

Some New Classifications of Soft Subsets and Soft Equalities with Soft Symmetric Difference-Difference Product of Groups

Aslıhan Sezgin¹ , İbrahim Durak² , Zeynep Ay³ 

Article Info

Received: 13 May 2025

Accepted: 14 Jun 2025

Published: 30 Jun 2025

Research Article

Abstract – Soft sets provide a comprehensive mathematical framework for tackling uncertainty. Soft set operations and products are fundamental to soft set theory, offering innovative solutions to problems that involve parametric data. In this study, we first adapted the soft L-subsets/equality concept and soft J-subsets/equality for the revised soft set concept. Additionally, we defined some new types of soft subsets and equalities, called soft S-subsets/equality and soft A-subsets, along with their specific examples to clarify these concepts. We investigated the connections among these new concepts. This paper presents an innovative product for soft sets whose parameter sets are groups, called the "soft symmetric difference-difference-product". We thoroughly analyzed its fundamental algebraic properties, considering various soft subsets and equality relations to inspire future research. This may lead to a new soft group theory arising from this concept. Since soft algebraic structure theories are grounded in soft set operations and products, this study contributes significantly to the literature on soft sets.

Keywords – Soft sets, soft subsets, soft equalities, soft symmetric difference-difference product

1. Introduction

Many researchers have introduced different mathematical approaches to address and model complex issues of uncertainty, vagueness, and ambiguity across various fields, such as engineering, economics, social sciences, and healthcare. Molodtsov [1] highlighted certain limitations within these frameworks, for instance, fuzzy set theory [2] often faces difficulties in defining an appropriate membership function, while probability theory requires numerous trials to determine the existence of a mean value.

To address these challenges, Molodtsov [1] introduced soft set theory, a new mathematical framework, and explored its applicability in diverse areas, such as probability theory, game theory, and operations research. Unlike classical models, soft set theory offers a more flexible structure as it avoids constraints related to approximate descriptions. Following the application of soft sets to decision-making by Maji et al. [3], several scholars [4–10] expanded on this foundation by proposing advanced decision-making techniques rooted in soft set theory. Çağman and Enginoğlu [11] introduced a notable soft set-based decision-making approach. They presented the concept of soft matrices [12] and formulated decision-making strategies based on OR, AND, AND-NOT, and OR-NOT operations on these matrices. These strategies were applied to effectively tackle real-world problems involving uncertainty. The versatility of soft set theory has led to its extensive use in decision-making contexts [13–24], encompassing a wide range of developments, such as bijective soft sets, exclusive disjunctive soft sets, generalized uni-int decision-making frameworks, soft approximations, the

¹aslihan.sezgin@amasya.edu.tr (Corresponding Author); ²ibrahimdurak391919@gmail.com; ³ayzeynep875@gmail.com

¹Department of Mathematics and Science Education, Faculty of Education, Amasya University, Amasya, Türkiye

^{2,3}Department of Mathematics, Graduate School of Natural and Applied Sciences, Amasya University, Amasya, Türkiye

impact of various operators on decision processes, reduced soft matrices, cardinality inverse matrix concepts, semantics in soft sets, as well as mean operators and generalized operations on fuzzy soft matrices, and soft set-valued mappings.

In recent years, many researchers have explored the foundational aspects of soft set theory. Maji et al. [25] conducted a detailed theoretical analysis, addressing concepts, such as soft subsets and supersets, soft set equality, and fundamental operations, including union, intersection, AND-product, and OR-product. Pei and Miao [26] explored how soft sets relate to information systems and offered revised definitions for the intersection operation and soft subsets. Ali et al. [27] contributed by introducing new operations, including restricted union, restricted intersection, restricted difference, and extended intersection operations. Subsequent studies [28–41] delved into soft set operations and the algebraic frameworks governing collections of soft sets. These works proposed innovative methods and corrected conceptual misunderstandings in earlier literature. Progress in soft set research has been particularly notable in recent years, especially on soft set operations. Various new operations have since been formulated and rigorously studied [42–49].

Soft equal relations and soft subsets represent two fundamental notions in soft set theory. Maji et al. [25] were the first to introduce a reasonably precise definition of soft subsets. Building upon this foundation, Pei and Miao [26] and Feng et al. [29] expanded the idea by offering a more comprehensive formulation that can be viewed as an evolution of Maji's original concept. Qin and Hong [50] proposed two novel forms of congruence and soft equality relations within the framework of soft sets. To refine Maji's distributive laws, Jun and Yang [51] incorporated broader categories of soft subsets. They developed an enhanced version of soft equal relations, referred to as J-soft equal relations for consistency. Their work explored generalized distributive properties involving soft product operations. Inspired by Jun and Yang's contributions [51], Liu et al. [52] presented a concise study on soft L-subsets and soft L-equal relations. A notable outcome of their research is that distributive laws are not universally valid across all soft equality types discussed in prior literature.

Building upon the work presented in [52], Feng et al. [53] advanced the research by concentrating on various forms of soft subsets and the soft product operations initially introduced in [25]. Their study explored different classifications of soft subsets and the algebraic properties of soft product operations. In addition to discussing fundamental principles, such as commutativity, associativity, and distributivity, topics previously explored by multiple researchers, they conducted a detailed theoretical analysis of soft products, such as the AND-product and OR-product, within the context of soft L-subsets. Their work addressed and resolved several incomplete findings related to soft product operations in earlier literature, comprehensively investigating them through J-equality and L-equality. Furthermore, they demonstrated that soft L-equal relations form congruence relations in free soft algebras, and that the resulting quotient structures are commutative semigroups. For expanded discussions on other forms of soft equalities, including generalized soft equality, soft lattice structures, generalized operations under relaxed parameter constraints, g-soft and gf-soft equality, and T-soft equality, see [54–58].

Çağman and Enginoğlu [11] revised the concept of soft sets proposed by Maji et al. [25] and the soft set operations to enhance their practical applicability. Besides, they introduced four distinct product operations within soft set theory: the AND-product, OR-product, AND-NOT-product, and OR-NOT-product, along with the uni-int decision function. Leveraging these definitions, they formulated a unified decision-making framework to select optimal elements from a given set of alternatives. They illustrated their method's effectiveness through a practical example addressing uncertainty-related problems. Later, Sezgin et al. [59] conducted a theoretical analysis of the AND-product, which has been widely recognized as a fundamental operation in soft set-based decision-making. Although various researchers have investigated the AND-product in the context of different soft equalities, including soft L-equality and soft J-equality, Sezgin et al. [59] offered a comprehensive study of its algebraic characteristics. Their analysis encompassed key properties, including idempotency, commutativity, and associativity, and compared these with earlier findings involving soft F-subsets, soft M-equality, soft L-equality, and soft J-equality.

The concept of the soft union product was introduced for rings in [60], for semigroups in [61], and for groups in [62] earlier, and its basic properties were studied. Besides, soft union ring, soft union semigroup, and soft union group theory have been built upon this definition. Similarly, the concept of the soft intersection product was introduced for groups in [63], for semigroups in [64], and for rings in [65] earlier, and its basic properties were studied. Besides, soft intersection group, soft intersection semigroup, and soft intersection ring theory have been built upon this definition. However, due to the differences in the algebraic structures of semigroups and rings compared to groups, the definition and traits of the soft union product and soft intersection product adapt accordingly. In particular, since each group contains a unit element and every member possesses an inverse, the definitions of the soft union product and soft intersection product for groups have their own characteristics.

