On Inextensible Flows Developable Surfaces Associated Focal Curve According to Ribbon Frame

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ABSTRACT

In this paper, we investigate inextensible flows of focal curves in Euclidean 3-space E^3 associated to developable surfaces to ribbon frame. We present some new generalizations for torsion and curvature of focal curves in E^3 associated to developable surfaces. Finally, in case of having a flow of developable surface is inextensible we prove that this surface is not minimal.

Keywords: Developable surface, inextensible flows, ribbon frame

1. INTRODUCTION

One of the major research field of the differential geometry is the analysis of space curves. For any unit speed curve, the focal curve is defined as the centres of the osculating spheres. According to Frenet frame $\{t(s),n(s),b(s)\}$ of unity speed curve $\gamma(s)$, the focal curve is given as follows,

$$C_\psi(s) = (\gamma + c_1 n + c_2 b)(s),$$

where the coefficients c_1, c_2 are smooth functions that are called focal curvatures of γ. There have been several studies related to the focal curve using this definition Vergas, Arslan, Korpinar concerned with this isuse [1,2,3].

Any ribbon consists of two functions $p(s)$ and $\theta(s)$ are defined in interval $s \in 0,Z]$, where Z is considered as the ribbpn's intrinsic length under construction. If $A(s)$ is defined as a unit vector field, then it must have an angle $\theta(s)$ to the ribbon's center curve. On the other hand, assuming $p(s)$ is a generating function, $A(s)$ is the direction field for the Darboux vector $\mathbf{W}(s)$ together with $p(s)$. An orthonormal triple $\{\mathbf{c}, \mathbf{v}, \mathbf{h}\}$ has the following differential system:

$$\mathbf{c}' = pA \times \mathbf{c}$$
$$\mathbf{v}' = pA \times \mathbf{v}$$
$$\mathbf{h}' = pA \times \mathbf{h}$$

This orthonormal triple defines a ribbon frame. According to this frame, the focal curve can be given as follows:

$$C_\psi(s) = (\gamma + c_1 \mathbf{v} + c_2 \mathbf{h})(s)$$

In CAGD, usage of splines on the construction of curve design is highly significant. Inextensible flows of curves preserve their shape connection thanks to their control polygon. Thus, formulation of algorithms for processing can be calculated. Kwon, Park and Chi examined inextensible flows of curves and develople surfaces [6]. Korpınar, Turhan and Altay gave inextensible flows of develople surfaces associated focal curve [3]. Recently, Bohr and Markvosen examined the ribbon frame [4]. Giomi and Mahadevan gave develope ribbons [7]. Also some authors have studied focal curves and flows in [8-14].
In this paper, we give inextensible flows of focal curves together with ribbon frame. We give some

new generalizations for torsion and curvature of focal curves associated with developable surfaces.

2. PRELIMINARIES

For the construction of ribbon and the center curve of the ribbon we need two smooth functions, $p(s)$ and $\theta(s)$. We also assume that they are defined in the interval $s \in [0, Z]$, where Z is the intrinsic length of the ribbon under construction. We also suppose that $\theta(s) \in [0, \pi]$ $\forall s$ and $\sin(\theta(s)) > 0$ $\forall s$, througout the paper. In this construction unit vector field $A(s)$ is significant since it is defined by having the angle $\theta(s)$ to the center curve of the ribbon and it is also a field tangent to the ribbon. In fact, $A(s)$ will generate the Darboux vector $\mathbf{W}(s)$ since it is a direction field together with the generating function $p(s)$ as a multiplying factor, i.e. $\mathbf{W}(s) = p(s)A(s)$, [4].

