).

Letting $A=\sigma^2T_{p-r}B\,{T_{p-r}}'$, where $B=S_{p-r}(k)^{-1}S_{p-r}(-d){\Lambda_{p-r}}^{-1}S_{p-r}(-d)S_{p-r}(k)^{-1}$

$a_1=(k+d)\,TS(k)^{-1}T'\beta$ and $a_2=[-(k+d)T_rS_r(k)^{-1}{T_r}'-T_{p-r}{T_{p-r}}']\beta.$

The Moore-Penrose inverse A^+ of A, which is given by,

$A^+=\frac{1}{\sigma^2}T_{p-r}B^{-1}{T_{p-r}}'$ and $A^+=T_{p-r}{T_{p-r}}'$. $a_1\notin\Re(A)$ and $a_2\in\Re(A{:}\ a_1)$ because $s=-1$, where

$\eta_2=-\frac{1}{\sigma^2}T_{p-r}S_{p-r}(k)\,S_{p-r}(-d)^{-1}\Lambda_{p-r}{T_{p-r}}'\beta$

also $(a_2+a_1)'A^-(a_2+a_1)=\eta_2'A\eta_2$ then the $r-(k,d)$ class estimator dominates the Liu- type estimator if and only if

$\eta_2'A\eta_2=\frac{1}{\sigma^2}\beta'T_{p-r}\Lambda_{p-r}{T_{p-r}}'\beta\le 0$

However, it is obvious that $\eta_2'A\eta_2$ is always ≥ 0 so the condition turns out to be $\eta_2'A\eta_2=0$.

If we let $Q_2=W_2\beta$ where

$W_2 = \left(\frac{1}{\sigma}\right)\left(\Lambda_{p-r}\right)^{\frac{1}{2}}T_{p-r}'$, we obtain $\eta_2' A \eta_2 = Q_2' Q_2$

Then $MSEM\left(\hat{\beta}_r(k)\right) - MSEM(\hat{\beta}_r(k,d)$ is nnd if and only if $Q_2 = 0$; $Q_2 = W_2\,\beta = 0$ means that $\beta \in N(W_2)$, where $N(W_2)$ is the null space of W_2.

3.4. Criteria of the $\mathbf{r} - (\boldsymbol{k}, \boldsymbol{d})$ class estimator to dominate both the PCR and Liu-type estimator

In the following theorem, we have obtained a necessary and sufficient condition for the $r - (k,d)$ class estimator to be superior over the PCR and Liu-type estimator in the MSEM sense.

Theorem 5. *A necessary and sufficient condition for the $r - (k,d)$ class estimator to dominate both the PCR and Liu-type estimators simultaneously is given by $\beta \in N(W_3)$, where $N(W_3)$ is the null space of* $W_3 = \left(\frac{1}{\sigma}\right)\Lambda^*(k,d)^{1/2}T'$

Proof. If we define partitioned matrix $W_3 = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}_{p\times p}$ then $W_3 = \left(\frac{1}{\sigma}\right)\Lambda^*(k,d)^{1/2}T'$.

Multiplying W_3 by β from the right, we get $W_3\beta = 0$ which is the proof the theorem.

4. A numerical example

In order to illustrate our theoretical results, we use a widely popular Portland cement data analyzed in detail by Woods et al. [7] and Kaçıranlar et al. [8]. These data came from an experimental investigation of the heat created during the setting and hardening of Portland cements of varied composition and the dependence of this heat on the percentages of four compounds in the clinkers from which the cement was produced. The four compounds considered were tricalcium aluminate: $3CaO.Al_2O_3$, tricalcium silicate: $3CaO.SiO_2$, tetracalcium aluminoferrite: $4CaO.Al_2O_3.$ Fe_2O_3 and β-dicalcium silicate: $2CaO.SiO_2$, which are denoted by X_1, X_2, X_3, and X_4 respectively. The heat evolved after 180 days of curing, which is denoted by y. The data set is given in Table 1.