Although soft L-subsets/equality and soft J-subsets/equality were introduced for soft sets defined by Maji et al. [25], they have not been proposed for soft sets defined by Çağman and Enginoğlu [11]. In this paper, firstly, we conveyed these concepts for the revised soft set concept proposed by Çağman and Enginoğlu [11]. Furthermore, we defined some new soft subsets and equalities called soft S-subsets/equality and soft A-subsets and provided concrete examples to illustrate these concepts. Besides, we presented the relations between them. This paper introduces a novel product for soft sets whose parameter sets are groups, referred to as the "soft symmetric difference-difference-product," developed in the framework of the definition of soft sets by Çağman and Enginoğlu [11]. We conducted a detailed investigation into its basic algebraic characteristics, considering various soft subsets and equality relations to inspire future studies in which a new soft group theory can be built on this concept. The paper is structured as follows: Section 2 reviews key concepts in soft set theory. Section 3 introduces new soft subsets and soft equalities and explores their relations. Section 4 proposes a new soft-set product, the soft symmetric difference-difference product, and explores its complete algebraic structure regarding various soft subsets and equalities. The final section summarizes the findings herein.

2. Preliminaries

In this section, we review some fundamental definitions and characteristics to be required for the next section. Soft sets were initially introduced in [1]; however, we adopt the definitions and notations as in [11] throughout this paper.

Definition 2.1. [11] Let E be a parameter set, U be a universal set, $P(U)$ be the power set of U , and $Y \subseteq E$. Then, the soft set f_Y over U is a function such that $f_Y: E \rightarrow P(U)$, where for all $x \notin Y$, $f_Y(x) = \emptyset$. That is,

$$f_Y = \{(x, f_Y(x)) : x \in E\}$$

Definition 2.2. [11] Let $f_{\mathcal{H}}$ be a soft set over U . If $f_{\mathcal{H}}(x) = \emptyset$, for all $x \in E$, then $f_{\mathcal{H}}$ is called a null soft set and indicated by $\emptyset_E$, and if $f_{\mathcal{H}}(x) = U$, for all $x \in E$, then $f_{\mathcal{H}}$ is called an absolute soft set and indicated by U_E .

Definition 2.3. [11] Let $f_{\mathcal{H}}$ and $f_{\mathcal{K}}$ be two soft sets over U . If $f_{\mathcal{H}}(x) \subseteq f_{\mathcal{K}}(x)$, for all $x \in E$, then $f_{\mathcal{H}}$ is a soft subset of $f_{\mathcal{K}}$ and indicated by $f_{\mathcal{H}} \subseteq f_{\mathcal{K}}$. If $f_{\mathcal{H}}(x) = f_{\mathcal{K}}(x)$, for all $x \in E$, then $f_{\mathcal{H}}$ is called soft equal to $f_{\mathcal{K}}$ and denoted by $f_{\mathcal{H}} = f_{\mathcal{K}}$.

Definition 2.4. [11] Let $f_{\mathcal{H}}$ and $f_{\mathcal{K}}$ be two soft sets over U . Then, the union of $f_{\mathcal{H}}$ and $f_{\mathcal{K}}$ is the soft set $f_{\mathcal{H}} \tilde{\cup} f_{\mathcal{K}}$, where $(f_{\mathcal{H}} \tilde{\cup} f_{\mathcal{K}})(w) = f_{\mathcal{H}}(w) \cup f_{\mathcal{K}}(w)$, for all $w \in E$.

Definition 2.5. [11] Let $f_{\mathcal{H}}$ and $f_{\mathcal{K}}$ be two soft sets over U . Then, the intersection of $f_{\mathcal{H}}$ and $f_{\mathcal{K}}$ is the soft set $f_{\mathcal{H}} \tilde{\cap} f_{\mathcal{K}}$, where $(f_{\mathcal{H}} \tilde{\cap} f_{\mathcal{K}})(w) = f_{\mathcal{H}}(w) \cap f_{\mathcal{K}}(w)$, for all $w \in E$.

Definition 2.6. [11] Let $f_{\mathcal{H}}$ and $f_{\mathcal{K}}$ be two soft sets over U . Then, the symmetric difference of $f_{\mathcal{H}}$ and $f_{\mathcal{K}}$ is the soft set $f_{\mathcal{H}} \tilde{\Delta} f_{\mathcal{K}}$, where $(f_{\mathcal{H}} \tilde{\Delta} f_{\mathcal{K}})(w) = f_{\mathcal{H}}(w) \Delta f_{\mathcal{K}}(w)$, for all $w \in E$.

Definition 2.7. [11] Let $f_{\mathcal{H}}$ be a soft set over U . Then, the complement of $f_{\mathcal{H}}$ denoted by $f_{\mathcal{H}}^c$, is defined by the soft set $f_{\mathcal{H}}^c: E \rightarrow P(U)$ such that $f_{\mathcal{H}}^c(e) = U \setminus f_{\mathcal{H}}(e)$, for all $e \in E$.

From now on, Δ represents the classical symmetric difference operation, which is commutative and associative, in classical set theory. Then, the symmetric difference of the family $\mathfrak{B} = \{C_i: i \in I\}$ such that I is an index set, is denoted by

$$\Delta \mathfrak{B} = \bigtriangleup_{i \in I} C_i = C_1 \Delta C_2 \Delta \dots \Delta C_n$$

Definition 2.8. [66] Let f_G and g_G be two soft sets over U , where G is a group. Then, the soft symmetric difference-union product $f_G \otimes_{s/u} g_G$ is defined by

$$(f_G \otimes_{s/u} g_G)(x) = \bigtriangleup_{x=yz} (f_G(y) \cup g_G(z)), \quad y, z \in G$$

for all $x \in G$.

Definition 2.9. [66] Let f_G and g_G be two soft sets over U , where G is a group. Then, the soft union-difference product $f_G \otimes_{u/d} g_G$ is defined by

$$(f_G \otimes_{u/d} g_G)(x) = \bigcup_{x=yz} (f_G(y) \setminus g_G(z)), \quad y, z \in G$$

for all $x \in G$.

For additional information on soft algebraic structures and soft products, we refer to [67-78].

3. Some New Types of Soft Subsets and Soft Equalities

This section defines some new types of soft subsets and soft equalities and explores their relations. The concepts of soft L-subsets/equality and soft J-subsets/equality proposed in [51, 52] were conveyed for the revised soft set definition in [11].

Throughout this section, let K and W be two subsets of the parameter set E .

Definition 3.1. Let f_K and g_W be two soft sets over U . Then, f_K is called a soft L-subset of g_W , denoted by $f_K \subseteq_L g_W$, if, for all $e \in E$, there exists at least one $h \in E$ such that $f_K(e) = g_W(h)$. Moreover, two soft sets f_K and g_W are said to be soft L-equal, denoted by $f_K =_L g_W$, if $f_K \subseteq_L g_W$ and $g_W \subseteq_L f_K$.

It is evident that if $f_K \subseteq_L g_W$ holds in the sense of Liu et al. [52], then it also holds in the sense of Definition 3.1, provided that W is a proper subset of E .

Definition 3.2. Let f_K and g_W be two soft sets over U . Then, f_K is called a soft J-subset of g_W , denoted by $f_K \subseteq_J g_W$, if, for all $e \in E$, there exists at least one $h \in E$ such that $f_K(e) \subseteq g_W(h)$. Moreover, two soft sets f_K and g_W are said to be soft J-equal, denoted by $f_K =_J g_W$, if $f_K \subseteq_J g_W$ and $g_W \subseteq_J f_K$.

It is obvious that $f_K \subseteq_J g_W$ holds under Definition 3.2, which is based on the soft set definition of Çağman and Enginoğlu [11], if and only if it holds under the definition of J-subsets introduced by Jun and Yang [51], which relies on the soft set definition of Maji et al. [25].

Definition 3.3. Let f_K and g_W be two soft sets over U . Then, f_K is called a soft A-subset of g_W , denoted by $f_K \subseteq_A g_W$, if, for each $x, y \in E$, $f_K(x) \subseteq g_W(y)$.

In Definition 3.3, if W is a proper subset of E , then $f_K \subseteq_A g_W$ if and only if $f_K = \emptyset_E$.

Definition 3.4. Let f_K and g_W be two soft sets over U . Then, f_K is called a soft S-subset of g_W , denoted by $f_K \subseteq_S g_W$, if, for all $e \in E$, $f_K(e) = A$ and $g_W(e) = B$, where A and B are two fixed sets and $A \subseteq B$. Moreover, two soft sets f_K and g_W are said to be soft S-equal, denoted by $f_K =_S g_W$, if $f_K \subseteq_S g_W$ and $g_W \subseteq_S f_K$.