We assume that $\{\mathbf{c}(s), \mathbf{v}(s), \mathbf{h}(s)\}$ are unique orthonormal vectors such that they are solutions to the below system:

$$\mathbf{c}'(s) = p(s)A(s) \times \mathbf{c}(s),$$
$$\mathbf{v}'(s) = p(s)A(s) \times \mathbf{v}(s),$$
$$\mathbf{h}'(s) = p(s)A(s) \times \mathbf{h}(s),$$ (1)

where $A(s)$ is defined in terms of $\mathbf{h}(s)$ and $\mathbf{c}(s)$ as the following:

$$A(s) = \sin(\theta(s))\mathbf{h}(s) + \cos(\theta(s))\mathbf{c}(s). \quad (2)$$

For the purpose of uniqueness we also implement arbitrary initial conditions referring to a basis in $\mathbb{R}^3$ for given fixed coordinate system

$$\{\mathbf{c}(0), \mathbf{v}(0), \mathbf{h}(0)\} = \{(1,0,0),(0,1,0),(0,0,1)\}. \quad (3)$$

The above system is stated as follows explicitly:

$$\mathbf{c}'(s) = p(s)\sin(\theta(s))\mathbf{v}(s) \quad (4)$$

(2)

$$\mathbf{v}'(s) = -p(s)\sin(\theta(s))\mathbf{c}(s) + p(s)\cos(\theta(s))\mathbf{h}(s)$$
$$\mathbf{h}'(s) = -p(s)\cos(\theta(s))\mathbf{v}(s),$$

where the dot notation denotes differentiation according to s. If we consider notation of compact matrix, then we have

$$\mathbf{R}'(s) = \mathbf{R}(s)\Xi(s), \quad (5)$$

where $\mathbf{R}(s)$ is the orthogonal matrix so that it satisfies $2\det(\mathbf{R}(s)) = 1$. Further, columns of the $\mathbf{R}(s)$ are the coordinate functions of $\{\mathbf{c}(s), \mathbf{v}(s), \mathbf{h}(s)\}$ respectively [4], where

$$\Xi(s) = \begin{bmatrix} 0 & -p(s)\sin(\theta(s)) & 0 \\ p(s)\sin(\theta(s)) & 0 & -p(s)\cos(\theta(s)) \\ 0 & p(s)\cos(\theta(s)) & 0 \end{bmatrix}.$$
(6)

3 Focal Curve According to Ribbon Frame

In the case of focal curve is denoted by F_ψ^B, we have the following expression

$$F_\psi^R(s) = (\psi + c_1\mathbf{v} + c_2\mathbf{h})(s), \quad (7)$$

where the coefficients c_1, c_2 are smooth functions of the curve ψ. They are named as first and second focal curvatures of ψ.

Lemma 1. Let $\psi : I \to E^3$ be a unit speed curve and F_ψ^R its focal curve on E^3. Then, the focal curvatures of F_ψ^R are

$$c_1 = \frac{1}{p(s)\sin\theta(s)} \quad (8)$$
$$c_2 = \frac{p'(s)\sin\theta(s) + \theta'(s)p(s)\cos\theta(s)}{p(s)^3\sin^2\theta(s)\cos\theta(s)} \quad (9)$$

Lemma 2. Let F_ψ^R be the focal curve of the unit speed curve $\psi : I \to E^3$ on E^3. If $p(s)$ is constant then, the focal curvatures of F_ψ^R are

$$c_1 = \frac{1}{p(s)\sin\theta(s)} \quad (10)$$
$$c_2 = \frac{\theta'(s)}{\sin^2\theta(s)} \quad (11)$$

Lemma 3. Let F_ψ^R be the focal curve of the unit speed curve $\psi : I \to E^3$ on E^3. If $\theta(s)$ is constant then, the focal curvatures of F_ψ^R are

$$c_1 = \frac{1}{p(s)\sin\theta(s)} \quad (12)$$

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$$c_2 = \frac{p'(s)}{2p(s)^3 \sin 2\theta(s)} \quad (13)$$

4. Inextensible Flows of Developable Surfaces Associated with Focal Curve According to Ribbon Frame

A ruled surface in E^3 is defined by

$$\Omega_{(\psi,\delta)}(s,u) = \psi(s) + u\delta(s).$$

Here, we have following smooth maps $\Omega_{(\psi,\delta)} : I \times R \to E^3$, $\psi : I \to E^3$, $\delta : I \to E^3 \setminus \{0\}$ where I is unit circle S^1 or an open interval. From the defininition ψ and δ are called base and director curve.