Table 1. Data Set

y	X_1	X_2	X_3	X_4
78.5	7	26	6	60
74.3	1	29	15	52
104.3	11	56	8	20
87.6	11	31	8	47
95.9	7	52	6	33
109.2	11	55	9	22
102.7	3	71	17	6
72.5	1	31	22	44
93.1	2	54	18	22
115.9	21	47	4	26
83.8	1	40	23	34
113.3	11	66	9	12
109.4	10	68	8	12

All the results below were computed by Matlab 7.13. We standardized the data so that $X'X$ matrix is in the form of a correlation matrix. According to eigenvalues the standardized data are

$\lambda_1 = 2.2357, \lambda_2 = 1.5760, \lambda_3 = 0.1866, \lambda_4 = 0.001624$

And the condition number is

$$\kappa = \frac{\lambda_{max}}{\lambda_{min}} = \frac{2.2357}{0.001624} = 1376.81$$

We can use the following formula to choose k and optimal value of d as given by Liu (9)

$$\hat{k} = \frac{\lambda_1 - 100 * \lambda_p}{99}$$

$$d_{opt} = \frac{\sum_{i=1}^{p}\left((\sigma^2 - k\gamma_i^2)/(\lambda_i + k)^2\right)}{\sum_{i=1}^{p}\left((\lambda_i\gamma_i^2 + \sigma^2)/\lambda_i(\lambda_i + k)^2\right)}$$

Values of d_{opt}, $\hat{k}$ and *MSE* estimates were obtained by replacing the corresponding theoretical expressions in all unknown model parameters with their *OLS* estimates.

Numerical results of the comparison of $r-(k,d)$ class estimator with the $r-k$ estimator and PCR estimator are summarized in Table 2 and 3 . The estimated MSE values for the *PCR*, $r-k$ and $r-(k,d)$ class estimator were obtained by replacing the corresponding theoretical MSE expressions in all unknown model parameters with their *OLS*.

Table 2. Values of estimates and MSE for d_{opt} and various values of k

	$k=0$	$\hat{k}$	$k=0.3$	$k=0.5$	$k=1$
PCR	0.1691	0.1691	0.1691	0.1691	0.1691
$r-k$	0.1632	0.1621	0.1621	0.1628	0.1668
$r-(k,d)$	0.1510	0.1518	0.1521	0.1539	0.1599

Table 3. Values of estimates and MSE for $\hat{k}_{HK}$ and various values of d

	$d=-0.5$	$d=-0.3$	d_{opt}	$d=0$	$d=0.5$	$d=0.7$	$d=1$
PCR	0.1691	0.1691	0.1691	0.1691	0.1691	0.1691	0.1691
$r-k$	0.1621	0.1621	0.1621	0.1621	0.1621	0.1621	0.1621
$r-(k,d)$	0.1517	0.1509	0.1518	0.1519	0.1598	0.1652	0.1757

We observed that under some conditions on d and k, $r-(k,d)$ class estimator performed well compared to others. We can see that $r-(k,d)$ class estimator is better than the $r-k$ and PCR, except for big values of d *(*$d = 0.70, 1$*)*. We see that in practice we can choose small d and regulate k. At the same time, we can choose big k and regulate d. Therefore, we believe that our estimator is meaningful in practice.

5. Monte Carlo simulation

In order to further the MSE performances of the PCR, $r-k$ and $r-(k,d)$ we performed a Monte Carlo simulation study by considering different levels of multicollinearity. Following Liu [9] and Kibria [10] , we obtain the explanatory variables while the response variables were generated using the following equations:

$$x_{ij} = (1-\gamma^2)^{1/2} z_{ij} + \gamma z_{ip}, \quad y_i = (1-\gamma^2)^{1/2} z_{ij} + \gamma z_{ip} \quad i = 1,2,\dots,n, \quad j = 1,2,\dots,p$$

where z_{ij} are independent standard normal pseudo-random numbers and p is specified so that correlation between any two explanatory variables is given by γ^2. In this simulation, three different sets of correlations namely, $\gamma = 0.90, 0.95$ *and* 0.99 were considered to show collinearity between the explanatory variables. By applying the variance inflation factors and condition indices it can easily be shown that the explanatory variables are weak, strong and severely collinear when $\gamma = 0.90, 0.95\ and\ 0.99$, respectively. In this experiment, we selected $p = 4$ for $n = 30$, 50, 100. Then, the experiment was replicated 1000 times by generating new error terms.