It is obvious that if $f_K =_S g_W$, then f_K and g_W are the same constant function, that is, for all $e \in E$, $f_K(e) = g_W(e) = A$, where A is a fixed set.

Note 3.5. In Definition 3.4, if W is a proper subset of E , then $f_K \subseteq_S g_W$ if and only if $f_K = g_W = \emptyset_E$. Besides, if K is a proper subset of E and $W = E$, then $f_K \subseteq_S g_E$ if and only if g_E is any constant function from E to $P(U)$. Furthermore, if K is a proper subset of E and $f_K \subseteq_S g_W$, then $f_K = \emptyset_E$. In addition, in Definition 3.4, if K or W is a proper subset of E , then $f_K =_S g_W$ if and only if $f_K = g_W = \emptyset_E$.

Definition 3.6. Let f_K and g_W be two soft sets over U . Then, g_K is called a soft S-complement of g_W , denoted by $f_K =_S (g_W)^c$, if, for all $e \in E$, $f_K(e) = A$ and $g_W(e) = B$, where A and B are two fixed sets and $A = B'$. Here, $B' = U \setminus B$.

Note 3.7. In Definition 3.6, if K is a proper subset of E , then $f_K =_S (g_W)'$ if and only if $f_K = \emptyset_E$ and $g_W = U_E$. Similarly, if W is a proper subset of E , then $f_K =_S (g_W)'$ if and only if $g_W = \emptyset_E$ and $f_K = U_E$.

The following theorem can be observed from the related definitions.

Theorem 3.8. Let f_K , g_W , f_E , and g_E be four soft sets over U . Then,

- i. $f_E \subseteq_S g_E \Rightarrow f_E \subseteq_A g_E \Rightarrow f_E \subseteq g_E$
- ii. $f_K \subseteq g_W \Rightarrow f_K \subseteq_J g_W$
- iii. $f_K \subseteq_L g_W \Rightarrow f_K \subseteq_J g_W$
- iv. $f_E =_S g_E \Rightarrow f_E = g_E$
- v. $f_K = g_W \Rightarrow f_K =_L g_W \Rightarrow f_K =_J g_W$

However, the converses may not be true.

Example 3.9. Let $E = \{r_1, r_2, r_3, r_4, r_5\}$ be a parameter set, $K = \{r_1, r_4\}$ and $W = \{r_1, r_4, r_5\}$ be two subsets of E , and $U = \{z_1, z_2, z_3, z_4, z_5\}$ be a universal set. Moreover, let

$$f_K = \{(r_1, \{z_1, z_3\}), (r_4, \{z_2, z_3, z_5\})\},$$

$$g_W = \{(r_1, \{z_1, z_3\}), (r_4, \{z_2, z_3\}), (r_5, \{z_1, z_2, z_3, z_5\})\},$$

$$h_W = \{(r_1, \{z_2, z_3, z_5\}), (r_4, \{z_1, z_3\}), (r_5, \{z_1, z_2, z_3, z_5\})\},$$

$$t_K = \{(r_1, \{z_1, z_3\}), (r_4, \{z_3, z_4\})\},$$

$$z_E = \{(r_1, \{z_1, z_3, z_4\}), (r_2, \{z_1, z_3, z_4, z_5\}), (r_3, \{z_1, z_3, z_4, z_5\}), (r_4, \{z_1, z_3, z_4\}), (r_5, \{z_1, z_2, z_3, z_4\})\},$$

$$r_E = \{(r_1, \{z_1, z_3, z_5\}), (r_2, \{z_1, z_3, z_4, z_5\}), (r_3, \{z_2, z_3, z_4\}), (r_4, \{z_2, z_3, z_5\}), (r_5, \{z_2, z_3, z_4, z_5\})\},$$

$$m_E = \{(r_1, \{z_2, z_3\}), (r_2, \{z_3\}), (r_3, \{z_2\}), (r_4, \{z_1\}), (r_5, \{z_1, z_3\})\},$$

$$y_E = \{(r_1, \{z_1, z_2, z_3\}), (r_2, \{z_1, z_2, z_3\}), (r_3, \{z_1, z_2, z_3\}), (r_4, \{z_1, z_2, z_3\}), (r_5, \{z_1, z_2, z_3\})\},$$

and

$$d_E = \{(r_1, \{z_1, z_2, z_3, z_4\}), (r_2, \{z_1, z_2, z_3, z_4\}), (r_3, \{z_1, z_2, z_3, z_4\}), (r_4, \{z_1, z_2, z_3, z_4\}), (r_5, \{z_1, z_2, z_3, z_4\})\}$$

be soft sets over U .

Since $f_K(r_1) \subseteq g_W(r_1)$, (or $f_K(r_1) \subseteq g_W(r_5)$), $f_K(r_1) \subseteq g_W(r_5)$, $f_K(r_2) = f_K(r_3) = f_K(r_5) = \emptyset$, and the empty set is a subset of any set, it is obvious that $f_K \subseteq_J g_W$. However, since $f_K(r_4) \neq g_W(r_1)$, $f_K(r_4) \neq$

$g_W(r_2), f_K(r_4) \neq g_W(r_3), f_K(r_4) \neq g_W(r_4)$, and $f_K(r_4) \neq g_W(r_5)$, we can deduce that $f_K \not\subseteq_L g_W$. Since $f_K(r_1) = h_W(r_4), f_K(r_2) = h_W(r_3), f_K(r_3) = h_W(r_2), f_K(r_4) = h_W(r_1)$, and $f_K(r_5) = h_W(r_2)$, it is obvious that $f_K \subseteq_L h_W$.

Since $t_K(r_1) \subseteq z_E(r_1), t_K(r_1) \subseteq z_E(r_2), t_K(r_1) \subseteq z_E(r_3), t_K(r_1) \subseteq z_E(r_4), t_K(r_1) \subseteq z_E(r_5), t_K(r_4) \subseteq z_E(r_1), t_K(r_4) \subseteq z_E(r_2), t_K(r_4) \subseteq z_E(r_3), t_K(r_4) \subseteq z_E(r_4), t_K(r_4) \subseteq z_E(r_5), t_K(r_2) = t_K(r_3) = t_K(r_5) = \emptyset$, and the empty set is a subset of any set, $t_K \subseteq_A z_E$. However, since $t_K(r_1) \not\subseteq r_E(r_3)$, then $t_K \not\subseteq_A r_E$.

Since $m_E(r_1) \subseteq y_E(r_1), m_E(r_1) \subseteq y_E(r_2), m_E(r_1) \subseteq y_E(r_3), m_E(r_1) \subseteq y_E(r_4), m_E(r_1) \subseteq y_E(r_5), m_E(r_2) \subseteq y_E(r_1), m_E(r_2) \subseteq y_E(r_2), m_E(r_2) \subseteq y_E(r_3), m_E(r_2) \subseteq y_E(r_4), m_E(r_2) \subseteq y_E(r_5), m_E(r_3) \subseteq y_E(r_1), m_E(r_3) \subseteq y_E(r_2), m_E(r_3) \subseteq y_E(r_3), m_E(r_3) \subseteq y_E(r_4), m_E(r_3) \subseteq y_E(r_5), m_E(r_4) \subseteq y_E(r_1), m_E(r_4) \subseteq y_E(r_2), m_E(r_4) \subseteq y_E(r_3), m_E(r_4) \subseteq y_E(r_4), m_E(r_4) \subseteq y_E(r_5), m_E(r_5) \subseteq y_E(r_1), m_E(r_5) \subseteq y_E(r_2), m_E(r_5) \subseteq y_E(r_3), m_E(r_5) \subseteq y_E(r_4)$, and $m_E(r_5) \subseteq y_E(r_5)$, then $m_E \subseteq_A y_E$. Thus, by Theorem 3.8, $m_E \subseteq y_E$ and $m_E \subseteq_J y_E$.