Definition 4. Let assume that Gaussian curvature U vanishes everywhere on the surface then a smooth surface $\Omega_{(\psi,\delta)}$ is called a developable surface.

Definition 5. Let $\psi : I \to E^3$ be a unit speed curve. We define the developable surface as follows

$$\Omega_{\left(F_\psi^R,\psi'\right)}(s,u) = F_\psi^R(s) + u\psi'(s), \quad (14)$$

where $F_\psi^R(s)$ is focal curve.

Definition 6. Let first fundamental form $\{E, F, G\}$ of a surface evolution satisfies following.

$$\frac{\partial E}{\partial t} = \frac{\partial F}{\partial t} = \frac{\partial G}{\partial t} = 0 \quad (15)$$

Then the surface evolution $\Omega(s,u,t)$ and its flow $\frac{\partial \Omega}{\partial t}$ are inextensible.

This definition says that the surface $\Omega(s,u,t)$ is the isometric image of the original surface $\Omega(s,u,t_0)$ which is determined for initial time t_0. $\Omega(s,u,t)$ can be visualized as a waving flag for a developable surface. There is no any nontrivial inextensible evolution for a given rigid surface.

Definition 7. We can define one-parameter family of developable ruled surface as follows:

$$\Omega(s,u,t) = F_\psi^R(s,t) + u\psi'(s,t). \quad (16)$$

Theorem 8. Let we assume that Ω is developable surface associated to a focal curve in E^3. If $\frac{\partial \Omega}{\partial t}$ is inextensible, then

$$\frac{\partial}{\partial t}[(up\sin\theta)^2 + (c_1 p\cos\theta)^2] = 0 \quad (17)$$

Proof. Let $\Omega(s,u,t)$ be a one-parameter family of developable surface. Then, we say that Ω is inextensible.

$$\Omega_s = c_1 p \cos\theta \, \mathbf{h} + up\sin\theta \, \mathbf{v},$$
$$\Omega_u = \mathbf{c}.$$

Calculating first fundamental form gives rise to obtain following results.

$$E = \langle \Omega_s, \Omega_s \rangle = u^2 (p\sin\theta)^2 + c_1^2 (p\cos\theta)^2,$$
$$F = 0,$$
$$G = 1.$$

Considering above results, we have

$$\frac{\partial E}{\partial t} = 0,$$
$$\frac{\partial F}{\partial t} = 0,$$
$$\frac{\partial G}{\partial t} = 0.$$

If $\frac{\partial \Omega}{\partial t}$ is inextensible, then we have (17).

Theorem 9. Let the flow of the developable surface Ω which is associated to focal curve in E^3, be inextensible then this surface is not minimal.

Proof. Assume that $\Omega(s,u,t) = F_\psi^R(s,t) + u\psi'(s,t)$ be a one-parameter family of developable ruled surface. Second fundamental form's component of developable surface are

$$h_{11} = \sqrt{[\tan\theta]^2 + u^2 (p\sin\theta)^2},$$
$$h_{12} = -p\cos\theta,$$

$$h_{22} = 0.$$

We also have components of metric

$$g_{11} = [\tan\theta]^2 + u^2 (p\sin\theta)^2,$$
$$g_{12} = 0,$$
$$g_{22} = 1.$$

So, one-parameter family of developable ruled surface $\Omega(s,u,t) = \mathsf{F}_{\psi}^{R}(s,t) + u\psi'(s,t)$ has the following mean curvature

$$H = g^{ij}h_{ij}$$

$$= \frac{p\cos\theta}{\sqrt{[\tan\theta]^2 + u^2(w\sin\theta)^2}}.$$

Ω is a minimal ruled surface in E^3 if and only if $p\cos\theta = 0$.

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