Let us consider the PCR, $r - k$ and $r - (k, d)$ and compute their respective estimated MSE values with the different levels of multicollinearity. Based on the simulation results shown in Tables 4, we can see that with the increase of the levels of multicollinearity, the estimated MSE values of the PCR, $r - k$ and $r - (k, d)$ increase in general. For fixed k and d the estimated MSE of estimators increase with the increasing level of multicollinearity. We can see that $r - (k, d)$ is much better than the competing estimators when the explanatory variables are severely collinear.

Table 4. MSE values for three estimator

γ	n	PCR	$r - k$	$(r - k, d)$
	30	0.1454	0.1304	0.1181
0.90	50	0.1860	0.1824	0.1818
	100	0.0800	0.0809	0.0645
	30	0.3462	0.4318	0.1627
0.95	50	0.1054	0.0821	0.0816
	100	0.2587	0.2530	0.2530
	30	0.1543	0.3523	0.1955
0.99	50	0.2721	0.1754	0.2542
	100	0.2659	0.1695	0.1688

6. Conclusion

In this paper, we used the *MSEM* criterion to compare the $r - (k, d)$ class estimator with the *OLS*, *PCR* and Liu-type estimators. We obtained necessary and sufficient conditions for the $r - (k, d)$ class estimator that showed it being superior over the other three estimators.

Furthermore, we saw that the conditions obtained on the comparisons of the $r - (k, d)$ class estimator with OLS, PCR and Liu-type estimators depend on the unknown parameters. We constructed tests to decide whether or not these conditions hold in given situations. Moreover, we also mentioned that if we use unknown parameters for the comparisons, we should use unbiased estimates or priori information for those parameters in order to obtain practical results. Finally, we illustrated our findings with a numerical example and a Monte Carlo simulation. Both numerical example and simulation results which agrees with our theoretical findings.

References

[1] C. Stein, 1956, Inadmissibility of the usual estimator for mean of multivariate normal distribution. In Neyman J (ed), *Proceedings of the third Berkley symposium on mathematical and statistics probability* 1, 197–206.

[2] D. Inan 2015, Combining the Liu-type estimator and the principal component regression estimator, *Statistical Papers* 56,147-156.

[3] C. Rao, H. Toutenburg, 1995, *Linear Models: Least Squares and Alternatives.* New York:Springer-Verlag Inc.

[4] G. Trenkler, 1985, Mean Square Error Matrix Comparisons of Estimators in Linear Regression. *Communications in Statistics Theory and Methods A*, 14, 2495–2509.

[5] G. Trenkler, H. Toutenburg, 1990, Mean squared error matrix comparisons between biased estimators an overview of recent results, *Statistical Papers*, 31,165–179.

[6] J.K. Baksalary, G. Trenkler, 1991. Nonnegative and positive definiteness of matrices modified by two matrices of rank one, *Linear Algebra and Its Application* 151,169–184.

[7] H.Woods, , H.H. Steinour , H.R. Starke, 1932. Effect of composition of Portland cement on heat evolved during hardening, *Industrial and Engineering Chemistry* 24, 1207–1214.

[8] S.Kaçıranlar, S. Sakallioglu, F. Akdeniz, G.P.H. Styan, H.J. Werner, 1999, A new biased estimator in linear regression and a detailed analysis of the widely-analysed dataset on Portland Cement, *Sankhya Indian J Stat* 61(B),443–459.

[9] K. Liu, 2003, Using Liu-type estimator to combat collinearity, *Communications in Statistics Theory and Methods* 32, (5),1009–1020.

[10] B.M.G. Kibria, 2003, Performance of some new ridge regression estimators, *Communications in Statistics - Simulation and Computation*, 32,2389-2413.