Furthermore, since y_E and d_E are constant functions such that $y_E(e) = \{z_1, z_2, z_3\} = A$ and $d_E(e) = \{z_1, z_2, z_3, z_4\} = B$, for all $e \in E$, where $A \subseteq B$, it is obvious that $y_E \subseteq_S d_E$.

Example 3.10. Let $E = \{r_1, r_2, r_3, r_4, r_5\}$ be a parameter set, $K = \{r_1, r_4\}$ and $W = \{r_1, r_4, r_5\}$ be two subsets of E , and $U = \{z_1, z_2, z_3, z_4, z_5\}$ be a universal set. Moreover, let

$$f_K = \{(r_1, \{z_1, z_3\}), (r_4, \{z_1, z_2, z_3, z_5\})\},$$

$$g_W = \{(r_1, \{z_1, z_2, z_3\}), (r_4, \{z_1, z_2, z_3, z_5\}), (r_5, \{z_1\})\},$$

$$x_E = \{(r_1, \{z_1, z_2, z_4\}), (r_2, \{z_1, z_2, z_4\}), (r_3, \{z_1, z_2, z_4\}), (r_4, \{z_1, z_2, z_4\}), (r_5, \{z_1, z_2, z_4\})\},$$

$$d_E = \{(r_1, \{z_1, z_2, z_4\}), (r_2, \{z_1, z_2, z_4\}), (r_3, \{z_1, z_2, z_4\}), (r_4, \{z_1, z_2, z_4\}), (r_5, \{z_1, z_2, z_4\})\},$$

$$h_E = \{(r_1, \{z_3, z_5\}), (r_2, \{z_3, z_5\}), (r_3, \{z_3, z_5\}), (r_4, \{z_3, z_5\}), (r_5, \{z_3, z_5\})\},$$

$$n_W = \{(r_1, \{z_2, z_3\}), (r_4, \{z_1, z_5\}), (r_5, \{z_4\})\},$$

and

$$l_W = \{(r_1, \{z_2, z_3\}), (r_4, \{z_1, z_5\}), (r_5, \{z_4\})\}$$

be soft sets over U .

Since $f_K(r_1) \neq g_W(r_1), f_K(r_1) \neq g_W(r_2), f_K(r_1) \neq g_W(r_3), f_K(r_1) \neq g_W(r_4)$, and $f_K(r_1) \neq g_W(r_5)$, it is obvious that $f_K \not\subseteq_L g_W$. However, since $f_K(r_1) \subseteq g_W(r_1), f_K(r_4) \subseteq g_W(r_4)$, and $f_K(r_2) = f_K(r_3) = f_K(r_5) = \emptyset$, we can deduce that $f_K \subseteq_J g_W$. Moreover, since $g_W(r_1) \subseteq f_K(r_4), g_W(r_4) \subseteq f_K(r_4), g_W(r_5) \subseteq f_K(r_1)$, and $g_W(r_2) = g_W(r_3) = \emptyset$, we can conclude that $g_W \subseteq_J f_K$. Therefore, $f_K =_J g_W$.

Since x_E and d_E are constant functions such that $x_E(e) = d_E(e) = \{z_1, z_2, z_4\} = A$, for all $e \in E$, it is obvious that $x_E =_S d_E$. Thus, $x_E = d_E$. Moreover, since x_E and h_E are constant functions such that $x_E(e) = \{z_1, z_2, z_4\} = A$ and $h_E(e) = \{z_3, z_5\} = B$, for all $e \in E$, where $A = B'$, it is obvious that $x_E =_S (h_E)^c$. It is clear that $n_W = l_W$; however, $n_W \neq_S l_W$.

4. Soft Symmetric Difference-Difference Product

In this section, we present a new product for soft sets whose parameter sets are groups, called the soft symmetric difference-difference product for soft sets. We thoroughly investigate its algebraic properties, focusing on various types of soft equalities and soft subsets and providing illustrative examples.

From now on, G denotes a group, $S_G(U)$ is the collection of all soft sets over U , whose parameter sets are G , and all soft sets in this section are elements of $S_G(U)$.

Definition 4.1. Let f_G and g_G be two soft sets over U . Then, the soft symmetric difference-difference product $f_G \otimes_{s/d} g_G$ is defined by

$$(f_G \otimes_{s/d} g_G)(x) = \bigtriangleup_{x=yz} (f_G(y) \setminus g_G(z)), \quad y, z \in G$$

for all $x \in G$.

Note here that since G is a group, there always exist $y, z \in G$ such that $x = yz$, for all $x \in G$. Let the order of the group G be n , that is, $|G| = n$. Then, it is obvious that there exist n different combinations of writing styles for each $x \in G$ such that $x = yz$, where $y, z \in G$.

Note 4.2. The soft symmetric difference-difference product is well-defined in $S_G(U)$. In fact, let $f_G, g_G, m_G, k_G \in S_G(U)$ such that $(f_G, g_G) = (m_G, k_G)$. Then, $f_G = m_G$ and $g_G = k_G$, implying that $f_G(x) = m_G(x)$ and $g_G(x) = k_G(x)$, for all $x \in G$. Thereby, for all $x \in G$,

$$\begin{aligned} (f_G \otimes_{s/d} g_G)(x) &= \bigtriangleup_{x=yz} (f_G(y) \setminus g_G(z)) \\ &= \bigtriangleup_{x=yz} (m_G(y) \setminus k_G(z)) \\ &= (m_G \otimes_{s/d} k_G)(x) \end{aligned}$$

Hence, $f_G \otimes_{s/d} g_G = m_G \otimes_{s/d} k_G$.

Example 4.3. Consider the group $G = \{a, b\}$ with the following operation:

$\cdot$	a	b
a	a	b
b	b	a

Let f_G and g_G be two soft sets over $U = D_2 = \{< x, y >: x^2 = y^2 = e, xy = yx\} = \{e, x, y, yx\}$ as follows:

$$f_G = \{(a, \{e, x, yx\}), (b, \{x, yx\})\} \text{ and } g_G = \{(a, \{e, y, yx\}), (b, \{e, y\})\}$$

Since $a = aa = bb$, $(f_G \otimes_{s/d} g_G)(a) = (f_G(a) \setminus g_G(a)) \Delta (f_G(b) \setminus g_G(b)) = \{yx\}$, and since $b = ab = ba$, $(f_G \otimes_{s/d} g_G)(b) = (f_G(a) \setminus g_G(b)) \Delta (f_G(b) \setminus g_G(a)) = \{yx\}$ is obtained. Hence,

$$f_G \otimes_{s/d} g_G = \{(a, \{yx\}), (b, \{yx\})\}$$

Proposition 4.4. The set $S_G(U)$ is closed under the soft symmetric difference-difference product. That is, if f_G and g_G are two soft sets over U , then so is $f_G \otimes_{s/d} g_G$.

PROOF. It is obvious that the soft symmetric difference-difference product is a binary operation in $S_G(U)$. Thereby, $S_G(U)$ is closed under the soft symmetric difference-difference product. ■

Proposition 4.5. The soft symmetric difference-difference product is not associative in $S_G(U)$.

PROOF. Let f_G, g_G , and h_G be three soft sets over $U = \{e, x, y, yx\}$ such that

$$f_G = \{(a, \{e, yx\}), (b, \{x, y\})\},$$

$$g_G = \{(a, \{e, x, y\}), (b, \{e, yx\})\},$$

and

$$h_G = \{(a, \{y, yx\}), (b, \{y\})\}$$

Since $f_G \otimes_{s/d} g_G = \{(a, \{x, y, yx\}), (b, \emptyset)\}$, then

$$(f_G \otimes_{s/d} g_G) \otimes_{s/d} h_G = \{(a, \{x\}), (b, \{x, yx\})\}$$

Moreover, since $g_G \otimes_{s/d} h_G = \{(a, \{x, yx\}), (b, \{x\})\}$, then

$$f_G \otimes_{s/d} (g_G \otimes_{s/d} h_G) = \{(a, \{e, y\}), (b, \{e, y, yx\})\}$$

Thereby, $(f_G \otimes_{s/d} g_G) \otimes_{s/d} h_G \neq f_G \otimes_{s/d} (g_G \otimes_{s/d} h_G)$. ■

Proposition 4.6. The soft symmetric difference-difference product is not commutative in $S_G(U)$.

PROOF. Consider the soft sets f_G and g_G in Example 4.3. Then,

$$f_G \otimes_{s/d} g_G = \{(a, \{yx\}), (b, \{yx\})\} \text{ and } g_G \otimes_{s/d} f_G = \{(a, \{e\}), (b, \{e\})\}$$

implying that $f_G \otimes_{s/d} g_G \neq g_G \otimes_{s/d} f_G$. ■

Proposition 4.7 The soft symmetric difference-difference product is not idempotent in $S_G(U)$.

PROOF. Consider the soft set f_G in Example 4.3. Then,

$$f_G \otimes_{s/d} f_G = \{(a, \emptyset), (b, \{e\})\}$$

implying that $f_G \otimes_{s/d} f_G \neq f_G$. ■

Proposition 4.8. $\emptyset_G$ is the left absorbing element of the soft symmetric difference-difference product in $S_G(U)$.

PROOF. Let $x \in G$. Then,

$$\begin{aligned} (\emptyset_G \otimes_{s/d} f_G)(x) &= \bigtriangleup_{x=yz} (\emptyset_G(y) \setminus f_G(z)) \\ &= \bigtriangleup_{x=yz} (\emptyset \setminus f_G(z)) \\ &= \emptyset_G(x) \end{aligned}$$

Thus, $\emptyset_G \otimes_{s/d} f_G = \emptyset_G$. ■

Proposition 4.9. $\emptyset_G$ is not the right absorbing element of the soft symmetric difference-difference product in $S_G(U)$.

PROOF. Consider the soft set f_G in Example 4.3. Then,

$$f_G \otimes_{s/d} \emptyset_G = \{(a, \{e\}), (b, \{e\})\}$$

implying that $f_G \otimes_{s/d} \emptyset_G \neq \emptyset_G$. ■

Remark 4.10. $\emptyset_G$ is not the absorbing element of the soft symmetric difference-difference product in $S_G(U)$.

Proposition 4.11. Let f_G and g_G be two soft sets over U . If one of the following assertions is satisfied, then $f_G \otimes_{s/d} g_G = \emptyset_G$:

- i. $f_G \subseteq_A g_G$
- ii. $g_G \subseteq_S f_G$ and $|G| = k$, where k is a positive even integer
- iii. $f_G =_S g_G$
- iv. $f_G =_S (g_G)^c$ and $|G| = k$, where k is a positive even integer
- v. $g_G = U_G$

PROOF. Let f_G and g_G be two soft sets over U .

i. Suppose that $f_G \tilde{\subseteq}_A g_G$. Then, $f_G(a) \subseteq g_G(b)$, for each $a, b \in G$. Thus, for all $x \in G$,

$$(f_G \otimes_{s/d} g_G)(x) = \bigtriangleup_{x=yz} (f_G(y) \setminus g_G(z)) = \emptyset = \emptyset_G(x)$$

Thereby, $f_G \otimes_{s/d} g_G = \emptyset_G$.

ii. Let $g_G \tilde{\subseteq}_S f_G$ and $|G| = k$, where k is a positive even integer. Hence, for all $a \in G$, $f_G(a) = A$ and $g_G(a) = B$, where A and B are two fixed sets and $B \subseteq A$. Thus, for all $x \in G$,

$$(f_G \otimes_{s/d} g_G)(x) = \bigtriangleup_{x=yz} (f_G(y) \setminus g_G(z)) = \underbrace{(A \setminus B) \Delta (A \setminus B) \Delta \dots \Delta (A \setminus B)}_{k \text{ times } A \setminus B, \text{ where } k \text{ is even}} = \emptyset = \emptyset_G(x)$$

Hence, $f_G \otimes_{s/d} g_G = \emptyset_G$.

iii. Let $f_G =_S g_G$. Then, for all $x \in G$, $f_G(x) = A$ and $g_G(x) = B$, where A and B are two fixed sets and $A = B$. Thus, for all $x \in G$,

$$(f_G \otimes_{s/d} g_G)(x) = \bigtriangleup_{x=yz} (f_G(y) \setminus g_G(z)) = \emptyset = \emptyset_G(x)$$

Hence, $f_G \otimes_{s/d} g_G = \emptyset_G$.

iv. Let $f_G =_S (g_G)^c$ and $|G| = k$, where k is a positive even integer. Then, for all $x \in G$, $f_G(x) = A$ and $g_G(x) = B$, where A and B are two fixed sets and $A = B'$. Thus, for all $x \in G$,

$$\begin{aligned} (f_G \otimes_{s/d} g_G)(x) &= \bigtriangleup_{x=yz} (f_G(y) \setminus g_G(z)) \\ &= \underbrace{A \Delta A \Delta \dots \Delta A}_{k \text{ times } A, \text{ where } k \text{ is even}} \\ &= \emptyset_G(x) \end{aligned}$$

Thereby, $f_G \otimes_{s/d} g_G = \emptyset_G$.

v. Let $g_G = U_G$. Then, for all $x \in G$, $g_G(x) = U$. Thus, for all $x \in G$,

$$\begin{aligned} (f_G \otimes_{s/d} g_G)(x) &= \bigtriangleup_{x=yz} (f_G(y) \setminus g_G(z)) \\ &= \bigtriangleup_{x=yz} (f_G(y) \setminus U_G(z)) \\ &= \bigtriangleup_{x=yz} (f_G(y) \setminus U) \\ &= \emptyset_G(x) \end{aligned}$$

Hence, $f_G \otimes_{s/d} g_G = \emptyset_G$. ■

Proposition 4.12. Let f_G and g_G be two soft sets over U . Then, $f_G \otimes_{s/d} g_G \tilde{\subseteq} f_G \otimes_{u/d} g_G$.

PROOF. Let f_G and g_G be two soft sets over U . Then, for all $x \in G$,

$$\begin{aligned} (f_G \otimes_{s/d} g_G)(x) &= \bigtriangleup_{x=yz} (f_G(y) \setminus g_G(z)) \\ &\subseteq \bigcup_{x=yz} (f_G(y) \setminus g_G(z)) \\ &= (f_G \otimes_{u/d} g_G)(x) \end{aligned}$$

Thus, $f_G \otimes_{s/d} g_G \tilde{\subseteq} f_G \otimes_{u/d} g_G$. ■

Proposition 4.13. Let f_G and g_G be two soft sets over U . If one of the following assertions is satisfied, then $f_G \otimes_{s/d} g_G = f_G \otimes_{u/d} g_G$:

- i. $f_G \subseteq_A g_G$
- ii. $g_G \subseteq_S f_G$ and $|G| = k$, where k is a positive odd integer
- iii. $f_G =_S g_G$

PROOF. Let f_G and g_G be two soft sets over U .

- i. Assume that $f_G \subseteq_A g_G$. Then, $f_G(a) \subseteq g_G(b)$, for each $a, b \in G$. Thus, for all $x \in G$,

$$\begin{aligned} (f_G \otimes_{s/d} g_G)(x) &= \bigtriangleup_{x=yz} (f_G(y) \setminus g_G(z)) \\ &= \emptyset \\ &= \bigcup_{x=yz} (f_G(y) \setminus g_G(z)) \\ &= (f_G \otimes_{u/d} g_G)(x) \end{aligned}$$

Hence, $f_G \otimes_{s/d} g_G = f_G \otimes_{u/d} g_G$.

- ii. $g_G \subseteq_S f_G$ and $|G| = k$, where k is a positive odd integer. Hence, for all $a \in G$, $f_G(a) = A$ and $g_G(a) = B$, where A and B are two fixed sets and $B \subseteq A$. Thus, for all $x \in G$,

$$\begin{aligned} (f_G \otimes_{s/d} g_G)(x) &= \bigtriangleup_{x=yz} (f_G(y) \setminus g_G(z)) \\ &= \bigcup_{x=yz} (f_G(y) \setminus g_G(z)) \\ &= (f_G \otimes_{u/d} g_G)(x) \end{aligned}$$

Then, $f_G \otimes_{s/d} g_G = f_G \otimes_{u/d} g_G$.

- iii. Let $f_G =_S g_G$. Thus, for all $x \in G$, $f_G(x) = A$ and $g_G(x) = B$, where A and B are two fixed sets and $A = B$. Hence, for all $x \in G$,

$$\begin{aligned} (f_G \otimes_{s/d} g_G)(x) &= \bigtriangleup_{x=yz} (f_G(y) \setminus g_G(z)) \\ &= \bigcup_{x=yz} (f_G(y) \setminus g_G(z)) \\ &= (f_G \otimes_{u/d} g_G)(x) \end{aligned}$$

Thereby, $f_G \otimes_{s/d} g_G = f_G \otimes_{u/d} g_G$. ■

Proposition 4.14. Let f_G and g_G be two soft sets over U . If one of the following assertions is satisfied, then $f_G \otimes_{s/d} g_G = f_G \otimes_{s/u} g_G$.

- i. $f_G \subseteq_S g_G$ and $|G| = k$, where k is a positive even integer
- ii. $g_G \subseteq_S f_G$ and $|G| = k$, where k is a positive even integer
- iii. $g_G = \emptyset_G$
- iv. $f_G =_S (g_G)^c$ and $|G| = k$, where k is a positive even integer
- v. $g_G = U_G$ and $|G| = k$, where k is a positive even integer

PROOF. Let f_G and g_G be two soft sets over U .

i. Suppose that $f_G \tilde{\subseteq}_S g_G$ and $|G| = k$, where k is a positive even integer. Hence, for all $a \in G$, $f_G(a) = A$ and $g_G(a) = B$, where A and B are two fixed sets and $A \subseteq B$. Thus, for all $x \in G$,

$$\begin{aligned}(f_G \otimes_{s/d} g_G)(x) &= \bigtriangleup_{x=yz} (f_G(y) \setminus g_G(z)) \\ &= \emptyset \\ &= \bigtriangleup_{x=yz} (f_G(y) \cup g_G(z)) \\ &= (f_G \otimes_{s/u} g_G)(x)\end{aligned}$$

Thereby, $f_G \otimes_{s/d} g_G = f_G \otimes_{s/u} g_G$.

ii. Suppose that $g_G \tilde{\subseteq}_S f_G$ and $|G| = k$, where k is a positive even integer. Hence, for all $a \in G$, $f_G(a) = A$ and $g_G(a) = B$, where A and B are two fixed sets and $B \subseteq A$. Thus, for all $x \in G$,

$$\begin{aligned}(f_G \otimes_{s/d} g_G)(x) &= \bigtriangleup_{x=yz} (f_G(y) \setminus g_G(z)) \\ &= \emptyset \\ &= \bigtriangleup_{x=yz} (f_G(y) \cup g_G(z)) \\ &= (f_G \otimes_{s/u} g_G)(x)\end{aligned}$$

Thereby, $f_G \otimes_{s/d} g_G = f_G \otimes_{s/u} g_G$.

iii. Let $g_G = \emptyset_G$. Then, for all $x \in G$, $g_G(x) = \emptyset$. Thus, for all $x \in G$,

$$\begin{aligned}(f_G \otimes_{s/d} g_G)(x) &= \bigtriangleup_{x=yz} (f_G(y) \setminus g_G(z)) \\ &= \bigtriangleup_{x=yz} (f_G(y) \setminus \emptyset_G(z)) \\ &= \bigtriangleup_{x=yz} (f_G(y) \cup \emptyset_G(z)) \\ &= \bigtriangleup_{x=yz} (f_G(y) \cup g_G(z)) \\ &= (f_G \otimes_{s/u} g_G)(x)\end{aligned}$$

Hence, $f_G \otimes_{s/d} g_G = f_G \otimes_{s/u} g_G$.

iv. Let $f_G =_S (g_G)^c$ and $|G| = k$, where k is a positive even integer. Then, for all $x \in G$, $f_G(x) = A$ and $g_G(x) = B$, where A and B are two fixed sets and $A = B'$. Thus, for all $x \in G$,

$$\begin{aligned}(f_G \otimes_{s/d} g_G)(x) &= \bigtriangleup_{x=yz} (f_G(y) \setminus g_G(z)) \\ &= \emptyset \\ &= \bigtriangleup_{x=yz} (f_G(y) \cup g_G(z)) \\ &= (f_G \otimes_{s/u} g_G)(x)\end{aligned}$$

Thereby, $f_G \otimes_{s/d} g_G = f_G \otimes_{s/u} g_G$.

v. Let $g_G = U_G$ and $|G| = k$, where k is a positive even integer. Then, for all $x \in G$, $g_G(x) = U$. Thus, for all $x \in G$,

$$\begin{aligned}
(f_G \otimes_{s/d} g_G)(x) &= \bigtriangleup_{x=yz} (f_G(y) \setminus U_G(z)) \\
&= \bigtriangleup_{x=yz} (f_G(y) \setminus U) \\
&= \emptyset \\
&= \bigtriangleup_{x=yz} (f_G(y) \cup U_G(z)) \\
&= \bigtriangleup_{x=yz} (f_G(y) \cup g_G(z)) \\
&= (f_G \otimes_{s/u} g_G)(x)
\end{aligned}$$

Hence, $f_G \otimes_{s/d} g_G = f_G \otimes_{s/u} g_G$. ■

Proposition 4.15. The soft symmetric difference-difference product distributes over the soft symmetric difference operation from the right side.

PROOF. Let f_G, g_G , and h_G be three soft sets over U . Then, for all $x \in G$,

$$\begin{aligned}
((f_G \tilde{\Delta} g_G) \otimes_{s/d} h_G)(x) &= \bigtriangleup_{x=yz} \left((f_G \tilde{\Delta} g_G)(y) \setminus h_G(z) \right) \\
&= \bigtriangleup_{x=yz} \left((f_G(y) \Delta g_G(y)) \setminus h_G(z) \right) \\
&= \bigtriangleup_{x=yz} \left((f_G(y) \setminus h_G(z)) \Delta (g_G(y) \setminus h_G(z)) \right) \\
&= \left[\bigtriangleup_{x=yz} (f_G(y) \setminus h_G(z)) \right] \Delta \left[\bigtriangleup_{x=yz} (g_G(y) \setminus h_G(z)) \right] \\
&= (f_G \otimes_{s/d} h_G)(x) \Delta (g_G \otimes_{s/d} h_G)(x) \\
&= ((f_G \otimes_{s/d} h_G) \tilde{\Delta} (g_G \otimes_{s/d} h_G))(x)
\end{aligned}$$

Thus, $(f_G \tilde{\Delta} g_G) \otimes_{s/d} h_G = (f_G \otimes_{s/d} h_G) \tilde{\Delta} (g_G \otimes_{s/d} h_G)$. ■

Example 4.16. Consider the soft sets f_G and g_G in Example 4.3. Let h_G be a soft set over U as follows:

$$h_G = \{(a, \{e, yx\}), (b, \{x, y\})\}$$

Since $f_G \otimes_{s/d} h_G = \{(a, \{x, yx\}), (b, \{e, x, yx\})\}$ and $g_G \otimes_{s/d} h_G = \{(a, \{e, y\}), (b, \{e, y, yx\})\}$, then

$$(f_G \otimes_{s/d} h_G) \tilde{\Delta} (g_G \otimes_{s/d} h_G) = \{(a, U), (b, \{x, y\})\}$$

Moreover, since $f_G \tilde{\Delta} g_G = \{(a, \{x, y\}), (b, U)\}$,

$$(f_G \tilde{\Delta} g_G) \otimes_{s/d} h_G = \{(a, U), (b, \{x, y\})\}$$

Thus, $(f_G \tilde{\Delta} g_G) \otimes_{s/d} h_G = (f_G \otimes_{s/d} h_G) \tilde{\Delta} (g_G \otimes_{s/d} h_G)$. ■

5. Conclusion

The concepts of soft L-subsets/equality and soft J-subsets/equality were proposed for the soft sets defined by Maji et al. [25]; however, they have not been adapted for the revised soft set definition of Çağman and Enginoğlu [11]. In this study, we first conveyed these concepts for the revised soft set definition. Additionally, we defined some new types of soft subsets and equalities, called soft S-subsets/equality and soft A-subsets, along with their specific examples to clarify these concepts. We investigated the connections among these new concepts. This paper further presents an innovative product for soft sets whose parameter sets are groups,

called the "soft symmetric difference-difference-product". We thoroughly analyzed its fundamental algebraic properties, considering various soft subsets and equality relations to inspire future research, which may lead to a new soft group theory arising from this concept. Future research may focus on proposing additional types of soft product operations and further investigating the fundamental characteristics of various soft equal relations, thereby enriching the theoretical and practical understanding of soft sets by proposing new soft group theory.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflict of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

References

- [1] D. A. Molodtsov, *Soft set theory-first results*, Computers and Mathematics with Applications 37 (4-5) (1999) 19–31.
- [2] L. A. Zadeh, *Fuzzy sets*, Information Control 8 (1965) 338–353.
- [3] P. K. Maji, A. R. Roy, R. Biswas, *An application of soft sets in a decision making problem*, Computers and Mathematics with Applications 44 (8-9) (2002) 1077–1083.
- [4] D. G. Chen, E. C. C. Tsang, D. S. Yeung, *Some notes on the parameterization reduction of soft sets*, in: J. Chen (Ed.), Proceedings of the 2003 International Conference on Machine Learning and Cybernetics, Xi'an, 2003, pp. 1442–1445.
- [5] D. G. Chen, E. C. C. Tsang, X. Wang, *The parametrization reduction of soft sets and its applications*, Computers and Mathematics with Applications 49 (5–6) (2005) 757–763.
- [6] Z. Xiao, L. Chen, B. Zhong, S. Ye, *Recognition for soft information based on the theory of soft sets*, in: J. Chen (Ed.), IEEE proceedings of International Conference on Services Systems and Services Management, Ghent, 2005, pp. 1104–1106.
- [7] M. M. Mushrif, S. Sengupta, A. K. Ray, *Texture classification using a novel, soft-set theory based classification algorithm*, in: P. J. Narayanan, S. K. Nayar, H. T. Shum (Eds.), Computer Vision -ACCV 2006, Lecture Notes in Computer Science, vol 3851. Springer, Berlin.
- [8] M. T. Herawan, M. M. Deris, *A direct proof of every rough set is a soft set*, Third Asia International Conference on Modelling and Simulation, Bundang, 2009, pp. 119–124.
- [9] M. T. Herawan, M. M. Deris, *Soft decision making for patients suspected influenza*, In: D. Taniar, O. Gervasi, B. Murgante, E. Pardede, B. O. Apduhan (Eds.), Computational Science and Its Applications - ICCSA 2010. Lecture Notes in Computer Science, vol 6018, Springer, Berlin.
- [10] T. Herawan, *Soft set-based decision making for patients suspected influenza-like illness*, International Journal of Modern Physics: Conference Series 1, 2005, 1–5.

- [11] N. Çağman, S. Enginoğlu, *Soft set theory and uni-int decision making*, European Journal of Operational Research 207 (2) (2010) 848–855.
- [12] N. Çağman, S. Enginoğlu, *Soft matrix theory and its decision making*, Computers and Mathematics with Applications 59 (10) (2010) 3308–3314.
- [13] X. Gong, Z. Xiao, X. Zhang, *The bijective soft set with its operations*, Computers and Mathematics with Applications 60 (8) (2010) 2270–2278.
- [14] Z. Xiao, K. Gong, S. Xia, Y. Zou, *Exclusive disjunctive soft sets*, Computers and Mathematics with Applications 59 (6) (2010) 2128–2137.
- [15] F. Feng, Y. Li, N. Çağman, *Generalized uni-int decision making schemes based on choice value soft sets*, European Journal of Operational Research 220 (1) (2012) 162–170.
- [16] Q. Feng, Y. Zhou, *Soft discernibility matrix and its applications in decision making*, Applied Soft Computing (24) (2014) 749–756.
- [17] A. Kharal, *Soft approximations and uni-int decision making*, The Scientific World Journal 4 (2014) 327408.
- [18] M. K. Dauda, M. Mamat, M. Y. Waziri, *An application of soft set in decision making*, Jurnal Teknologi 77 (13) (2015) 119–122.
- [19] V. Inthumathi, V. Chitra, S. Jayasree, *The role of operators on soft set in decision making problems*, International Journal of Computational and Applied Mathematics 12 (3) (2017) 899–910.
- [20] A. O. Atagün, H. Kamacı, O. Oktay, *Reduced soft matrices and generalized products with applications in decision making*, Neural Computing and Applications (29) (2018) 445–456.
- [21] H. Kamacı, K. Saltık, H. F. Akız, A. O. Atagün, *Cardinality inverse soft matrix theory and its applications in multicriteria group decision making*, Journal of Intelligent and Fuzzy Systems 34 (3) (2018) 2031–2049.
- [22] J. L. Yang, Y. Y. Yao, *Semantics of soft sets and three-way decision with soft sets*, Knowledge-Based Systems 194 (2020) 105538.
- [23] S. Petchimuthu, H. Garg, H. Kamacı, A. O. Atagün, *The mean operators and generalized products of fuzzy soft matrices and their applications in MCGDM*, Computational and Applied Mathematics, 39 (2) (2020) 1–32.
- [24] İ. Zorlutuna, *Soft set-valued mappings and their application in decision making problems*, Filomat 35 (5) (2021) 1725–1733.
- [25] P. K. Maji, R. Biswas, A. R. Roy, *Soft set theory*, Computers and Mathematics with Applications 45 (1) (2003) 555–562.
- [26] D. Pei, D. Miao, *From soft sets to information systems*, in: X. Hu, Q. Liu, A. Skowron, T.Y. Lin, R.R. Yager, B. Zhang (Eds.), Proceedings of Granular Computing (2) IEEE, Beijing, 2005, pp. 617–621.
- [27] M. I. Ali, F. Feng, X. Liu, W. K. Min, M. Shabir, *On some new operations in soft set theory*, Computers and Mathematics with Applications 57 (9) (2009) 1547–1553.
- [28] C. F. Yang, *A note on: "Soft set theory"*, Computers and Mathematics with Applications 56 (7) (2008) 1899–1900.
- [29] F. Feng, Y. M. Li, B. Davvaz, M. I. Ali, *Soft sets combined with fuzzy sets and rough sets: A tentative approach*, Soft Computing 14 (2010) 899–911.

- [30] Y. Jiang, Y. Tang, Q. Chen, J. Wang, S. Tang, *Extending soft sets with description logics*, Computers and Mathematics with Applications 59 (6) (2010) 2087–2096.
- [31] M. I. Ali, M. Shabir, M. Naz, *Algebraic structures of soft sets associated with new operations*, Computers and Mathematics with Applications 61 (9) (2011) 2647–2654.
- [32] C. F. Yang, *A note on soft set theory*, Computers and Mathematics with Applications 56 (7) (2008) 1899–1900.
- [33] I. J. Neog, D. K. Sut, *A new approach to the theory of soft set*, International Journal of Computer Applications 32 (2) (2011) 1–6.
- [34] L. Fu, *Notes on soft set operations*, ARPN Journal of Systems and Software 1 (6) (2011) 205–208.
- [35] X. Ge, S. Yang, *Investigations on some operations of soft sets*, Engineering and Technology International Journal of Mathematical and Computational Sciences 5 (3) (2011) 370–373.
- [36] D. Singh, I. A. Onyeozili, *Some conceptual misunderstanding of the fundamentals of soft set theory*, ARPN Journal of Systems and Software 2 (9) (2012) 251–254.
- [37] D. Singh, I. A. Onyeozili, *Some results on distributive and absorption properties of soft operations*, IOSR Journal of Mathematics 4 (2) (2012) 18–30.
- [38] D. Singh, I. A. Onyeozili, *On some new properties of soft set operations*, International Journal of Computer Applications 59 (4) (2012) 39–44.
- [39] D. Singh, I. A. Onyeozili, *Notes on soft matrices operations*, ARPN Journal of Science and Technology 2 (9) (2012) 861–869.
- [40] P. Zhu, Q. Wen, *Operations on soft sets revisited*, Journal of Applied Mathematics 2013 (2013) 1–7.
- [41] J. Sen, *On algebraic structure of soft sets*, Annals of Fuzzy Mathematics and Informatics 7 (6) (2014) 1013–1020.
- [42] Ö. F. Eren, *On operations of soft sets*, Master's Thesis Ondokuz Mayıs University (2019) Samsun.
- [43] N. S. Stojanovic, *A new operation on soft sets: Extended symmetric difference of soft sets*, Military Technical Courier 69 (4) (2021) 779–791.
- [44] A. Sezgin, N. Çağman, A. O. Atagün, F. N. Aybek, *Complemental binary operations of sets and their application to group theory*, Matrix Science Mathematic 7 (2) (2023) 114–121.
- [45] A. Sezgin, K. Dagtoros, *Complementary soft binary piecewise symmetric difference operation: A novel soft set operation*, Scientific Journal of Mehmet Akif Ersoy University 6 (2) (2023) 31–45.
- [46] A. Sezgin, H. Çalışıcı, *A comprehensive study on soft binary piecewise difference operation*, Eskişehir Technical University Journal of Science and Technology B- Theoretical Sciences 12 (1) (2024) 32–54.
- [47] A. Sezgin, E. Yavuz, *Soft binary piecewise plus operation: A new type of operation for soft sets*, Uncertainty Discourse and Applications 1 (1) (2024) 79–100.
- [48] A. Sezgin, M. Sarılioğlu, *New soft set operation: Complementary soft binary piecewise star operation*, Natural and Applied Sciences Journal 7 (1) (2024) 15–44.
- [49] A. Sezgin, E. Şenyiğit, *A new product for soft sets with its decision-making: Soft star-product*, Big Data and Computing Visions 5 (1) (2025) 52–73.
- [50] K. Y. Qin, Z. Y. Hong, *On soft equality*, Journal of Computational and Applied Mathematics 234 (5) (2010) 1347–1355.

- [51] Y. B. Jun, X. Yang, *A note on the paper "Combination of interval-valued fuzzy set and soft set"*, Computers and Mathematics with Applications 61 (5) (2011) 1468–1470.
- [52] X. Y. Liu, F. F. Feng, Y. B. Jun, *A note on generalized soft equal relations*, Computers and Mathematics with Applications 64 (4) (2012) 572–578.
- [53] F. Feng, L. Yongming, *Soft subsets and soft product operations*, Information Sciences 232 (2013) 44–57.
- [54] M. Abbas, B. Ali, S. Romaguer, *On generalized soft equality and soft lattice structure*, Filomat 28 (6) (2014) 1191–1203.
- [55] M. Abbas, M. I. Ali, S. Romaguera, *Generalized operations in soft set theory via relaxed conditions on parameters*, Filomat 31 (19) (2017) 5955–5964.
- [56] T. Alshami, *Investigation and corrigendum to some results related to g-soft equality and g f-soft equality relations*, Filomat 33 (11) (2019) 3375–3383.
- [57] T. Alshami, M. El-Shafei, *T-soft equality relation*, Turkish Journal of Mathematics 44 (4) (2020) 1427–1441.
- [58] B. Ali, N. Saleem, N. Sundus, S. Khaleeq, M. Saeed, R. A. George, *Contribution to the theory of soft sets via generalized relaxed operations*, Mathematics 10 (15) (2022) 26–36.
- [59] A. Sezgin, A. O. Atagün, N. Çağman, *A complete study on and-product of soft sets*, Sigma Journal of Engineering and Natural Sciences 43 (1) (2025) 1–14.
- [60] A. S. Sezer, *A new view to ring theory via soft union rings, ideals and bi-ideals*, Knowledge-Based Systems 36 (2012) 300–314.
- [61] A. Sezgin, *A new approach to semigroup theory I: Soft union semigroups, ideals and bi-ideals*, Algebra Letters 2016 (2016) 3 1–46.
- [62] E. Muştuoğlu, A. Sezgin, Z. K. Türk, *Some characterizations on soft uni-groups and normal soft uni-groups*, International Journal of Computer Applications 155 (10) (2016) 1–8.
- [63] K. Kaygisiz, *On soft int-groups*, Annals of Fuzzy Mathematics and Informatics 4 (2) (2012) 363–375.
- [64] A. S. Sezer, N. Çağman, A. O. Atagün, M. I. Ali, E. Türkmen, *Soft intersection semigroups, ideals and bi-ideals; A new application on semigroup theory I*, Filomat 29 (5) (2015) 917–946.
- [65] A. Sezgin, N. Çağman, A. O. Atagün, *A completely new view to soft intersection rings via soft uni-int product*, Applied Soft Computing 54 (2017) 366–392.
- [66] İ. Durak, *Soft intersection-union, soft intersection-symmetric difference product and the constructions of soft symmetrical difference groups*, Unpublished Master's Thesis Amasya University (2025) Amasya.
- [67] A. Khan, M. Izhar, A. Sezgin, *Characterizations of Abel Grassmann's groupoids by the properties of double-framed soft ideals*, International Journal of Analysis and Applications 15 (1) (2017) 62–74.
- [68] A. O. Atagün, A. Sezgin, *Soft sets, soft semimodules and soft substructures of semimodules*, Mathematical Sciences Letters 4 (3) (2015) 235.
- [69] A. S. Sezer, A. O. Atagün, N. Çağman, *N-group SI-action and its applications to N-Group Theory*, Fasciculi Mathematici 52 (2014) 139–153.
- [70] A. O. Atagün, A. Sezgin, *Int-soft substructures of groups and semirings with applications*, Applied Mathematics and Information Sciences 11 (1) (2017) 105–113.

- [71] M. Gulistan, F. Feng, M. Khan, A. Sezgin, *Characterizations of right weakly regular semigroups in terms of generalized cubic soft sets*, Mathematics (6) (2018) 293.
- [72] A. S. Sezer, A. O. Atagün, N. Çağman. *A new view to N-group theory: Soft N-groups*, Fasciculi Mathematici 51 (2013) 123–140.
- [73] C. Jana, M. Pal, F. Karaaslan, A. Sezgin, *(α, β) -Soft intersectional rings and ideals with their applications*, New Mathematics and Natural Computation 15 (02) (2019) 333–350.
- [74] A. O. Atagün, H. Kamacı, İ. Taştekin, A. Sezgin, *P-properties in near-rings*, Journal of Mathematical and Fundamental Sciences 51 (2) (2019) 152–167.
- [75] A. Sezgin, M. Orbay, *Analysis of semigroups with soft intersection ideals*, Acta Universitatis Sapientiae, Mathematica, 14 (1) (2022) 166–210.
- [76] A. O. Atagün, A. Sezgin, *A new view to near-ring theory: Soft near-rings*, South East Asian Journal of Mathematics and Mathematical Sciences 14 (3) (2018) 1–14.
- [77] T. Manikantan, P. Ramasany, A. Sezgin, *Soft quasi-ideals of soft near-rings*, Sigma Journal of Engineering and Natural Science 41 (3) (2023) 565–574.
- [78] K. Naeem, *Soft set theory and Soft sigma algebras*, LAP LAMBERT Academic Publishing, 2017.